CSE462/562: Database Systems (Spring 22)

Lecture 9: Hashing Techniques

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Logistics updates

- Poll for final exam alternative date for those with conflicts
- 2-day extension to project 2
  - Code due on 3/10, 11:59 pm. Write-up due on 3/12, 11:59 pm.
  - Project 3 will still be released on 3/10

In this lecture

- Composite key in hash index
- How to design a good hash function?

Composite keys in hash index

- Composite key: multiple fields as the key (f1, f2, …, fk)

- How to handle composite keys in hash index?
  - Combine the hash values of each field together
  - Many libs available, e.g., boost::hash_combine, absl::Hash::combine(), etc …

    e.g., in boost:

    ```
    seed ^= hash_value + 0x9e3779b9 + (seed<<6) + (seed>>2);
    ```

- Search with composite keys
  - Must specify all the keys, equality search only
  - Can't perform partial key search
    - e.g., hash index on (sid, login)
      - may be used for predicate sid = 12345 AND login = 'alice'
      - but not sid = 12345, nor loging = 'Alice'

What might go wrong with hashing?

- Too many items with the same key
  - Extendible hashing and linear hashing will also *fail* when that happens
- Why can that happen?
  - Too many entries with the same key?
    - Not much that we can do, but we can try to incorporate other fields to make the keys distinct if it's possible from the user's perspective
    - Alternatively, consider using other types of index
  - Hash collision
    - Some hash functions are prone to too many hash collisions
      - For instance, you're hashing pointers of int64_t,
        - using modular hashing $h(x) = x \; mod \; m$ with $m = 2^d$ for some d is going to leave many buckets completely empty

Designing Good Hash Functions

- Formal set up: let U=[N] denote the numbers {0, 1, 2, . . . , N − 1}. For any set S ⊆ U, where |S|=n, we want to support:
  - add(x): add the key x to S
  - query(x): is the key q ∈ S?
  - delete(x): remove the key x from S

  efficiently!

  We consider the static case here (fixed set S). Note that even though S is fixed, we don't know S ahead of time. Imagine it's chosen by an adversary from $\binom{N}{n}$ ***possible choices.***

  Our hash function needs to work well for any such (fixed) set S.

Static vs Dynamic

- Static: Given a set S of items, we want to store them so that we can do lookups quickly. E.g., a fixed dictionary.

- Dynamic: here we have a sequence of insert, lookup, and perhaps delete requests. We want to do these all efficiently.

Hash Function Basics

- We will perform inserts and lookups by
  an array A of M buckets, and a hash function h : U → {0,... ,M − 1} (i.e., h : U → [M]).
  Given an element x, the idea of hashing is we want to store it in A[h(x)].
  - If N=|U| is small, this problem is trivial. But in practice, N is often big.

- Collision happens when $x \neq y \wedge h(x) = h(y)$
  - Open hashing with linked list/overflow pages
  - Extendible/linear hashing can be used to alleviate the problem
    but can't handle it well if there is skewness in hash values

Desirable Properties

- Small probability of distinct keys colliding: if $x \neq y \in S$ then $\Pr_{h \leftarrow H}[h(x) = h(y)]$ is "small".
  - $h \leftarrow H$ means the random choice over a family H of hash functions.
- Small range: we want M to be small. At odds with first desired property
  - ideally M=O(n) but it takes too much space.
- Small number of bits to store a hash function h. This is at least $\Omega(\log_2 |H|)$.
- h should be easy to compute
- Given this, the time to lookup an item x is O(length of list A[h(x)])

Bad News

- One way to spread elements out nicely is to spread them randomly. Unfortunately, we can't just use a random number generator to decide where the next element goes because then we would never be able to find it again. So, we want h to be something "pseudorandom" in some formal sense.

- (Bad news) For any deterministic hash function h (i.e., $|H|=1$), if $|U| \geq (n-1)M + 1$, there exists a set S of n elements that all hash to the same location.
  - simple pigeon hole argument.

Randomness to Rescue

- Introduce a family of hash functions, H with $|H|>1$, that h will be randomly chosen from for each key (but use the same choice for the same key).

- Universal Hashing: if $x \neq y \in S$ then $\Pr_{h \leftarrow H}[h(x) = h(y)] \leq 1/M$.

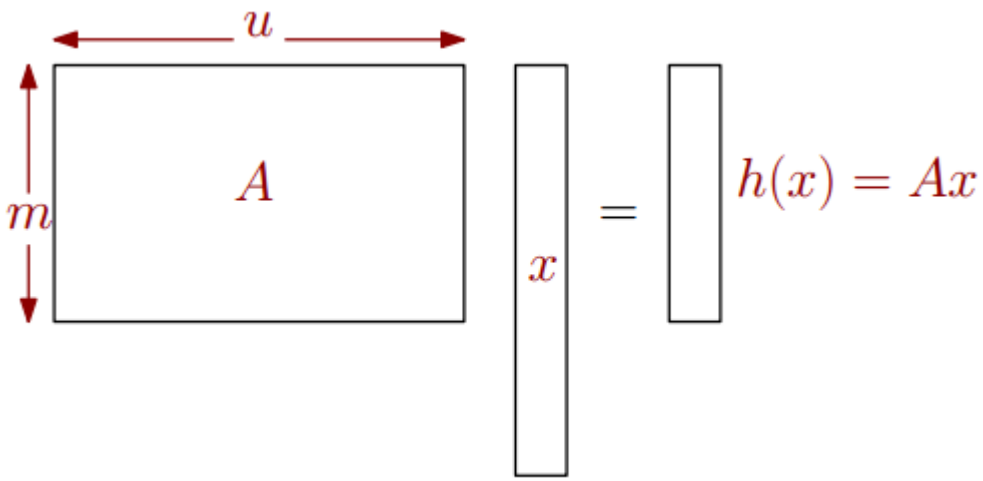
- If H is universal, then for any set $S \subseteq U$ of size n, for any $x \in U$ (e.g., that we might want to lookup, x may not come from S), if we choose h at random in a universal hash family H, the expected number of collisions between x and other elements in S is at most n/M.

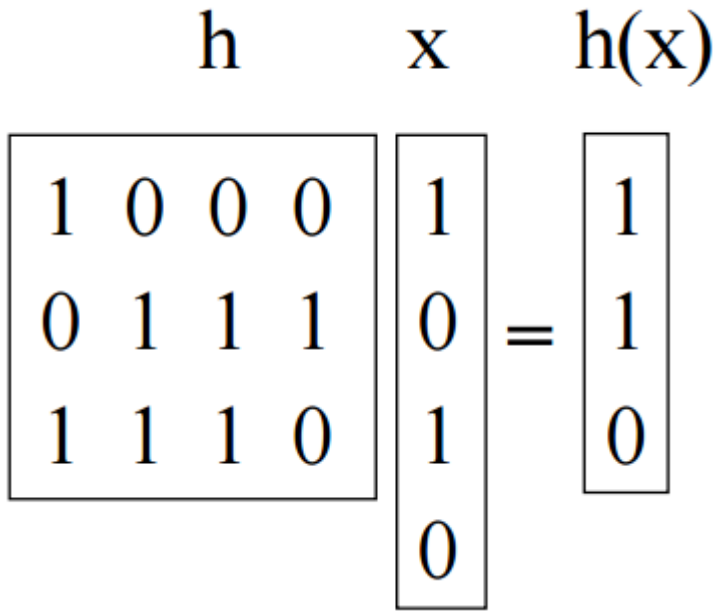
Property of Universal Hashing

- Proof:
  - Each y ∈ S (y ≠ x) has at most a 1/M chance of colliding with x by the definition of "universal". So
  - Let C_{xy}= 1 if x and y collide and 0 otherwise.
  - Let C_x denote the total number of collisions for x. So, $C_x = \sum_{y \in S \wedge y \neq x} C_{xy}$.
  - We know E[C_{xy}] = Pr(x and y collide) ≤ 1/M.
  - So, by linearity of expectation, E[C_x] = $\sum_{y \in S \wedge y \neq x} E\left[C_{xy}\right] \leq$ n/M.

How to Construct Universal Hashing?

- Consider the case where $|U| = 2^u$ and $M = 2^m$
- Take an u × m matrix A and fill it with random bits. For x ∈ U, view x as a u-bit vector in $\{0, 1\}^u$, and define h(x) := Ax where the calculations are done modulo 2.
- There are 2^{um} hash functions in this family H

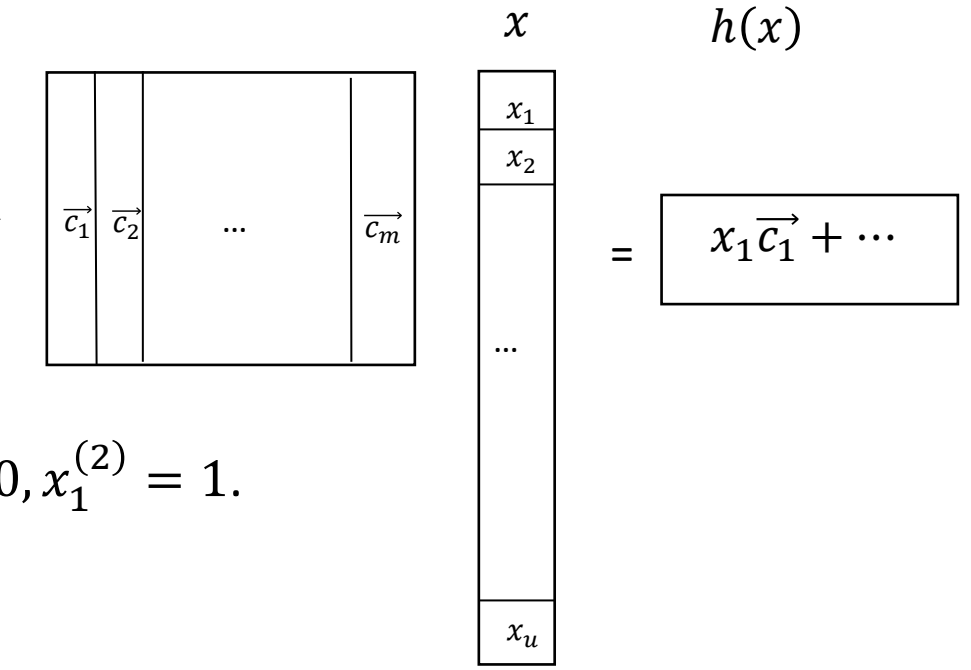




Note that $h(\vec{0}) = 0$, so picking a random function from H does not map each key to a random place

Why it is a universal hash family?

- Proof:
- Let $A = (\overrightarrow{c_1}, \overrightarrow{c_2}, \ldots, \overrightarrow{c_m})$, where $\overrightarrow{r_i}$ is the i^{th} row of the matrix A.
- Let's view $x \in U$ as a vector of {0,1}, e.g., x = (0, 0, 1, 0, …, 1).
- Then $\mathrm{h}(x) = x_1\overrightarrow{c_1} + x_2\overrightarrow{c_2} + \cdots + x_m\overrightarrow{c_m}$.



- Suppose we have $x^{(1)}, x^{(2)} \in U$, s.t., $x^{(1)} \neq x^{(2)}$.
  They will differ in at least one bit. WLOG, say it's bit 1 and $x_1^{(1)} = 0, x_1^{(2)} = 1$.

- For any $\overrightarrow{c_2}, \ldots \overrightarrow{c_m} \in \{0,1\}^u$, let's fix those vectors (except $\overrightarrow{c_1}$).
  - No matter how $\overrightarrow{c_1}$ changes, $h(x^{(1)}) = x_2\overrightarrow{c_2} + \cdots + x_m\overrightarrow{c_m}$ remain the same.
  - On the other hand, $h(x^{(2)}) = x_1\overrightarrow{c_1} + h(x^{(1)})$ are all different
    - because each $\overrightarrow{c_1}$ will be different from any other vectors by at least one bit and the corresponding bit in the hash value is flipped.
  - Thus we have only 1 out of 2^m different $\overrightarrow{c_1}$ so that $h(x^{(1)}) = h(x^{(2)})$.

- $\Pr\left(h(x^{(1)}) = h(x^{(2)})\right) = \sum_{\overrightarrow{c_2}, \ldots \overrightarrow{c_m}} \Pr(h(x^{(1)}) = h(x^{(2)}) | \overrightarrow{c_2}, \ldots \overrightarrow{c_m}) \Pr(\overrightarrow{c_2}, \ldots \overrightarrow{c_m})$

This is not the only way to construct universal hash family though.

Perfect Hashing (for static case)

- We say a hash function is perfect for S if all lookups involve O(1) work.
- Naïve method: an $O(n^2)$ space solution
- Let H be universal and $M = n^2$. Then just pick a random h from H and try it out!

- Claim: If H is universal and $M = n^2$, then $\Pr_{h \sim H}(no\ collisions\ in\ S) \geq 1/2$

Naïve method: $O(n^2)$ space

- Proof:
  - How many pairs (x,y) in S are there? Answer:
    - For each pair, the chance they collide is ≤ 1/M by definition of “universal”
    - So, $\Pr(exists\ a\ collision) \leq n(n-1)/2M = n(n-1)/2n^2 < 1/2.$ (by union bound)

An O(n) space solution (for static S)

- first hash into a table of size n using universal hashing. This will produce some collisions (unless we are extraordinarily lucky)
- then rehash each bin using Method 1, squaring the size of the bin to get zero collisions

Formally:

- a first-level hash function h and first-level table A,
- n second-level hash functions h1,... ,hn and n second-level tables A1,... ,An
- To lookup an element x, we first compute i = h(x) and then find the element in A_i [h_i(x)].
- We omit the analysis of this method.

Dynamic S?

- Cuckoo hashing
  - Linear space
  - Constant lookup time

  - Pagh, Rasmus; Rodler, Flemming Friche (2001). "Cuckoo Hashing". *Algorithms — ESA 2001*

Summary

- Today's lecture
  - Multi-field index key in hash index
  - How to construct a good hash function