Generalized Closed World Assumption is Π^0_2-complete *

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1 Introduction

Minker [10] has defined an inference rule called the Generalized Closed World Assumption (GCWA). According to this rule, a negated ground atom $\neg A$ can be inferred from a *non-Horn* (called also *disjunctive* or *indefinite*) logic program P iff A is not in any minimal Herbrand model of P. GCWA, as opposed to CWA ([13]), does not lead to inconsistency and has been adopted as a standard rule for inferring negative information from a disjunctive logic program [18, 4, 17, 8, 12, 7]).

In this note, we show that GCWA is Π^0_2-complete, i.e. at the second level of the arithmetical hierarchy [14]. The non-obvious part is Π^0_2-hardness. Therefore, GCWA is strictly harder than CWA which is Π^0_1-complete [2] for both Horn and non-Horn logic programs. Moreover, GCWA is strictly harder than a weaker inference rule called Weak GCWA in [11] and Disjunctive Database Rule in [15] which, like CWA, is Π^0_1-complete. Weak GCWA is Π^0_1-hard, because it reduces to CWA for Horn programs and is in Π^0_1, because of its syntactic characterization [11]. On the other hand, GCWA is strictly easier than general circumscription which is Π^1_2-complete as shown by Schlipf [16].

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2 Preliminaries

Definite Horn logic programs [9] consist of universally-quantified clauses with exactly one positive literal. Non-Horn logic programs [10, 6] consist of universally-quantified clauses with at least one positive literal.

Reiter [13] defined the *Closed World Assumption* (CWA) as the following inference rule:

$$\neg A \in CWA(P) \text{ iff } P \nvdash A$$

where P is a logic program and A is a ground atom belonging to the Herbrand base of P. For any definite Horn logic program P, $CWA(P)$ can be defined equivalently as:

$$\neg A \in CWA(P) \text{ iff } M_P \not\models A$$

where M_P is the least Herbrand model of P. In that case, $P \cup CWA(P)$ is consistent. Moreover, it is maximally consistent. However, in the case of non-Horn logic programs, $P \cup CWA(P)$ may be inconsistent.

Minker [10] defined the *Generalized Closed World Assumption* (GCWA) as the following inference rule:

$$\neg A \in GCWA(P) \text{ iff } \forall K,\ P \vdash A \vee K \Rightarrow P \vdash K$$

where P is a logic program, A is a ground atom, K is a (possibly empty) disjunction of ground atoms, and A and all the disjuncts in K belong to the Herbrand base of P. He also showed that the above definition has a model-theoretic counterpart:

$$\neg A \in GCWA(P) \text{ iff } A \text{ does not belong to any minimal Herbrand model of } P.$$

For any (Horn or non-Horn) logic program P, $P \cup GCWA(P)$ is consistent. Moreover, it is maximally consistent in the sense that adding more negative literals would result either in an inconsistency or in a new positive conclusion being derivable.

3 Main result

Theorem 3.1 *$GCWA(P)$ is a Π^0_2-complete set.*

Proof: We show first that $GCWA(P)$ is in Π^0_2. The complementary problem:

$$\neg A \notin GCWA(P) \text{ iff } \exists K,\ P \vdash A \vee K \textit{ and } P \nvdash K$$

is recursively enumerable in Π^0_1, and therefore is in Σ^0_2.

We now show Π^0_2-hardness. Take an arbitrary Π^0_2 subset Q of some finitely generated Herbrand universe U that contains at least one constant and one function symbol. We will exhibit a non-Horn logic program P over the same Herbrand universe U such that

$$\neg c(x) \in GCWA(P) \text{ iff } Q(x)$$

for all x and some predicate symbol c.

By the definition of Π^0_2, there is a recursively enumerable relation R over U such that:

$$Q(x) \text{ iff } \forall y,\ R(x,y).$$

For this relation R, there is a definite Horn logic program S such that:

$$R(x,y) \text{ iff } S \vdash r(x,y)$$

by a result of Andreka and Nemeti [1] (also [2, Theorem 7] and [5]).

We define the non-Horn logic program P postulated at the beginning of the proof in several steps.

First, we include the program S in P. Second, we introduce three new predicate symbols a,b and c, not appearing in S and defined by the following rules, the second of which is non-Horn:

$a(X,Y) \leftarrow r(X,Y).$
$a(X,Y) \vee b(X,Y).$
$c(X) \leftarrow b(X,Y).$

We will show now that for all x:

$$c(x) \text{ is not in any minimal Herbrand model of } P \text{ iff } \forall y,\ S \vdash r(x,y)$$

which is equivalent to:

$$c(x) \text{ is in a minimal Herbrand model of } P \text{ iff } \exists y,\ S \not\vdash r(x,y)$$

and establishes the hardness result.

Assume first that

$$\forall y,\ S \vdash r(x,y).$$

We will assume that $c(x)$ is in a minimal Herbrand model M_0 of P and derive a contradiction. If $c(x)$ is in a minimal Herbrand model M_0 of P, then:

$$\exists y,\ M_0 \models b(x,y).$$

Let y_0 be such a y. Now because S is definite Horn, the original assumption implies that

$$\forall y,\ M_S \models r(x,y)$$

where M_S is the least Herbrand model of S. Thus also:

$$\forall y,\ M \models r(x,y)$$

for any minimal Herbrand model M of P, because every such model has to contain M_S. Consequently:

$$\forall y,\ M_0 \models a(x,y)$$

and because M_0 is a minimal model:

$\forall y,\ M_0 \models \neg b(x, y).$

This contradicts the fact that

$M_0 \models b(x, y_0).$

We prove now the opposite direction. Assume that:

$\exists y,\ S \not\models r(x, y).$

Let y_1 be such a y. We will prove that $c(x)$ is in a minimal Herbrand model of P.
Let P' result from P by syntactically replacing

$a(X, Y) \vee b(X, Y).$

by

$b(X, Y) \leftarrow \neg a(X, Y).$

The program P' is stratified [3] with two strata and has a unique perfect Herbrand model $M_{P'}$ which is a superset of the least Herbrand model M_P. The perfect model $M_{P'}$ is a minimal model of P [3], thus:

$M_{P'} \not\models a(x, y_1).$

and because $M_{P'}$ is a model of P:

$M_{P'} \models b(x, y_1).$

Therefore, also:

$M_{P'} \models c(x).$

It is easy to see that the transformation leading from S to P is total and recursive. Therefore, given any Π_2^0 set, we have shown how to construct a non-Horn logic program P such that:

$\neg A \in GCWA(P)$ iff $A \in X$.

End of proof.

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