

CSE 350: Advanced Data Structures and Indexes (Spring 2026)

Lecture 3: External Memory Model
Common Techniques & Cost Analysis

1/29/2026

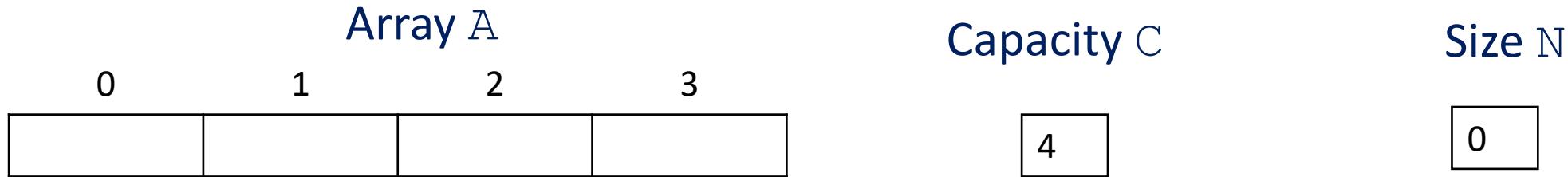
Example: from list to heap file

- Example: a collection of unordered items -- assuming everything is unique
 - Need to support
 - `insert(x)`
 - `delete(x)`
 - `lookup(x)`: reports whether the item x is found
 - In-memory solution:

size	0	1	2	3	4	5	6
4	100	200	300	400			

Algorithms of list

```
// List L: (A, C, N)
```



```
def init():
    return (array(4), 4, 0)
```

```
def insert(L, x):
    if L.N >= L.C then
        resize(L, L.C * 2)
    A[L.N] <- x
    L.N <- L.N + 1
```

```
def resize(L, C2):
    A2 <- array(C2)
    N2 <- min(L.N, C2)
    A2[0..N-1] <- A[0..N-1]
    L <- (A2, C2, N2)
```

Complexity analysis of list

Assumptions:

In-memory: count # of steps

What counts as a step:

- assignment of each word \leftarrow
- arithmetic operations ($+, -, *, /, \%, \dots$)
- allocation/deallocation of memory -- regardless of size

What about branch & loop? Depends

- Worst-case analysis
 - Count the maximum number of steps
- Average-case analysis
 - average number of stepsmay need to derive/assume distribution

complexity for size N	In-memory List
insert(x)	
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lookup(x)	

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Asymptotic analysis

- Only trend matters, how to formalize this?
 - Mathematical tool: **limit**

$$f: N \rightarrow R$$

$$\lim_{n \rightarrow +\infty} f(n) = L \text{ iff} \\ \forall \varepsilon > 0, \exists n_0 \in N, \forall n \geq n_0, |f(n) - L| \leq \varepsilon$$

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- How to use it to express trends? Intuition:

- constants c_0 and c_1 has the same trend, regardless of how much they differ
 - polynomials of the same degree have the same trend, regardless of their parameters
 - polynomials of a higher degree has a faster trend to increase in cost

Suppose $f_1(n) = c_0, f_2(n) = c_1, f_3(n) = 3n + 2, f_4(n) = 5n - 4, f_5(n) = n^2 + n - 3$

Want to define (\prec, \equiv) such that $f_1 \equiv f_2 \prec f_3 \equiv f_4 \equiv f_5$

Does this work: two function have the same trend if their difference is bounded by constant?

No! Counter example: $\lim_{n \rightarrow +\infty} |f_3(n) - f_4(n)| = \lim_{n \rightarrow +\infty} |2n - 6|$ which does not exist

Asymptotic analysis

- Only trend matters, how to formalize this?

- Mathematical tool: **limit**

$$f: N \rightarrow R^+$$

$$\lim_{n \rightarrow +\infty} f(n) = L \text{ iff}$$

$$\forall \varepsilon > 0, \exists n_0 \in N, \forall n \geq n_0, |f(n) - L| \leq \varepsilon$$

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Want to define (" $<$ ", " \equiv ") such that $f_1 \equiv f_2 < f_3 \equiv f_4 \equiv f_5$

Does this work: two function have the same trend if their difference is bounded by constant?

No! Counter example: $\lim_{n \rightarrow +\infty} |f_3(n) - f_4(n)| = \lim_{n \rightarrow +\infty} |2n - 6|$ which does not exist

Asymptotic analysis

- Whether the limit of the ratio of two functions $\lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)}$
 - does not exist (i.e., tends to $+\infty$) => $f(n)$ grows faster than $g(n)$
 - is constant => same trend
 - is zero => $f(n)$ grows slower than $g(n)$

Suppose $f_1(n) = c_0, f_2(n) = c_1, f_3(n) = 3n + 2, f_4(n) = 5n - 4, f_5(n) = n^2 + n - 3$

Want to define (\prec, \equiv) such that $f_1 \equiv f_2 \prec f_3 \equiv f_4 \equiv f_5$

$$\lim_{n \rightarrow +\infty} \frac{f_1(n)}{f_2(n)} = \lim_{n \rightarrow +\infty} \frac{c_0}{c_1} = \frac{c_0}{c_1} \quad \Rightarrow f_1 \text{ and } f_2 \text{ has the same trend}$$

$$\lim_{n \rightarrow +\infty} \frac{f_1(n)}{f_3(n)} = \lim_{n \rightarrow +\infty} \frac{c_0}{3n+2} = 0 \quad \Rightarrow f_1 \text{ grows slower than } f_3$$

$$\lim_{n \rightarrow +\infty} \frac{f_5(n)}{f_3(n)} = \lim_{n \rightarrow +\infty} \frac{n^2+n-3}{3n+2} \rightarrow +\infty \quad \Rightarrow f_5 \text{ grows faster than } f_3$$

Asymptotic analysis

- Formally, we define following sets of all functions $g: N \rightarrow R^+$ for $f: N \rightarrow R^+$

- $$o(f) = \left\{ g \in R^{+N} \mid \lim_{n \rightarrow +\infty} \frac{g(n)}{f(n)} = 0 \right\}$$

- $$\omega(f) = \left\{ g \in R^{+N} \mid \lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = 0 \right\}$$

- $$\Theta(f) = \left\{ g \in R^{+N} \mid \exists c \in R^+, \lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = c \right\}$$

- Big oh and Big Oemga:

- $$O(f) = \Theta(f) \cup o(f)$$

- $$\Omega(f) = \Theta(f) \cup \omega(f)$$

Examples:

$$2n^2 - 4 \in O(n^2)$$

means

$2n^2 - 4$ grows at most as fast as n^2

For convenience, we often write “=” instead of “ \in ”

i.e., $2n^2 - 4 = O(n^2)$

Complexity analysis of list

Assumptions:

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What counts as a step:

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What about branch & loop? Depends

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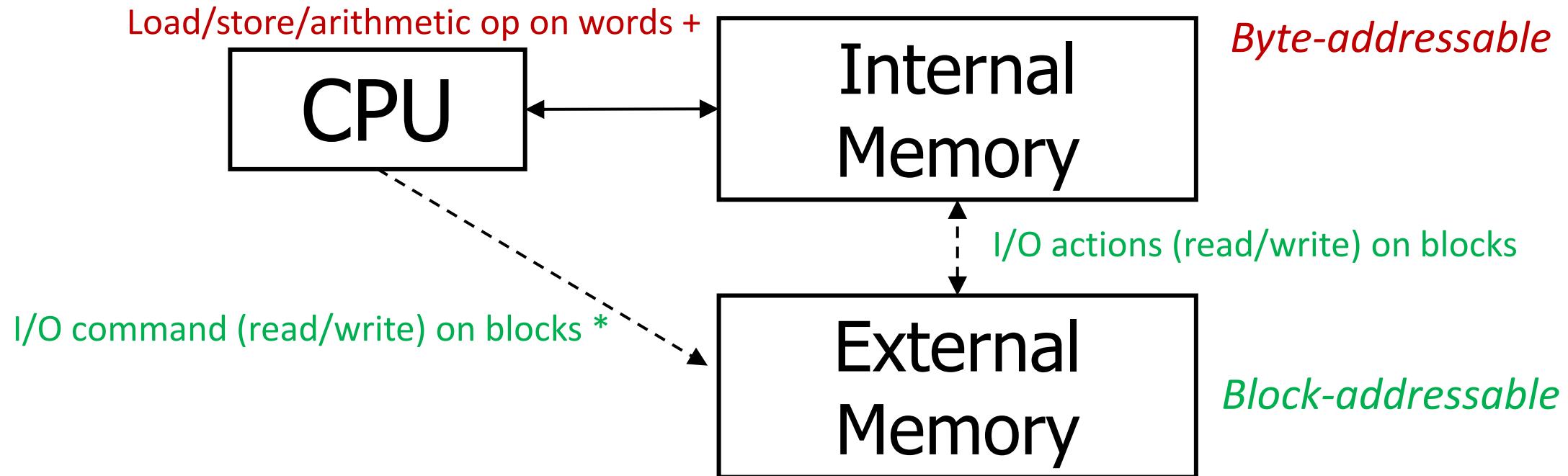
complexity for size N	In-memory List
$\text{insert}(x)$	$\Theta(N)$ (worst case) $\Theta(1)$ (average case)
$\text{delete}(x)$	
$\text{lookup}(x)$	

```
def insert(L, x):  
    if L.N >= L.C then  
        resize(L, L.C * 2)  
    A[L.N] <- x  
    L.N <- L.N + 1
```

```
def resize(L, C2):  
    A2 <- array(C2)  
    N2 <- min(L.N, C2)  
    A2[0..N-1] <- A[0..N-1]  
    L <- (A2, C2, N2)
```

External Memory (EM) Model

- Two levels in storage hierarchy
 - I/O latency dominates computation/memory access latencies
 - Complexity analysis will focus on # of I/Os (i.e., # of blocks read/written)



- * One block is a fixed number of consecutive bytes (e.g., 512 B, 4 KB), aligned to modulo = 0 boundaries.
- + A word is a unit of consecutive bytes for operations (e.g., a 4-byte integer).

Complexity Analysis for EM model

Assumptions:

EM model: count # of I/O operations

What counts as an I/O operation?

- reading up to a page
- writing up to a page
- If a read/write crosses page boundaries, count both

Assuming 4 KiB pages

```
ssize_t pread(int fd, void *buf, size_t count, off_t offset);  
ssize_t pwrite(int fd, const void *buf, size_t count, off_t offset);
```

Operation	# I/O
<code>pread(fd, buf, 4096, 0)</code>	1
<code>pwrite(fd, buf, 4096, 0)</code>	1
<code>pread(fd, buf, 4096, 10 * 4096)</code>	1
<code>pread(fd, buf, 8, 512)</code>	1
<code>pread(fd, buf, 4096, 2048)</code>	2

Example: from list to heap file

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 - In-memory solution:

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External memory solution?

How about replace all memory loads with 4-byte disk read, and all memory stores with 4-byte disk writes?

Random access is expensive

How about replace all memory loads with 4-byte disk read, and all memory stores with 4-byte disk writes?

```
def insert(L, x):  
    if L.N >= L.C then  
        resize(L, L.C * 2)  
    A[L.N] <- x  
    L.N <- L.N + 1  
  
def resize(L, C2):  
    A2 <- array(C2)  
    N2 <- min(L.N, C2)  
    A2[0..N-1] <- A[0..N-1]  
    L <- (A2, C2, N2)
```



L is a file descriptor to a file
Bytes 0 - 3: N
Bytes 4(i + 1) - 4(i + 1) + 3: ith value

```
def insert_heap(L, x):  
    pread(L, &N, 4, 0)  
    N <- N + 1  
    pwrite(L, &N, 4, 0)  
    pwrite(L, &x, 4, 4 * N)
```

complexity for size N	In-memory List	Naïve heap file
insert(x)	$\Theta(N)$ (worst case) $\Theta(1)$ (average case)	$5 = \Theta(1)$ (worst case)
delete(x)		
lookup(x)		

But, 5 I/O is around
50 ms for magnetic disks
or 500 μ s for SATA NAND SSD

in-memory: \leq hundreds of ns

Caching and page-granular I/O

`L` is a file descriptor to a file

Bytes 0 - 3: N

Bytes $4(i + 1) - 4(i + 1) + 3$: i th value

```
def insert_heap(L, x):  
    pread(L, &N, 4, 0)  
    N <- N + 1  
    pwrite(L, &N, 4, 0)  
    pwrite(L, &x, 4, 4 * N)
```



```
def init_heap_with_cache(fd):  
    pread(L, p, 4096, 0)  
    pn <- 0  
    N <- p[0:4]  
    return (fd, N, pn, p) // L  
def insert_heap_with_cache(L, x):  
    pn2 <- (L.N + 1) / 4096  
    off <- (L.N + 1) % 4096  
    if pn2 != L.pn then  
        pwrite(L.fd, L.p, 4096, L.pn * 4096)  
        pread(L.fd, L.p, 4096, pn2 * 4096)  
        L.pn = pn2  
    L.p[off:off + 4] <- x
```

```
def close_heap_with_cache(L):  
    // write cached page L.p  
    // read page 0  
    // update N  
    // write page 0
```

complexity for size N	In-memory List	Naïve heap file
<code>insert(x)</code>	$\Theta(N)$ (worst case) $\Theta(1)$ (average case)	$5 = \Theta(1)$ (worst case)
<code>delete(x)</code>		
<code>lookup(x)</code>		

Caching and page-granular I/O

`L` is a file descriptor to a file

Bytes 0 - 3: `N`

Bytes $4(i + 1) - 4(i + 1) + 3$: `i`th value

```
def insert_heap(L, x):  
    pread(L, &N, 4, 0)  
    N <- N + 1  
    pwrite(L, &N, 4, 0)  
    pwrite(L, &x, 4, 4 * N)
```



```
def init_heap_with_cache(fd):  
    pread(L, p, 4096, 0)  
    pn <- 0  
    N <- p[0:4]  
    return (fd, N, pn, p) // L  
def insert_heap_with_cache(L, x):  
    pn2 <- (L.N + 1) / 4096  
    off <- (L.N + 1) % 4096  
    if pn2 != L.pn then  
        pwrite(L.fd, L.p, 4096, L.pn * 4096)  
        pread(L.fd, L.p, 4096, pn2 * 4096)  
        L.pn = pn2  
    L.p[off:off + 4] <- x
```

complexity for size <code>N</code>	In-memory List	Naïve heap file	Heap file with one page cache
<code>insert(x)</code>	$\Theta(N)$ (worst case) $\Theta(1)$ (average case)	$5 = \Theta(1)$ (worst case)	$2 = \Theta(1)$ (worst case) $2/1024 = \Theta(1)$ (average case)
<code>delete(x)</code>			
<code>lookup(x)</code>			

Additional problems?

- Concurrency?
- Crash & recovery?
- How to further improve performance?

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def init_heap_with_cache(fd):
    pread(L, p, 4096, 0)
    pn <- 0
    N <- p[0:4]
    return (fd, N, pn, p) // L
def insert_heap_with_cache(L, x):
    pn2 <- (L.N + 1) / 4096
    off <- (L.N + 1) % 4096
    if pn2 != L.pn then
        pwrite(L.fd, L.p, 4096, L.pn * 4096)
        pread(L.fd, L.p, 4096, pn2 * 4096)
        L.pn = pn2
    L.p[off:off + 4] <- x

def close_heap_with_cache(L):
    // write cached page L.p
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