Bayesian Decision Theory

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Overview and Plan

- Covering Chapter 2 of DHS.
- Bayesian Decision Theory is a fundamental statistical approach to the problem of pattern classification.
- Quantifies the tradeoffs between various classifications using probability and the costs that accompany such classifications.
- Assumptions:
  - Decision problem is posed in probabilistic terms.
  - All relevant probability values are known.

Recall the Fish!

- Recall our example from the first lecture on classifying two fish as salmon or sea bass.
- And recall our agreement that any given fish is either a salmon or a sea bass; DHS call this the **state of nature** of the fish.
- Let's define a (probabilistic) variable ω that describes the state of nature.

$$\omega = \omega_1 \quad \text{for sea bass} \tag{1}$$
$$\omega = \omega_2 \quad \text{for salmon} \tag{2}$$

- Let's assume this two class case.

Salmon

Sea Bass

Prior Probability

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- Note: The prior may vary depending on the situation.
    - If we get equal numbers of salmon and sea bass in a catch, then the priors are equal, or **uniform**.
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- Note: The prior may vary depending on the situation.
  - If we get equal numbers of salmon and sea bass in a catch, then the priors are equal, or **uniform**.
  - Depending on the season, we may get more salmon than sea bass, for example.
- We write $P(\omega = \omega_1)$ or just $P(\omega_1)$ for the prior the next is a sea bass.
- The priors must exhibit exclusivity and exhaustivity. For c states of nature, or classes:

$$1 = \sum_{i=1}^{c} P(\omega_i) \tag{3}$$

Decision Rule From Only Priors

- A **decision rule** prescribes what action to take based on observed input.
- IDEA CHECK: What is a reasonable Decision Rule if
  - the only available information is the prior, and
  - the cost of any incorrect classification is equal?

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- Decide ω_1 if $P(\omega_1) > P(\omega_2)$; otherwise decide ω_2.
- What can we say about this decision rule?
  - Seems reasonable, but it will **always** choose the same fish.
  - If the priors are uniform, this rule will behave poorly.
  - Under the given assumptions, no other rule can do better! (We will see this later on.)

Features and Feature Spaces

- A **feature** is an observable variable.
- A **feature space** is a set from which we can sample or observe values.
- Examples of features:
  - Length
  - Width
  - Lightness
  - Location of Dorsal Fin
- For simplicity, let's assume that our features are all continuous values.
- Denote a scalar feature as x and a vector feature as $\mathbf{x}$. For a d-dimensional feature space, $\mathbf{x} \in \mathbb{R}^d$.

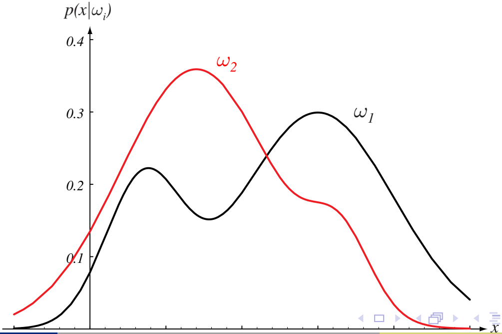
Class-Conditional Density

or Likelihood

- The **class-conditional probability density** function is the probability density function for $\mathbf{x}$, our feature, given that the state of nature is ω:

$$p(\mathbf{x}|\omega) \tag{4}$$

- Here is the hypothetical class-conditional density $p(x|\omega)$ for lightness values of sea bass and salmon.



Posterior Probability

Bayes Formula

- If we know the prior distribution and the class-conditional density, how does this affect our decision rule?
- **Posterior probability** is the probability of a certain state of nature given our observables: $P(\omega|\mathbf{x})$.
- Use Bayes Formula:

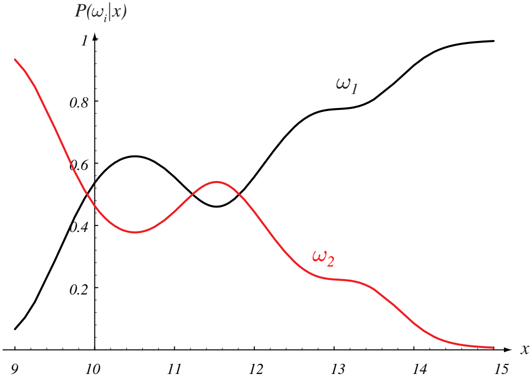
$$P(\omega, \mathbf{x}) = P(\omega|\mathbf{x})p(\mathbf{x}) = p(\mathbf{x}|\omega)P(\omega) \tag{5}$$

$$P(\omega|\mathbf{x}) = \frac{p(\mathbf{x}|\omega)P(\omega)}{p(\mathbf{x})} \tag{6}$$

$$= \frac{p(\mathbf{x}|\omega)P(\omega)}{\sum_i p(\mathbf{x}|\omega_i)P(\omega_i)} \tag{7}$$

Posterior Probability

- Notice the likelihood and the prior govern the posterior. The $p(x)$ evidence term is a scale-factor to normalize the density.
- For the case of $P(\omega_1) = 2/3$ and $P(\omega_2) = 1/3$ the posterior is



Probability of Error

- For a given observation x, we would be inclined to let the posterior govern our decision:

$$\omega^* = \arg\max_i P(\omega_i|\mathbf{x}) \tag{8}$$

- What is our **probability of error**?

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- What is our **probability of error**?
- For the two class situation, we have

$$P(\text{error}|\mathbf{x}) = \begin{cases} P(\omega_1|\mathbf{x}) & \text{if we decide } \omega_2 \\ P(\omega_2|\mathbf{x}) & \text{if we decide } \omega_1 \end{cases} \tag{9}$$

Probability of Error

- We can minimize the probability of error by following the posterior:

$$\text{Decide } \omega_1 \text{ if } P(\omega_1|\mathbf{x}) > P(\omega_2|\mathbf{x}) \tag{10}$$

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- And, this minimizes the average probability of error too:

$$P(\text{error}) = \int_{-\infty}^{\infty} P(\text{error}|\mathbf{x})p(\mathbf{x})d\mathbf{x} \tag{11}$$

(Because the integral will be minimized when we can ensure each $P(\text{error}|\mathbf{x})$ is as small as possible.)

Bayes Decision Rule (with Equal Costs)

- Decide ω_1 if $P(\omega_1|\mathbf{x}) > P(\omega_2|\mathbf{x})$; otherwise decide ω_2
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$$P(\text{error}|\mathbf{x}) = \min\left[P(\omega_1|\mathbf{x}), P(\omega_2|\mathbf{x})\right] \tag{12}$$

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- Conversely, if we have uniform priors, then the decision will rely exclusively on the likelihoods.

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- Conversely, if we have uniform priors, then the decision will rely exclusively on the likelihoods.
- Take Home Message: **Decision making relies on both the priors and the likelihoods and Bayes Decision Rule combines them to achieve the minimum probability of error.**

Loss Functions

- A **loss function** states exactly how costly each action is.
- As earlier, we have c classes $\{\omega_1, \ldots, \omega_c\}$.
- We also have a possible actions $\{\alpha_1, \ldots, \alpha_a\}$.
- The loss function $\lambda(\alpha_i|\omega_j)$ is the loss incurred for taking action α_i when the class is ω_j.

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- The loss function $\lambda(\alpha_i|\omega_j)$ is the loss incurred for taking action α_i when the class is ω_j.
- The **Zero-One Loss Function** is a particularly common one:

$$\lambda(\alpha_i|\omega_j) = \begin{cases} 0 & i = j \\ 1 & i \neq j \end{cases} \qquad i, j = 1, 2, \dots, c \tag{13}$$

It assigns no loss to a correct decision and uniform unit loss to an incorrect decision.

Expected Loss

a.k.a. Conditional Risk

- We can consider the loss that would be incurred from taking each possible action in our set.
- The **expected loss** or conditional risk is by definition

$$R(\alpha_i|\mathbf{x}) = \sum_{j=1}^{c} \lambda(\alpha_i|\omega_j) P(\omega_j|\mathbf{x}) \tag{14}$$

- The **zero-one conditional risk** is

$$\begin{aligned} R(\alpha_i|\mathbf{x}) &= \sum_{j \neq i} P(\omega_j|\mathbf{x}) && (15) \\ &= 1 - P(\omega_i|\mathbf{x}) && (16) \end{aligned}$$

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- Hence, for an observation x, we can minimize the expected loss by selecting the action that minimizes the conditional risk.
- (Teaser) You guessed it: this is what Bayes Decision Rule does!

Overall Risk

- Let $\alpha(x)$ denote a decision rule, a mapping from the input feature space to an action, $\mathbb{R}^d \mapsto \{\alpha_1, \ldots, \alpha_a\}$.
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- Let $\alpha(x)$ denote a decision rule, a mapping from the input feature space to an action, $\mathbb{R}^d \mapsto \{\alpha_1, \ldots, \alpha_a\}$.
  - This is what we want to learn.
- The **overall risk** is the expected loss associated with a given decision rule.

$$R = \oint R\left(\alpha(\mathbf{x})|\mathbf{x}\right) p\left(\mathbf{x}\right) d\mathbf{x} \tag{17}$$

Clearly, we want the rule $\alpha(\cdot)$ that minimizes $R(\alpha(\mathbf{x})|\mathbf{x})$ for all $\mathbf{x}$.

Bayes Risk

The Minimum Overall Risk

- Bayes Decision Rule gives us a method for minimizing the overall risk.
- Select the action that minimizes the conditional risk:

$$\alpha* = \arg\min_{\alpha_i} R\left(\alpha_i|\mathbf{x}\right) \tag{18}$$

$$= \arg\min_{\alpha_i} \sum_{j=1}^{c} \lambda(\alpha_i|\omega_j) P(\omega_j|\mathbf{x}) \tag{19}$$

- The Bayes Risk is the best we can do.

Two-Category Classification Examples

- Consider two classes and two actions, α_1 when the true class is ω_1 and α_2 for ω_2.
- Writing out the conditional risks gives:

$$R(\alpha_1|\mathbf{x}) = \lambda_{11}P(\omega_1|\mathbf{x}) + \lambda_{12}P(\omega_2|\mathbf{x}) \quad (20)$$

$$R(\alpha_2|\mathbf{x}) = \lambda_{21}P(\omega_1|\mathbf{x}) + \lambda_{22}P(\omega_2|\mathbf{x}) \ . \quad (21)$$

- Fundamental rule is decide ω_1 if

$$R(\alpha_1|\mathbf{x}) < R(\alpha_2|\mathbf{x}) \ . \quad (22)$$

- In terms of posteriors, decide ω_1 if

$$(\lambda_{21} - \lambda_{11})P(\omega_1|\mathbf{x}) > (\lambda_{12} - \lambda_{22})P(\omega_2|\mathbf{x}) \ . \quad (23)$$

  The more likely state of nature is scaled by the differences in loss (which are generally positive).

Two-Category Classification Examples

- Or, expanding via Bayes Rule, decide ω_1 if

$$(\lambda_{21} - \lambda_{11}) p(\mathbf{x}|\omega_1) P(\omega_1) > (\lambda_{12} - \lambda_{22}) p(\mathbf{x}|\omega_2) P(\omega_2) \tag{24}$$

- Or, assuming $\lambda_{21} > \lambda_{11}$, decide ω_1 if

$$\frac{p(\mathbf{x}|\omega_1)}{p(\mathbf{x}|\omega_2)} > \frac{\lambda_{12} - \lambda_{22}}{\lambda_{21} - \lambda_{11}} \frac{P(\omega_2)}{P(\omega_1)} \tag{25}$$

  - LHS is called the **likelihood ratio**.
- Thus, we can say the Bayes Decision Rule says to decide ω_1 if the likelihood ratio exceeds a threshold that is independent of the observation $\mathbf{x}$.

Pattern Classifiers Version 1: Discriminant Functions

- **Discriminant Functions** are a useful way of representing pattern classifiers.
- Let's say $g_i(\mathbf{x})$ is a discriminant function for the ith class.
- This classifier will assign a class ω_i to the feature vector $\mathbf{x}$ if

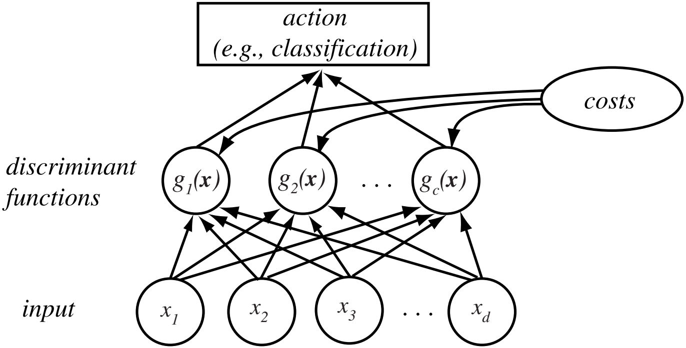
$$g_i(\mathbf{x}) > g_j(\mathbf{x}) \qquad \forall j \neq i \enspace , \tag{26}$$

or, equivalently

$$i^* = \arg\max_i g_i(x) \enspace , \quad \text{decide} \quad \omega_{i^*} \enspace .$$

Discriminants as a Network

- We can view the discriminant classifier as a network (for c classes and a d-dimensional input vector).



Bayes Discriminants

Minimum Conditional Risk Discriminant

- General case with risks

$$g_i(\mathbf{x}) = -R(\alpha_i|\mathbf{x}) \tag{27}$$

$$= -\sum_{j=1}^{c} \lambda(\alpha_i|\omega_j)P(\omega_j|\mathbf{x}) \tag{28}$$

- Can we prove that this is correct?

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- Can we prove that this is correct?
- **Yes!** The minimum conditional risk corresponds to the maximum discriminant.

Minimum Error-Rate Discriminant

- In the case of zero-one loss function, the Bayes Discriminant can be further simplified:

$$g_i(\mathbf{x}) = P(\omega_i|\mathbf{x}) \quad . \tag{29}$$

Uniqueness Of Discriminants

- Is the choice of discriminant functions unique?

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- Is the choice of discriminant functions unique?
- **No!**
- Multiply by some positive constant.
- Shift them by some additive constant.
- For monotonically increasing function $f(\cdot)$, we can replace each $g_i(\mathbf{x})$ by $f(g_i(\mathbf{x}))$ without affecting our classification accuracy.
  - These can help for ease of understanding or computability.
  - The following all yield the same exact classification results for minimum-error-rate classification.

$$g_i(\mathbf{x}) = P(\omega_i|\mathbf{x}) = \frac{p(\mathbf{x}|\omega_i)P(\omega_i)}{\sum_j p(\mathbf{x}|\omega_j)P(\omega_j)} \tag{30}$$

$$g_i(\mathbf{x}) = p(\mathbf{x}|\omega_i)P(\omega_i) \tag{31}$$

$$g_i(\mathbf{x}) = \ln p(\mathbf{x}|\omega_i) + \ln P(\omega_i) \tag{32}$$

Visualizing Discriminants

Decision Regions

- The effect of any decision rule is to divide the feature space into decision regions.
- Denote a decision region $\mathcal{R}_i$ for ω_i.
- One not necessarily connected region is created for each category and assignments is according to:

$$\text{If } g_i(\mathbf{x}) > g_j(\mathbf{x}) \ \forall j \neq i, \text{ then } \mathbf{x} \text{ is in } \mathcal{R}_i \quad . \tag{33}$$

- **Decision boundaries** separate the regions; they are ties among the discriminant functions.

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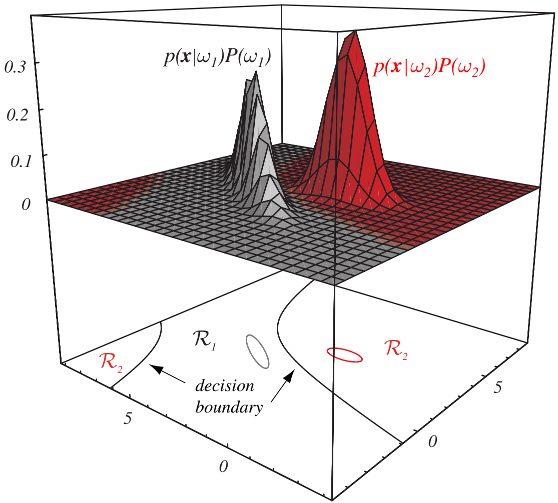
Decision Regions



Two-Category Discriminants

Dichotomizers

- In the two-category case, one considers single discriminant

$$g(\mathbf{x}) = g_1(\mathbf{x}) - g_2(\mathbf{x}) \quad . \tag{34}$$

- What is a suitable decision rule?

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- Various manipulations of the discriminant:

$$g(\mathbf{x}) = P(\omega_1|\mathbf{x}) - P(\omega_2|\mathbf{x}) \tag{36}$$

$$g(\mathbf{x}) = \ln \frac{p(\mathbf{x}|\omega_1)}{p(\mathbf{x}|\omega_2)} + \ln \frac{P(\omega_1)}{P(\omega_2)} \tag{37}$$

Background on the Normal Density

- This next section is a slight digression to introduce the Normal Density (most of you will have had this already).
- The Normal density is very well studied.
- It easy to work with analytically.
- Often in PR, an appropriate model seems to be a single typical value corrupted by continuous-valued, random noise.
- Central Limit Theorem (Second Fundamental Theorem of Probability).
  - The distribution of the sum of n random variables approaches the normal distribution when n is large.
  - E.g., `http://www.stattucino.com/berrie/dsl/Galton.html`

Expectation

- Recall the definition of expected value of any scalar function $f(x)$ in the continuous $p(x)$ and discrete $P(x)$ cases

$$\mathcal{E}[f(x)] = \int_{-\infty}^{\infty} f(x)p(x)dx \tag{38}$$

$$\mathcal{E}[f(x)] = \sum_{x} f(x)P(x) \tag{39}$$

where we have a set $\mathcal{D}$ over which the discrete expectation is computed.

Univariate Normal Density

- Continuous univariate normal, or **Gaussian**, density:

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \ . \tag{40}$$

- The **mean** is the expected value of x is

$$\mu \equiv \mathcal{E}[x] = \int_{-\infty}^{\infty} x p(x) dx \ . \tag{41}$$

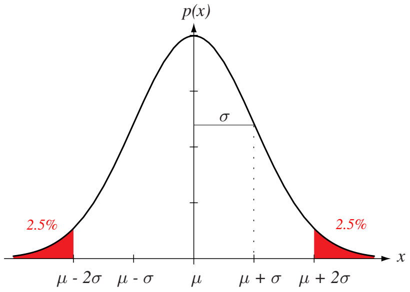
- The **variance** is the expected squared deviation

$$\sigma^2 \equiv \mathcal{E}[(x-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 p(x) dx \ . \tag{42}$$

Univariate Normal Density

Sufficient Statistics

- Samples from the normal density tend to cluster around the mean and be spread-out based on the variance.



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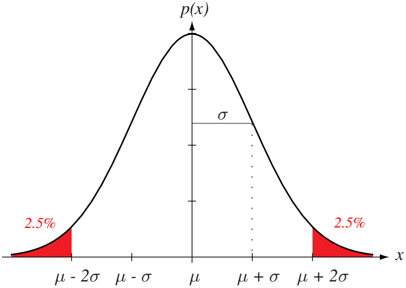

- The normal density is completely specified by the mean and the variance. These two are its **sufficient statistics**.
- We thus abbreviate the equation for the normal density as

$$p(x) \sim N(\mu, \sigma^2) \quad (43)$$

Entropy

- **Entropy** is the uncertainty in the random samples from a distribution.

$$H(p(x)) = -\int p(x) \ln p(x) dx \tag{44}$$

- The normal density has the maximum entropy for all distributions have a given mean and variance.
- What is the entropy of the uniform distribution?

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- What is the entropy of the uniform distribution?
- The uniform distribution has maximum entropy (on a given interval).

Multivariate Normal Density

And a test to see if your Linear Algebra is up to snuff.

- The multivariate Gaussian in d dimensions is written as

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}|\mathbf{\Sigma}|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathsf{T}}\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right] \quad . \tag{45}$$

- Again, we abbreviate this as $p(\mathbf{x}) \sim N(\boldsymbol{\mu}, \mathbf{\Sigma})$.
- The sufficient statistics in d-dimensions:

$$\boldsymbol{\mu} \equiv \mathcal{E}[\mathbf{x}] = \int \mathbf{x} p(\mathbf{x}) d\mathbf{x} \tag{46}$$

$$\mathbf{\Sigma} \equiv \mathcal{E}[(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^{\mathsf{T}}] = \int (\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^{\mathsf{T}} p(\mathbf{x}) d\mathbf{x} \tag{47}$$

The Covariance Matrix

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- Symmetric.
- Positive semi-definite (but DHS only considers positive definite so that the determinant is strictly positive).
- The diagonal elements σ_{ii} are the variances of the respective coordinate x_i.
- The off-diagonal elements σ_{ij} are the covariances of x_i and x_j.
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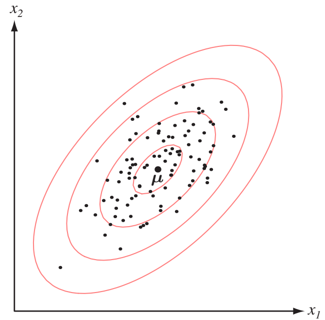
- What does $\boldsymbol{\Sigma}$ reduce to if all off-diagonals are 0?
- The product of the d univariate densities.

Mahalanobis Distance

- The shape of the density is determined by the covariance $\boldsymbol{\Sigma}$.
- Specifically, the eigenvectors of $\boldsymbol{\Sigma}$ give the principal axes of the hyperellipsoids and the eigenvalues determine the lengths of these axes.
- The loci of points of constant density are hyperellipsoids with constant **Mahalonobis distance**:

$$(\mathbf{x}-\boldsymbol{\mu})^{\mathsf{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) \qquad (48)$$

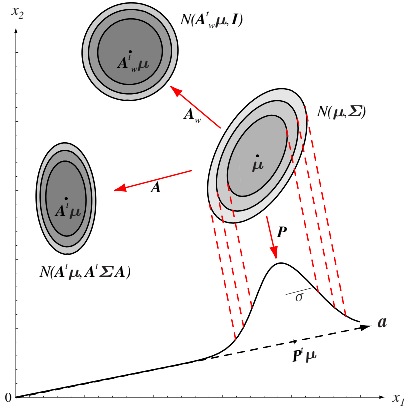


Linear Combinations of Normals

- Linear combinations of jointly normally distributed random variables, independent or not, are normally distributed.
- For $p(\mathbf{x}) \sim N((\mu), \mathbf{\Sigma})$ and $\mathbf{A}$, a d-by-k matrix, define $\mathbf{y} = \mathbf{A}^\mathsf{T}\mathbf{x}$. Then:

$$p(\mathbf{y}) \sim N(\mathbf{A}^\mathsf{T}\boldsymbol{\mu}, \mathbf{A}^\mathsf{T}\mathbf{\Sigma}\mathbf{A}) \quad (49)$$

- With the covariance matrix, we can calculate the dispersion of the data in any direction or in any subspace.



General Discriminant for Normal Densities

- Recall the minimum error rate discriminant, $g_i(\mathbf{x}) = \ln p(\mathbf{x}|\omega_i) + \ln P(\omega_i)$.
- If we assume normal densities, i.e., if $p(\mathbf{x}|\omega_i) \sim N(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$, then the general discriminant is of the form

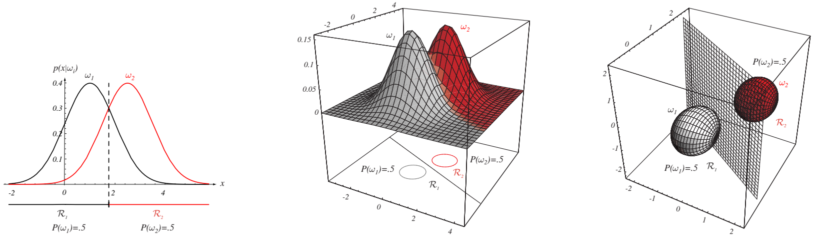
$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_i)^{\mathsf{T}}\boldsymbol{\Sigma}_i^{-1}(\mathbf{x}-\boldsymbol{\mu}_i) - \frac{d}{2}\ln 2\pi - \frac{1}{2}\ln|\boldsymbol{\Sigma}_i| + \ln P(\omega_i) \tag{50}$$

Simple Case: Statistically Independent Features with Same Variance

- What do the decision boundaries look like if we assume $\boldsymbol{\Sigma}_i = \sigma^2 \mathbf{I}$?

Simple Case: Statistically Independent Features with Same Variance

- What do the decision boundaries look like if we assume $\mathbf{\Sigma}_i = \sigma^2 \mathbf{I}$?
- They are hyperplanes.



- Let's see why...

Simple Case: $\Sigma_i = \sigma^2 \mathbf{I}$

- The discriminant functions take on a simple form:

$$g_i(\mathbf{x}) = -\frac{\|\mathbf{x} - \boldsymbol{\mu}_i\|^2}{2\sigma^2} + \ln P(\omega_i) \tag{51}$$

- Think of this discriminant as a combination of two things
  1. The distance of the sample to the mean vector (for each i).
  2. A normalization by the variance and offset by the prior.

Simple Case: $\Sigma_i = \sigma^2 \mathbf{I}$

- But, we don't need to actually compute the distances.
- Expanding the quadratic form $(\mathbf{x} - \boldsymbol{\mu})^{\mathsf{T}}(\mathbf{x} - \boldsymbol{\mu})$ yields

$$g_i(\mathbf{x}) = -\frac{1}{2\sigma^2}\left[\mathbf{x}^{\mathsf{T}}\mathbf{x} - 2\boldsymbol{\mu}_i^{\mathsf{T}}\mathbf{x} + \boldsymbol{\mu}_i^{\mathsf{T}}\boldsymbol{\mu}_i\right] + \ln P(\omega_i) \ . \qquad (52)$$

- The quadratic term $\mathbf{x}^{\mathsf{T}}\mathbf{x}$ is the same for all i and can thus be ignored.
- This yields the equivalent **linear discriminant functions**

$$g_i(\mathbf{x}) = \mathbf{w}_i^{\mathsf{T}}\mathbf{x} + w_{i0} \qquad (53)$$

$$\mathbf{w}_i = \frac{1}{\sigma^2}\boldsymbol{\mu}_i \qquad (54)$$

$$w_{i0} = -\frac{1}{2\sigma^2}\boldsymbol{\mu}_i^{\mathsf{T}}\boldsymbol{\mu}_i + \ln P(\omega_i) \qquad (55)$$

- w_{i0} is called the **bias**.

Simple Case: $\Sigma_i = \sigma^2 \mathbf{I}$

Decision Boundary Equation

- The decision surfaces for a linear discriminant classifiers are hyperplanes defined by the linear equations $g_i(\mathbf{x}) = g_j(\mathbf{x})$.
- The equation can be written as

$$\mathbf{w}^{\mathsf{T}}(\mathbf{x} - \mathbf{x}_0) = 0 \tag{56}$$

$$\mathbf{w} = \boldsymbol{\mu}_i - \boldsymbol{\mu}_j \tag{57}$$

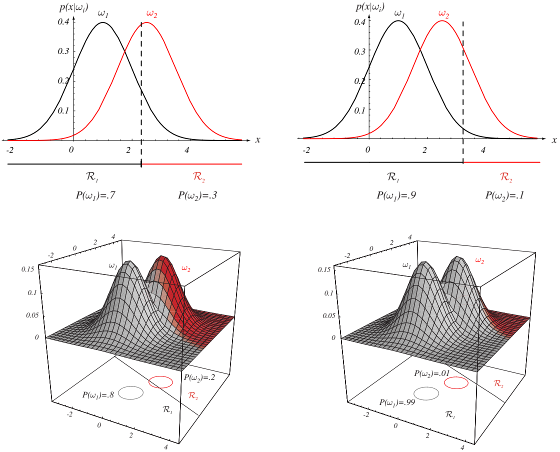
$$\mathbf{x}_0 = \frac{1}{2}(\boldsymbol{\mu}_i + \boldsymbol{\mu}_j) - \frac{\sigma^2}{\|\boldsymbol{\mu}_i - \boldsymbol{\mu}_j\|^2} \ln \frac{P(\omega_i)}{P(\omega_j)}(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j) \tag{58}$$

- These equations define a hyperplane through point x_0 with a normal vector $\mathbf{w}$.

Simple Case: $\Sigma_i = \sigma^2 \mathbf{I}$

Decision Boundary Equation

- The decision boundary changes with the prior.



General Case: Arbitrary Σ_i

- The discriminant functions are quadratic (the only term we can drop is the $\ln 2\pi$ term):

$$g_i(\mathbf{x}) = \mathbf{x}^{\mathsf{T}} \mathbf{W}_i \mathbf{x} + \mathbf{w}_i^{\mathsf{T}} \mathbf{x} + w_{i0} \tag{59}$$

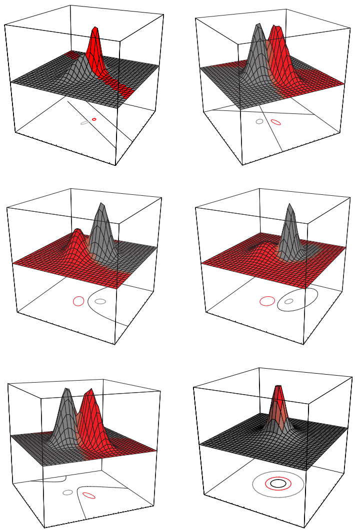
$$\mathbf{W}_i = -\frac{1}{2} \mathbf{\Sigma}_i^{-1} \tag{60}$$

$$\mathbf{w}_i = \mathbf{\Sigma}_i^{-1} \boldsymbol{\mu}_i \tag{61}$$

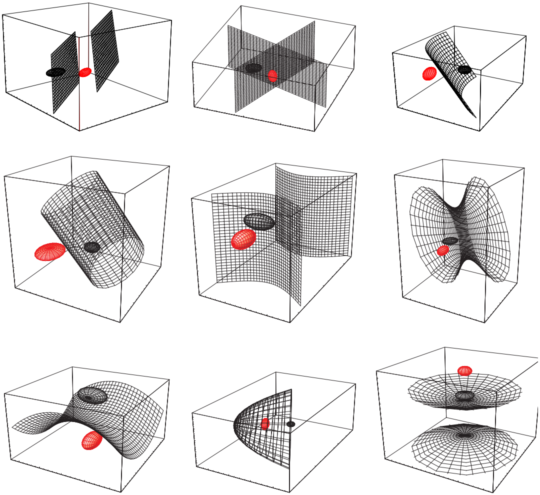
$$w_{i0} = -\frac{1}{2} \boldsymbol{\mu}_i^{\mathsf{T}} \mathbf{\Sigma}_i^{-1} \boldsymbol{\mu}_i - \frac{1}{2} \ln |\mathbf{\Sigma}_i| + \ln P(\omega_i) \tag{62}$$

- The decision surface between two categories are **hyperquadrics**.

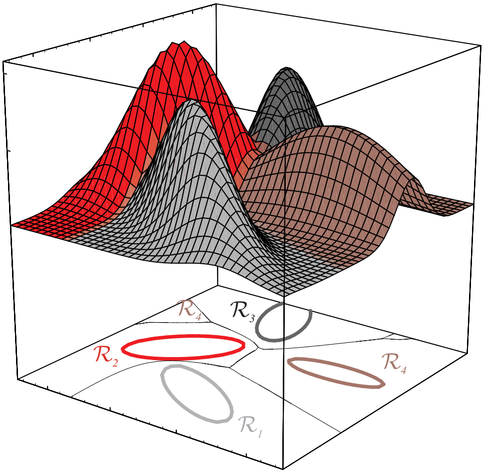
General Case: Arbitrary Σ_i



General Case: Arbitrary Σ_i



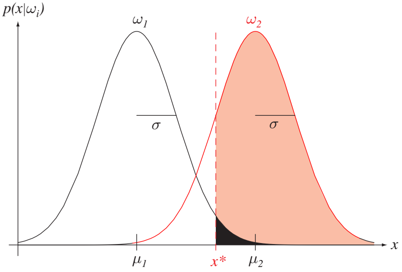
General Case for Multiple Categories



Quite A Complicated Decision Surface!

Signal Detection Theory

- A fundamental way of analyzing a classifier.
- Consider the following experimental setup:



- Suppose we are interested in detecting a single pulse.
- We can read an internal signal x.
- The signal is distributed about mean μ_2 when an external signal is present and around mean μ_1 when no external signal is present.
- Assume the distributions have the same variances, $p(x|\omega_i) \sim N(\mu_i, \sigma^2)$.

Signal Detection Theory

- The detector uses x^* to decide if the external signal is present.
- **Discriminability** characterizes how difficult it will be to decide if the external signal is present without knowing x^*.

$$d' = \frac{|\mu_2 - \mu_1|}{\sigma} \tag{63}$$

- Even if we do not know μ_1, μ_2, σ, or x^*, we can find d' by using a **receiver operating characteristic** or ROC curve, as long as we know the state of nature for some experiments

Receiver Operating Characteristics

Definitions

- A **Hit** is the probability that the internal signal is above x^* given that the external signal is present

$$P(x > x^* | x \in \omega_2) \tag{64}$$

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- A **False Alarm** is the probability that the internal signal is above x^* despite there being no external signal present.

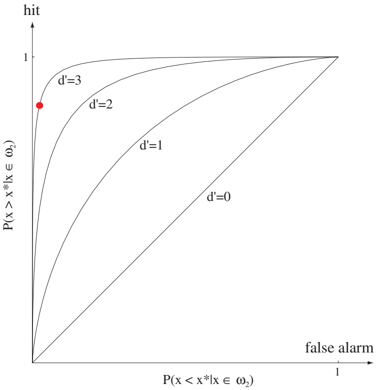
$$P(x > x^* | x \in \omega_1) \tag{66}$$

- A **Miss** is the probability that the internal signal is below x^* given that the external signal is present.

$$P(x < x^* | x \in \omega_2) \tag{67}$$

Receiver Operating Characteristics

- We can experimentally determine the rates, in particular the Hit-Rate and the False-Alarm-Rate.
- Basic idea is to assume our densities are fixed (reasonable) but vary our threshold x^*, which will thus change the rates.
- The receiver operating characteristic plots the hit rate against the false alarm rate.
- What shape curve do we want?



Missing Features

- Suppose we have built a classifier on multiple features, for example the lightness and width.
- What do we do if one of the features is not measurable for a particular case? For example the lightness can be measured but the width cannot because of occlusion.

Missing Features

- Suppose we have built a classifier on multiple features, for example the lightness and width.
- What do we do if one of the features is not measurable for a particular case? For example the lightness can be measured but the width cannot because of occlusion.
- **Marginalize!**
- Let $\mathbf{x}$ be our full feature feature and $\mathbf{x}_g$ be the subset that are measurable (or good) and let $\mathbf{x}_b$ be the subset that are missing (or bad/noisy).
- We seek an estimate of the posterior given **just the good features** $\mathbf{x}_g$.

Missing Features

$$P(\omega_i|\mathbf{x}_g) = \frac{p(\omega_i, \mathbf{x}_g)}{p(\mathbf{x}_g)} \tag{68}$$

$$= \frac{\int p(\omega_i, \mathbf{x}_g, \mathbf{x}_b) d\mathbf{x}_b}{p(\mathbf{x}_g)} \tag{69}$$

$$= \frac{\int p(\omega_i|\mathbf{x}) p(\mathbf{x}) d\mathbf{x}_b}{p(\mathbf{x}_g)} \tag{70}$$

$$= \frac{\int g_i(\mathbf{x}) p(\mathbf{x}) d\mathbf{x}_b}{\int p(\mathbf{x}) d\mathbf{x}_b} \tag{71}$$

- We will cover the Expectation-Maximization algorithm later.
- This is normally quite expensive to evaluate unless the densities are special (like Gaussians).

Statistical Independence

- Two variables x_i and x_j are independent if

$$p(x_i, x_j) = p(x_i)p(x_j) \tag{72}$$

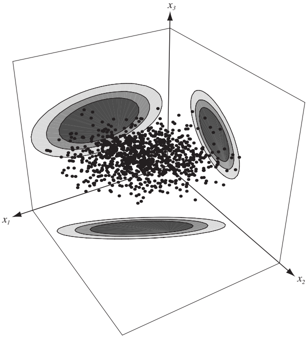
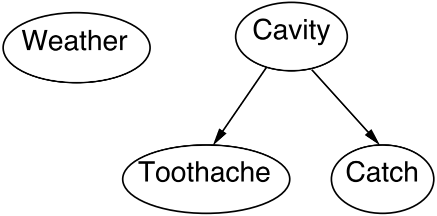


FIGURE 2.23. A three-dimensional distribution which obeys $p(x_1, x_3) = p(x_1)p(x_3)$; thus here x_1 and x_3 are statistically independent but the other feature pairs are not. From: Richard O. Duda, Peter E. Hart, and David G. Stork, *Pattern Classification*. Copyright © 2001 by John Wiley & Sons, Inc.

Simple Example of Conditional Independence

From Russell and Norvig

- Consider a simple example consisting of four variables: the weather, the presence of a cavity, the presence of a toothache, and the presence of other mouth-related variables such as dry mouth.
- The weather is clearly independent of the other three variables.
- And the toothache and catch are conditionally independent given the cavity (one as no effect on the other given the information about the cavity).



Naïve Bayes Rule

- If we assume that all of our individual features $x_i, i = 1, \dots, d$ are conditionally independent given the class, then we have

$$p(\omega_k|\mathbf{x}) \propto \prod_{i=1}^{d} p(x_i|\omega_k) \tag{73}$$

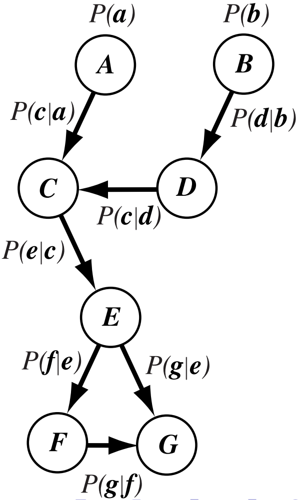
- Circumvents issues of dimensionality.
- Performs with surprising accuracy even in cases violating the underlying independence assumption.

An Early Graphical Model

- We represent these statistical dependencies graphically.
- Bayesian Belief Networks, or Bayes Nets, are **directed acyclic graphs**.
- Each link is directional.
- No loops.
- The Bayes Net factorizes the distribution into independent parts (making for more easily learned and computed terms).

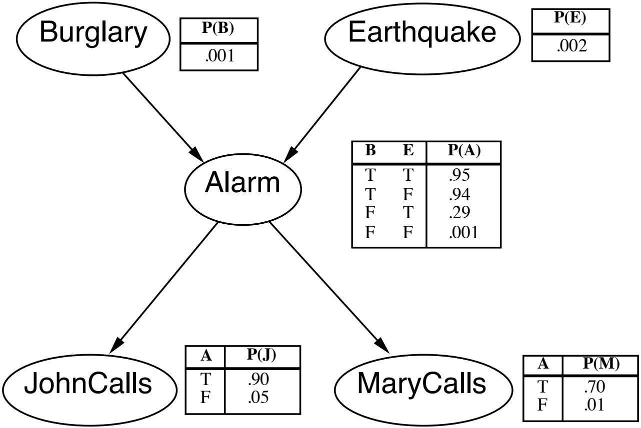
Bayes Nets Components

- Each **node** represents one variable (assume discrete for simplicity).
- A **link** joining two nodes is directional and it represents **conditional probabilities**.
- The intuitive meaning of a link is that the source has a direct influence on the sink.
- Since we typically work with discrete distributions, we evaluate the conditional probability at each node given its parents and store it in a lookup table called a **conditional probability table**.



A More Complex Example

From Russell and Norvig



- Key: given knowledge of the values of some nodes in the network, we can apply Bayesian inference to determine the maximum posterior values of the unknown variables!

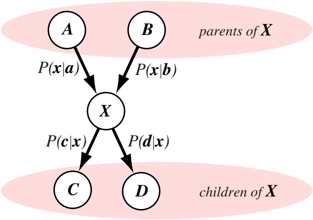
Full Joint Distribution on a Bayes Net

- Consider a Bayes network with n variables $x_1, \ldots, x_n$.
- Denote the parents of a node x_i as $\mathcal{P}(x_i)$.
- Then, we can decompose the joint distribution into the product of conditionals

$$P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P\left(x_i | \mathcal{P}(x_i)\right) \tag{74}$$

Belief at a Single Node

- What is the distribution at a single node, given the rest of the network and the evidence $\mathbf{e}$?
- **Parents** of $\mathbf{X}$, the set $\mathcal{P}$ are the nodes on which $\mathbf{X}$ is conditioned.
- **Children** of $\mathbf{X}$, the set $\mathcal{C}$ are the nodes conditioned on $\mathbf{X}$.



- Use the Bayes Rule, for the case on the right:

$$P(a,b,x,c,d) = P(a,b,x|c,d)P(c,d) \tag{75}$$

$$= P(a,b|x)P(x|c,d)P(c,d) \tag{76}$$

or more generally,

$$P(\mathcal{C}(x),x,\mathcal{P}(x)|\mathbf{e}) = P(\mathcal{C}(x)|x,\mathbf{e})P(x|\mathcal{P}(x),\mathbf{e})P(\mathcal{P}(x)|,\mathbf{e}) \tag{77}$$