CSE 250
Data Structures

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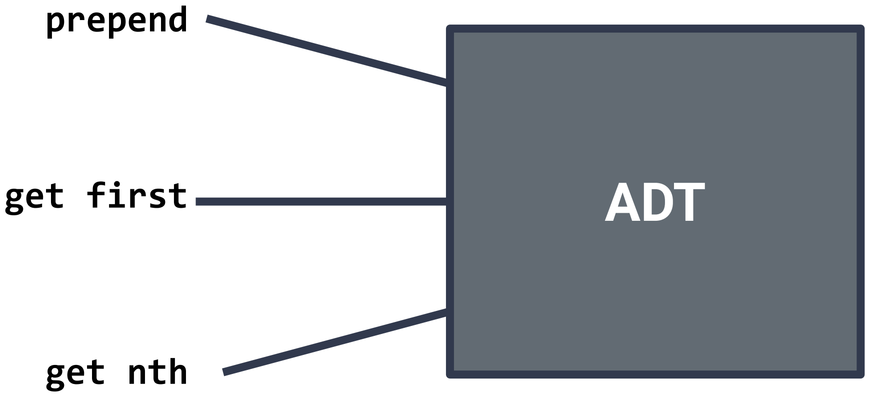
Lec 04: Intro to Complexity

Announcements and Feedback

- Office hours start this week
- Normal recitations begin next week
- Academic Integrity Quiz due Tonight @ 11:59PM
- PA0 due Friday @ 11:59PM
- WA1 due Friday @ 11:59PM

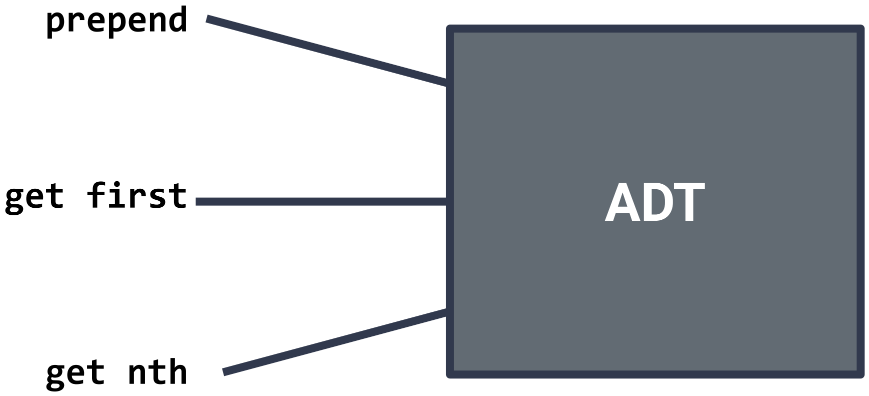
Thought Experiment

An Abstract Data Type is a specification of **what** a data structure can do



Thought Experiment

Often, many data structures can satisfy a given ADT…how do you choose?



Thought Experiment

Data Structure 1
- Very fast `prepend`, `get first`
- Very slow `get nth`

Data Structure 2
- Very fast `get nth`, `get first`
- Very slow `prepend`

Data Structure 3
- Very fast `get nth`, `get first`
- Occasionally slow `prepend`

Which is better?

Thought Experiment

Data Structure 1 (Linked List)

- Very fast `prepend`, `get first`
- Very slow `get nth`

Data Structure 2 (Array)

- Very fast `get nth`, `get first`
- Very slow `prepend`

Data Structure 3 (Array Buffer…in reverse)

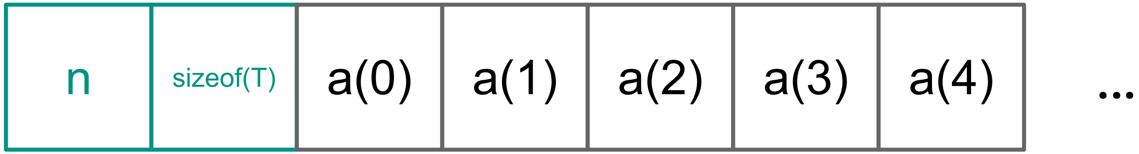
- Very fast `get nth`, `get first`
- Occasionally slow `prepend`

Which is better?

IT DEPENDS!

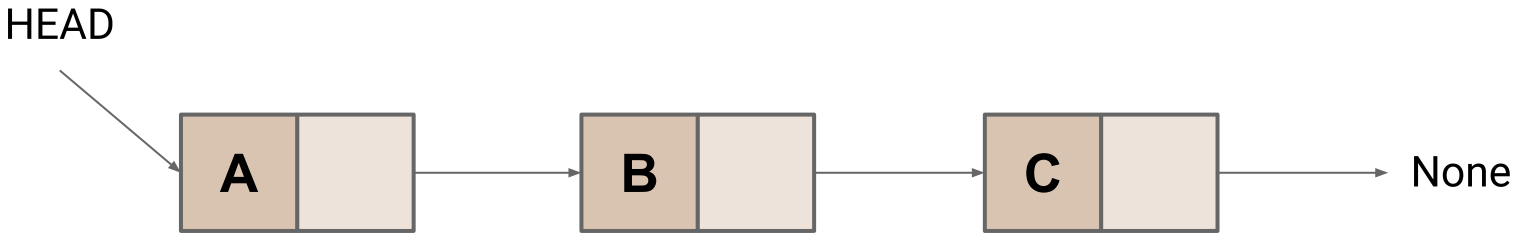
A (very) Brief Refresher: Array

- An array is an ordered container (elements stored one after another)
- Array elements are all stored in a contiguous block of memory



A (very) Brief Refresher: Linked Lists

- Also an ordered container
- Each element stores a pointer to the next element
    - …not necessarily in a contiguous block of memory



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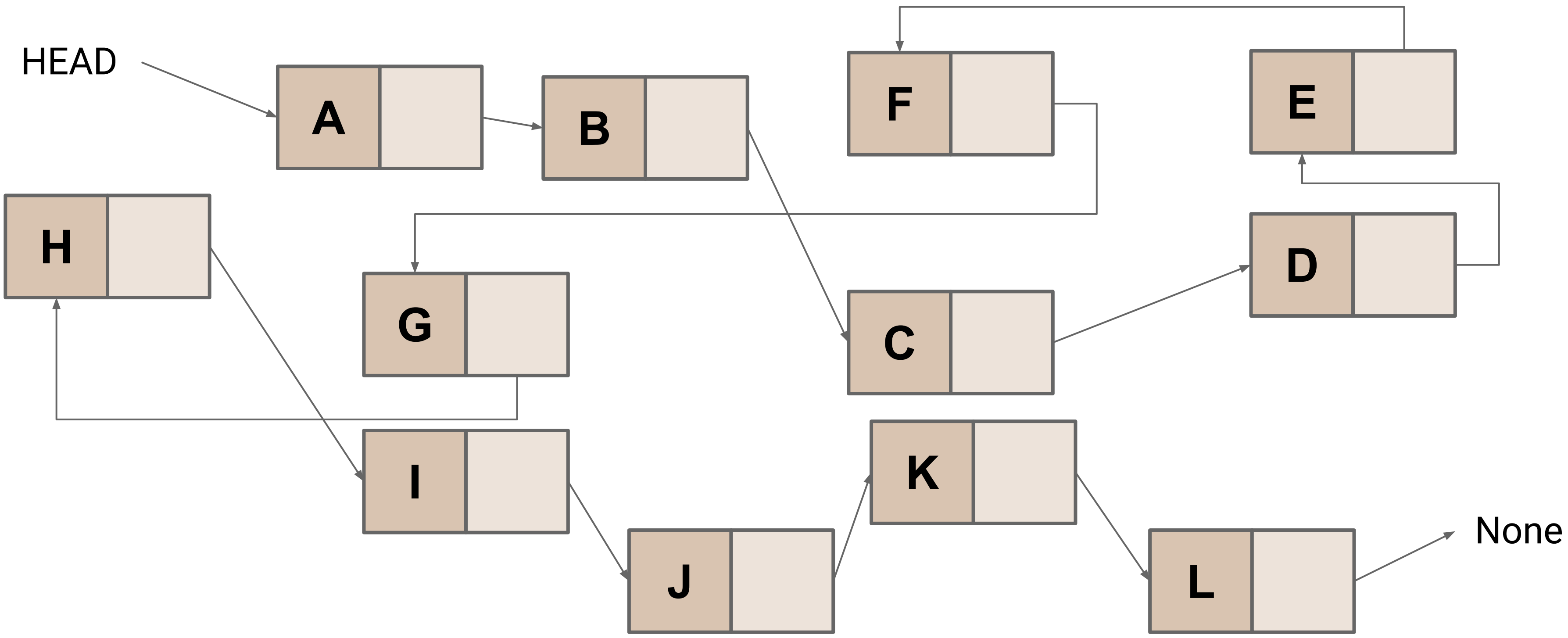
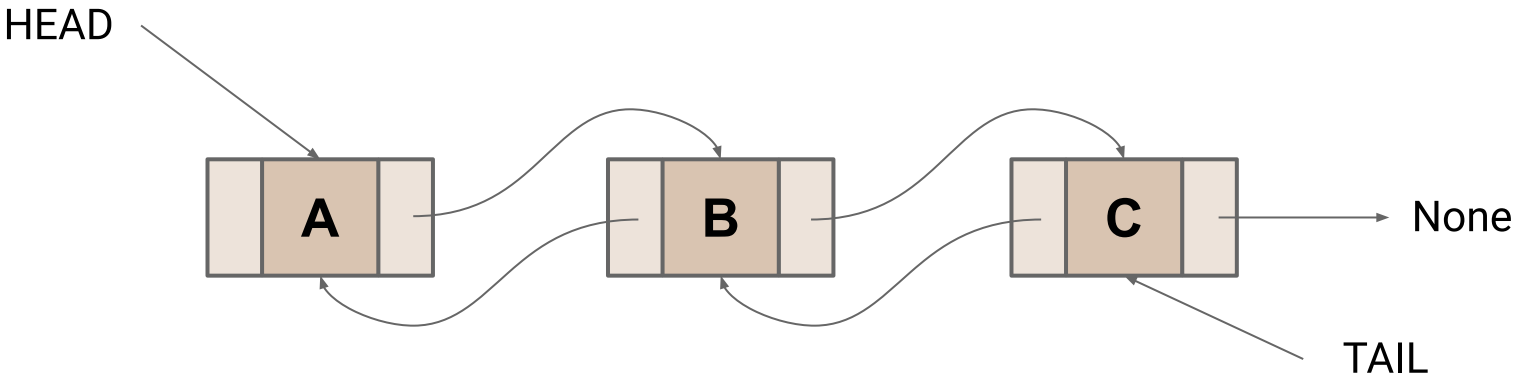


A (very) Brief Refresher: Linked Lists

- Can also be doubly linked (a next AND a prev pointer per node)
- PA1 will have you implementing a **Sorted Doubly Linked List** with some minor twists



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What is "fast"? "slow"?

Attempt #1: Wall-clock time?

- What is fast?
  - 10s? 100ms? 10ns?
  - …it depends on the task
- Algorithm vs Implementation
  - Compare Grace Hopper’s implementation to yours
- What machine are you running on?
  - Your old laptop? A lab machine? The newest, shiniest processor on the market?
- What bottlenecks exist? CPU vs IO vs Memory vs Network…

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Wall-clock time is not terribly useful…

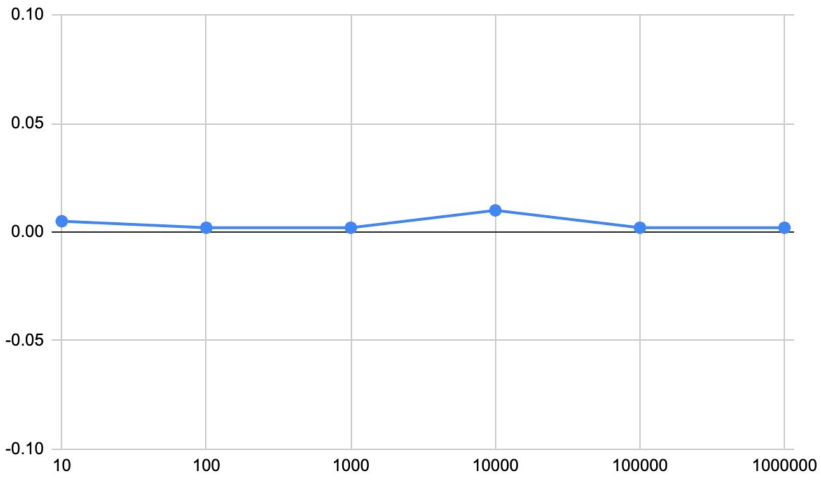
Analysis Checklist

1. Don’t think in terms of wall-time, think in terms of “number of steps”

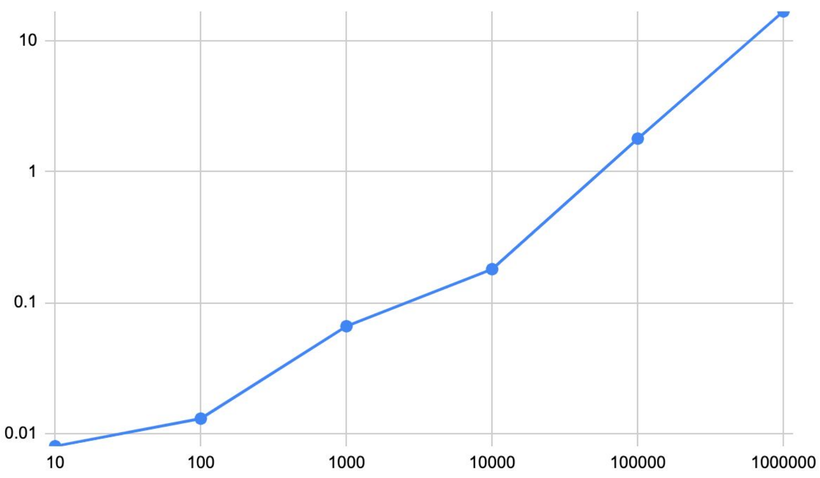
Let’s do a quick demo…

Comparing Random Access for Array vs List

Array



List



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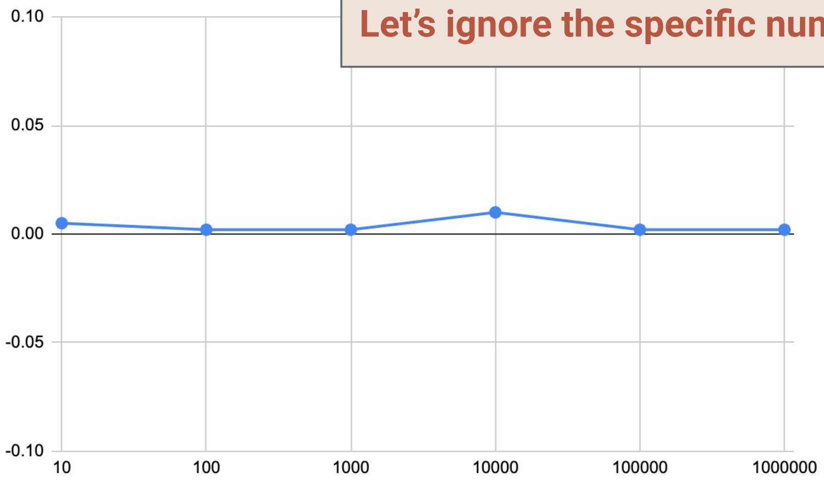
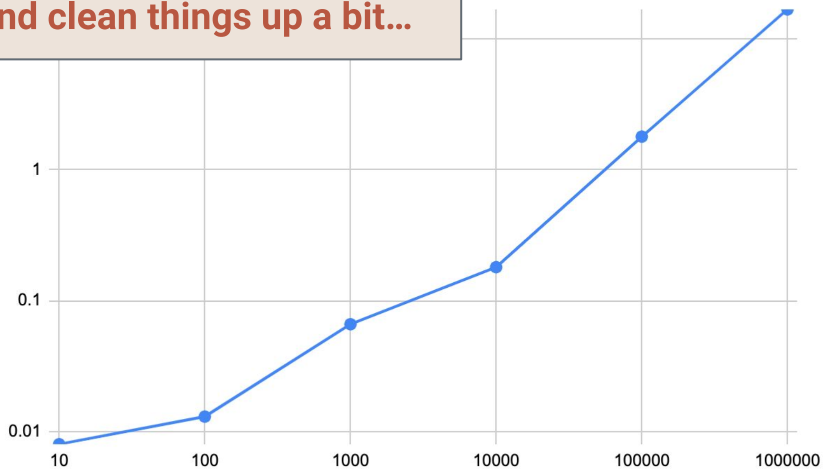


List

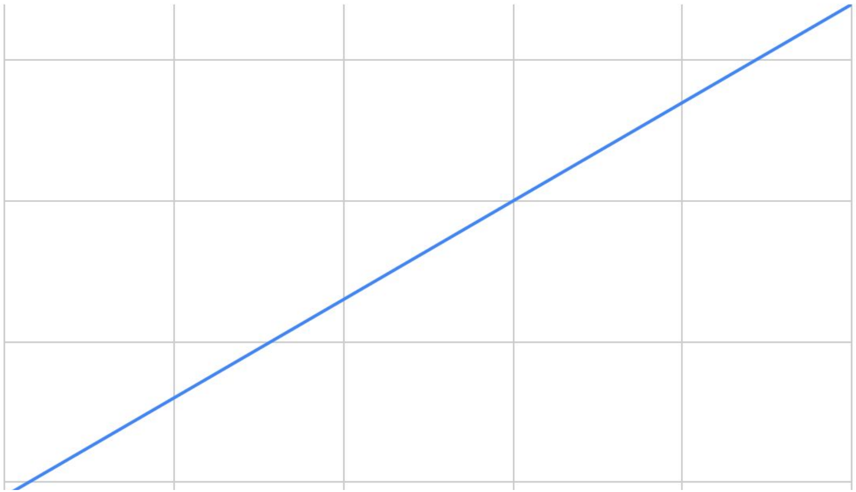


Let’s ignore the specific numbers and clean things up a bit…

Comparing Random Access for Array vs List

Array

List



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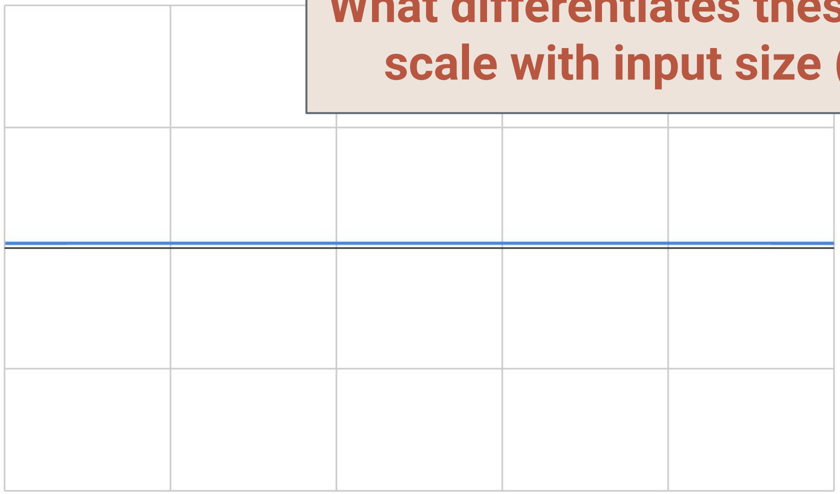

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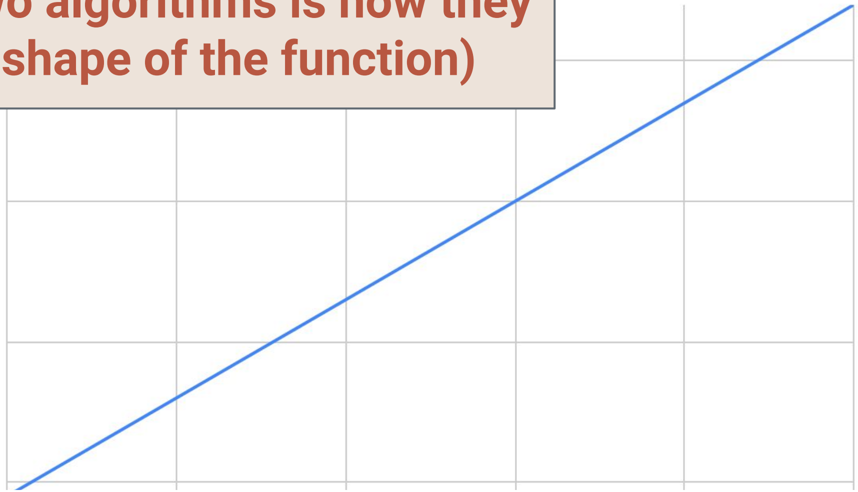

What differentiates these two algorithms is how they scale with input size (the shape of the function)

Analysis Checklist

1. Don’t think in terms of wall-time, think in terms of “number of steps”
2. **To give a useful solution, we should take “scale” into account**
    - **How does the runtime change as we change the size of the input?**

Counting Steps

```
public void updateUsers(User[] users) {
  x = 1;
  for(user : users) {
    user.id = x;
    x = x + 1;
  }
}
```

Counting Steps

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1

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$$1 + \sum_{user \in users}$$

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$$1 + \sum_{user \in users} 4$$

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Counting Steps

```java
public void userFullName(User[] users, int id) {
  User user = users[id];
  String fullName = user.firstName + user.lastName;
  return fullName;
}
```

Counting Steps

```java
public void userFullName(User[] users, int id) {
  User user = users[id];
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  return fullName;
}
```

3 steps…(sort of, more details later)

Counting Steps

```java
public void totalReads(User[] users, Post[] posts) {
  int totalReads = 0;
  for(post : posts) {
    int userReads = 0;
    for(user : users) {
      if(user.readPost(post)){ userReads += 1; }
    }
    totalReads += userReads;
  }
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1 .

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$$1 + \sum_{post \in posts} \Bigg(3$$

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$$1 + \sum_{post \in posts} \left(3 + \sum_{user \in users} \right.$$

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  }
}
```

$$1 + \sum_{post \in posts} \left(3 + \sum_{user \in users} 2 \right)$$

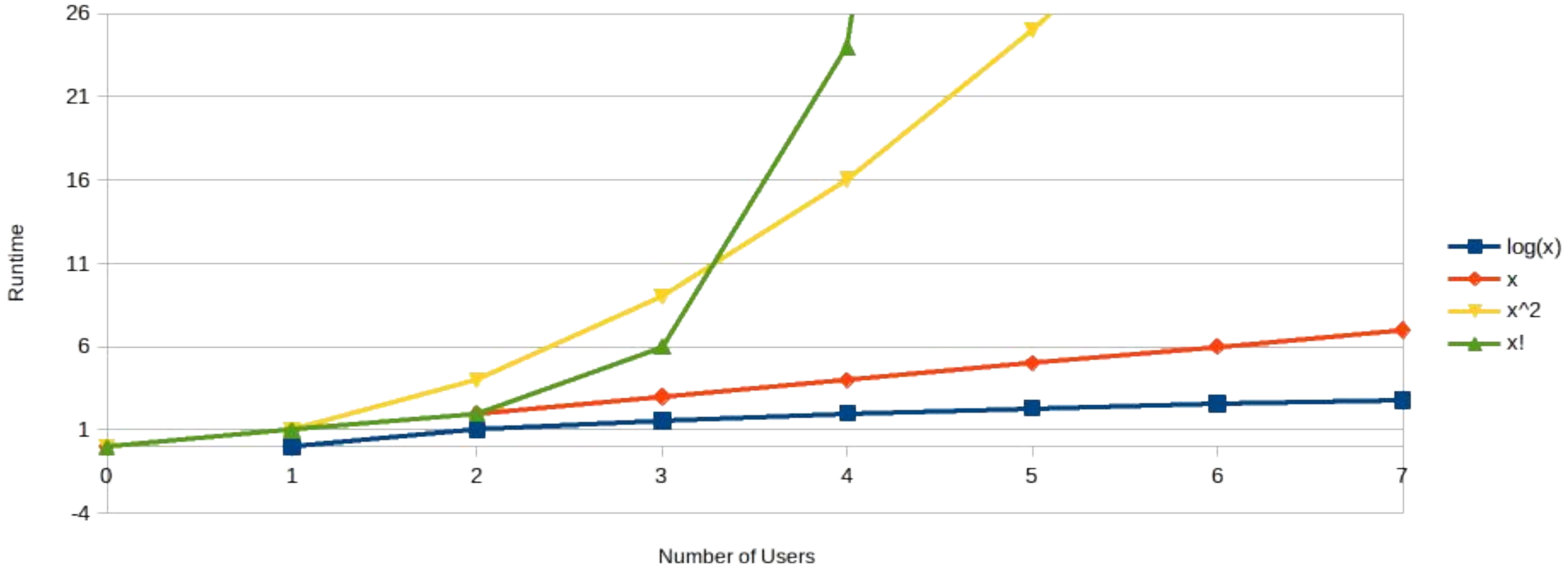
Steps to "Functions"

Now that we have number of steps in terms of summations…

…which we can simplify (like in WA1) into mathematical functions…

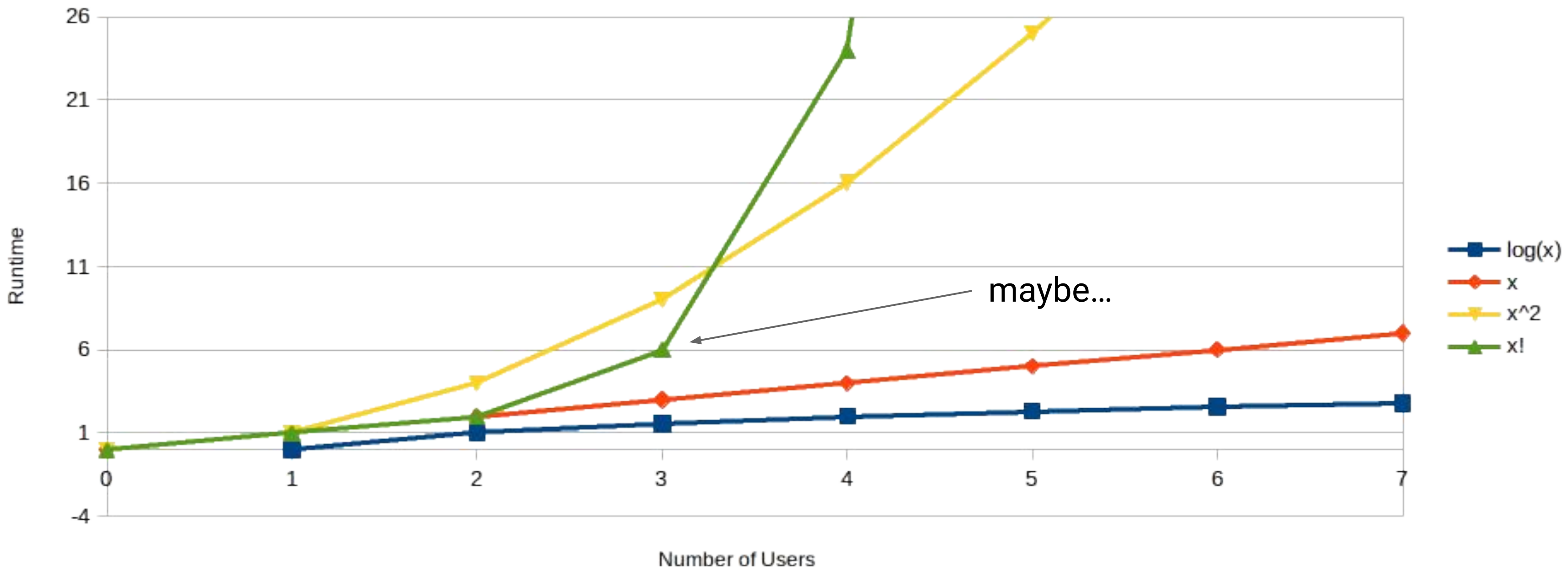
We can start analyzing runtime as a function

Runtime as a Function



Would you consider an algorithm that takes |Users|! number of steps?

Runtime as a Function



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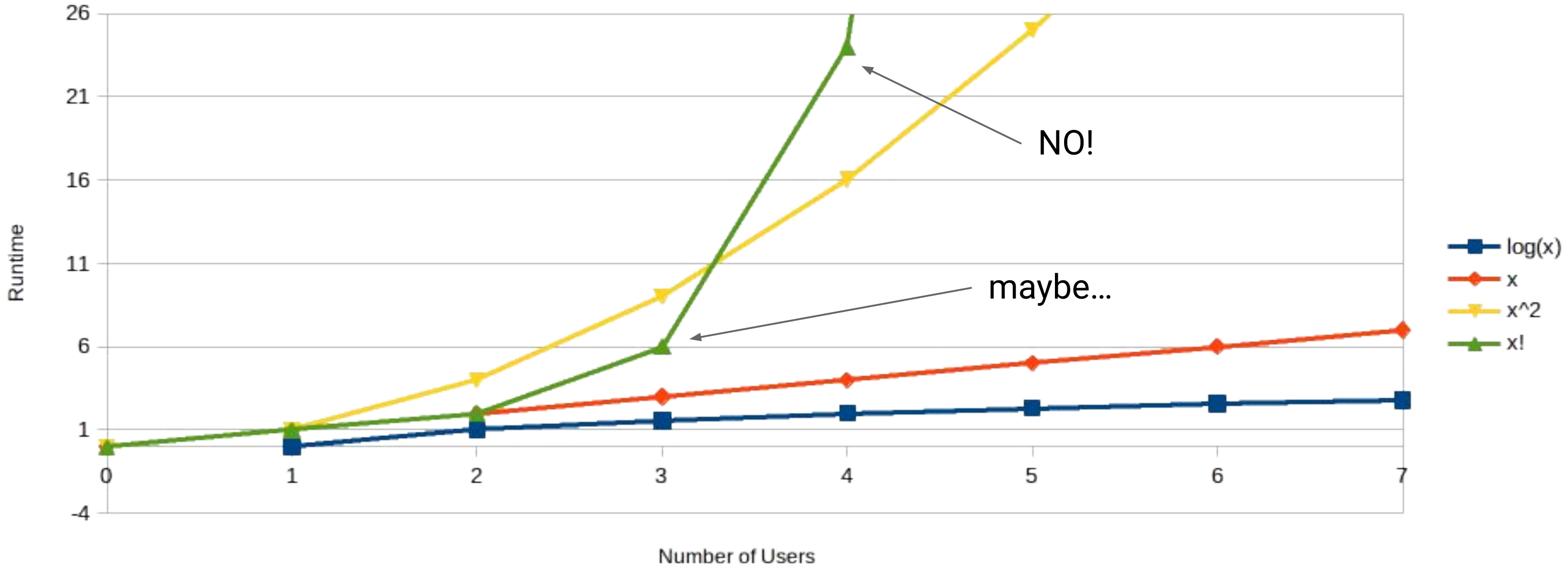

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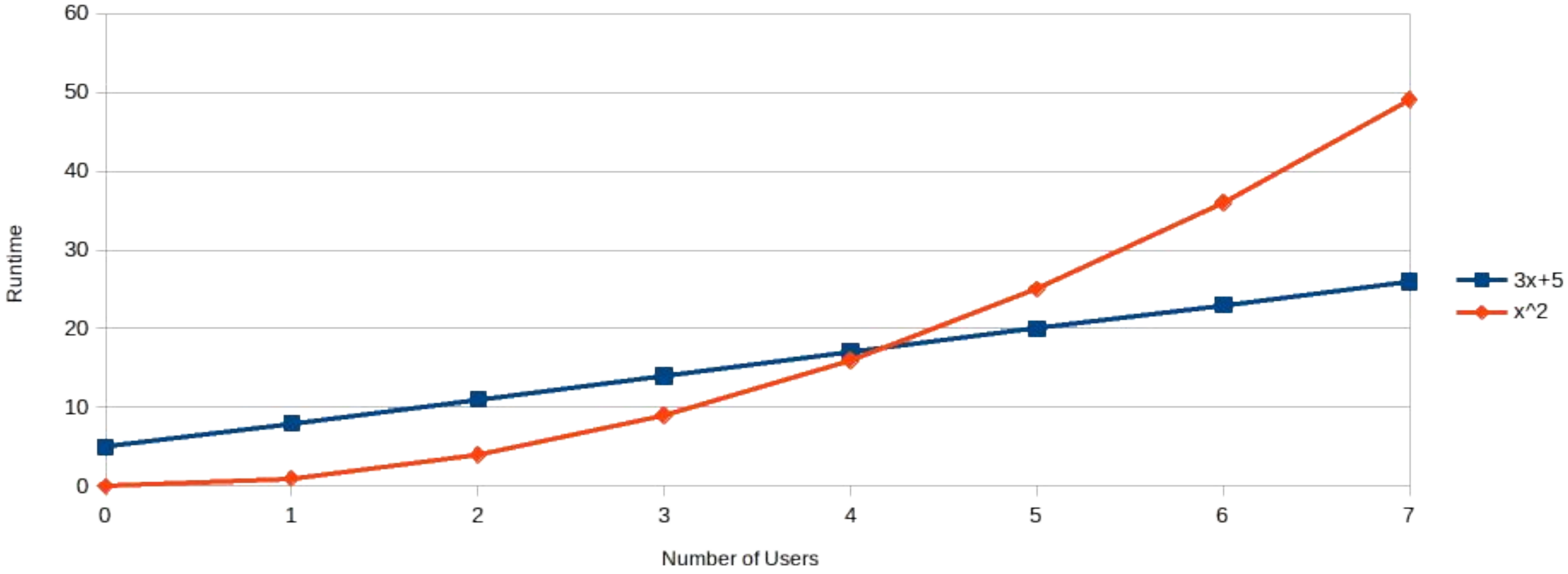

Which is better? 3x|Users|+5 or $|Users|^2$

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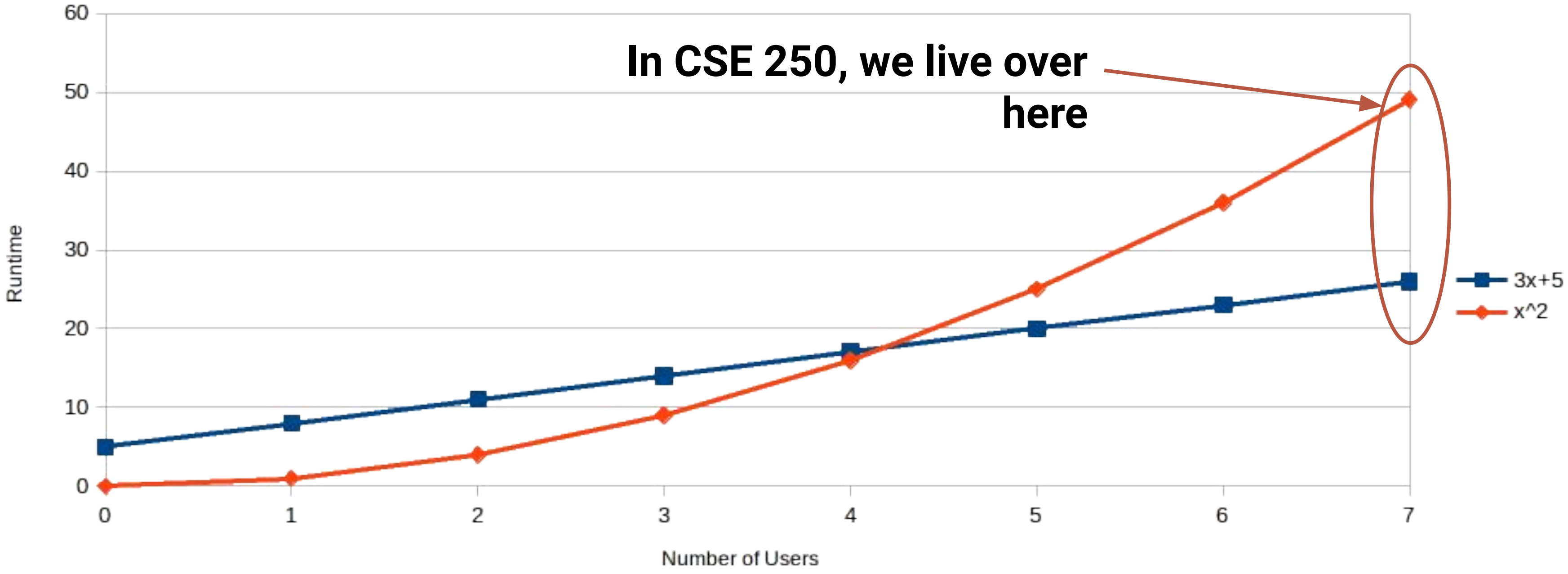
3. **Focus on “large” inputs**
    - **Rank functions based on how they behave at large scales**

Runtime as a Function



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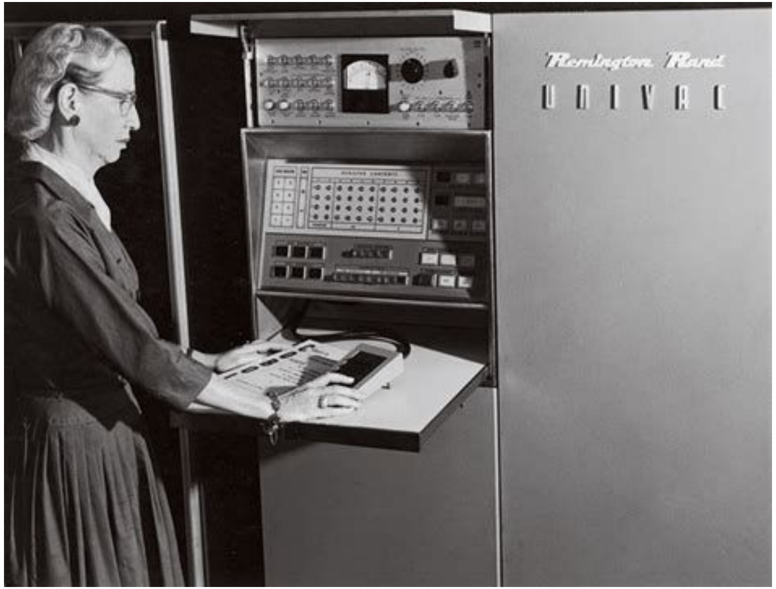
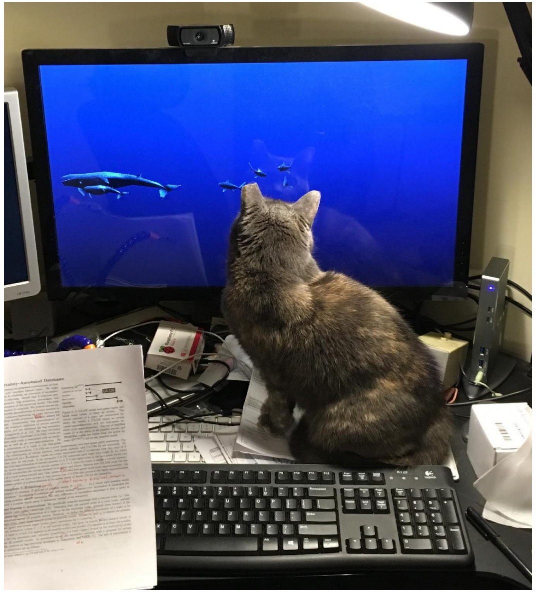
Goal: Ignore implementation details



vs

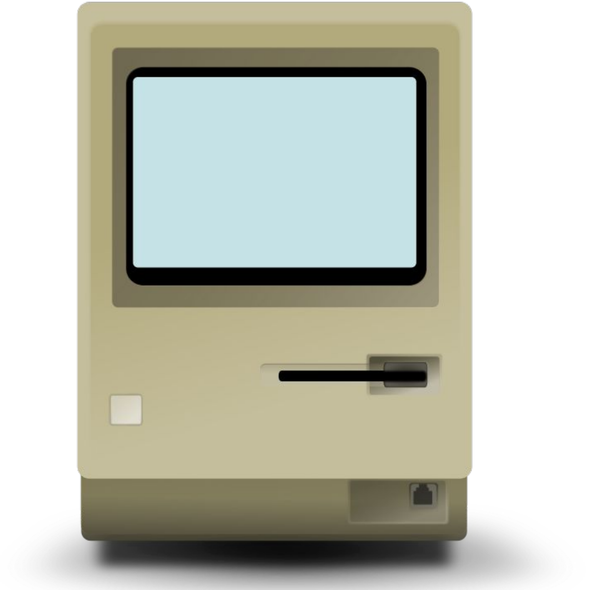


Seasoned Pro Implementation

Error 23: Cat on Keyboard

Goal: Ignore execution environment

vs



Intel i9

Motorola 68000

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Goal: Judge the Algorithm Itself

- How fast is a step? Don’t care
  - Only count number of steps
- Can this be done in two steps instead of one?
  - “3 steps per user” vs “some number of steps per user”
  - Sometimes we don’t care…sometimes we do

Analysis Checklist

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    - Rank functions based on how they behave at large scales
4. **Decouple algorithm from infrastructure/implementation**
    - **Asymptotic notation…?**

Attempt #2: Growth Functions

Not a function in code…but a mathematical function:

$$T(n)$$

n: The "size" of the input

ie: number of users,rows, pixels, etc

***T*(n): The number of "steps" taken for input of size n**

ie: 20 steps per user, where n = |Users|, is 20 x n

Some Basic Assumptions:

Problem sizes are non-negative integers

$$n \in \{0, 1, 2, 3, \ldots\} = \{0\} \cup \mathbb{Z}^+$$

We can’t reverse time…(obviously)

$$T(n) > 0$$

Smaller problems aren’t harder than bigger problems

$$n_1 < n_2 \Rightarrow T(n_1) \leq T(n_2)$$

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$T: \{0\} \cup \mathbb{Z}^+ \to \mathbb{R}^+$

T is non-decreasing

First Problem...

We are still implementation dependent...

$$T_1(n) = 19n$$

$$T_2(n) = 20n$$

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Does 1 extra step per element really matter...?

Is this just an implementation detail?

First Problem...

We are still implementation dependent...

$$T_1(n) = 19n$$

$$T_2(n) = 20n$$

$$T_3(n) = 2n^2$$

T_1 and T_2 are much more "similar" to each other than they are to T_3

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We are still implementation dependent...

$$T_1(n) = 19n$$

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$$T_3(n) = 2n^2$$

T_1 and T_2 are much more "similar" to each other than they are to T_3

How do we capture this idea formally?

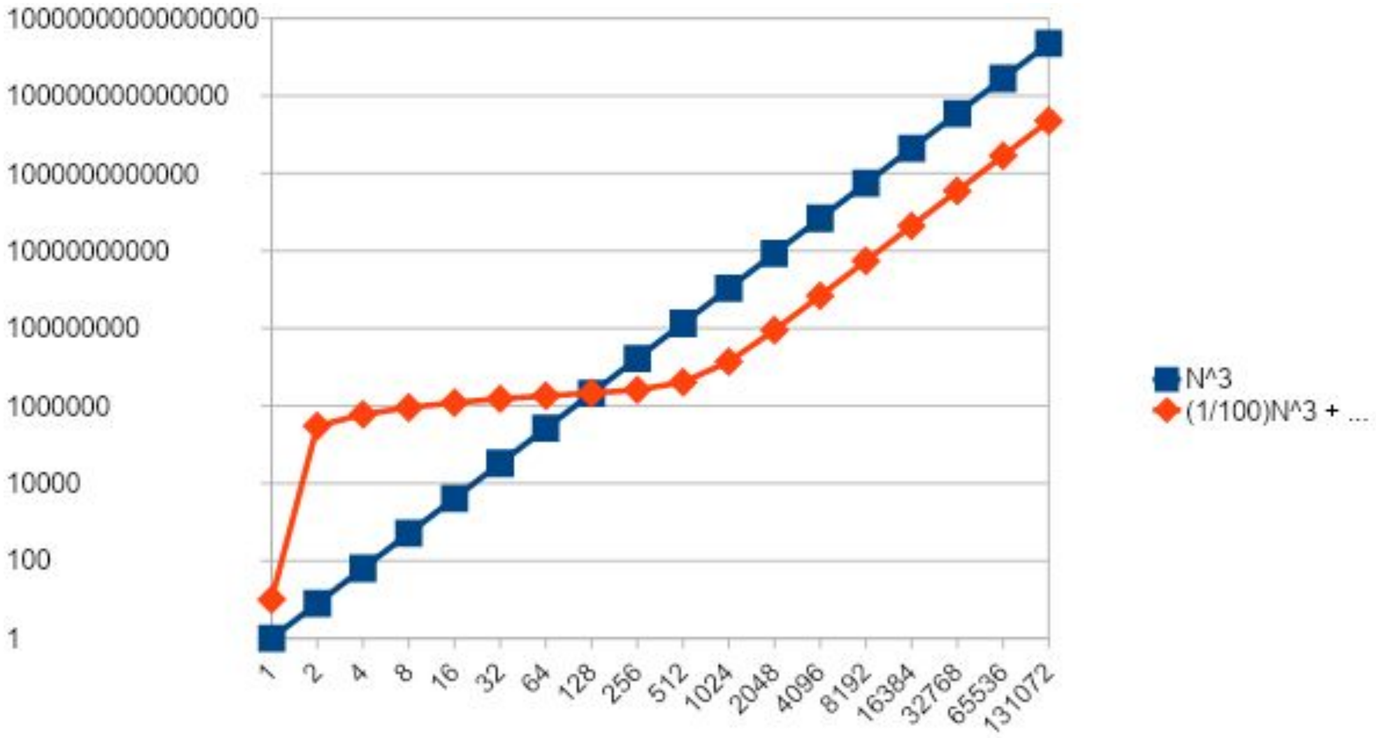
How Do We Capture Behavior at Scale?

Consider the following two functions:

$$\frac{1}{100}n^3 + 10n + 1000000\log(n)$$

$$n^3$$

How Do We Capture Behavior at Scale?



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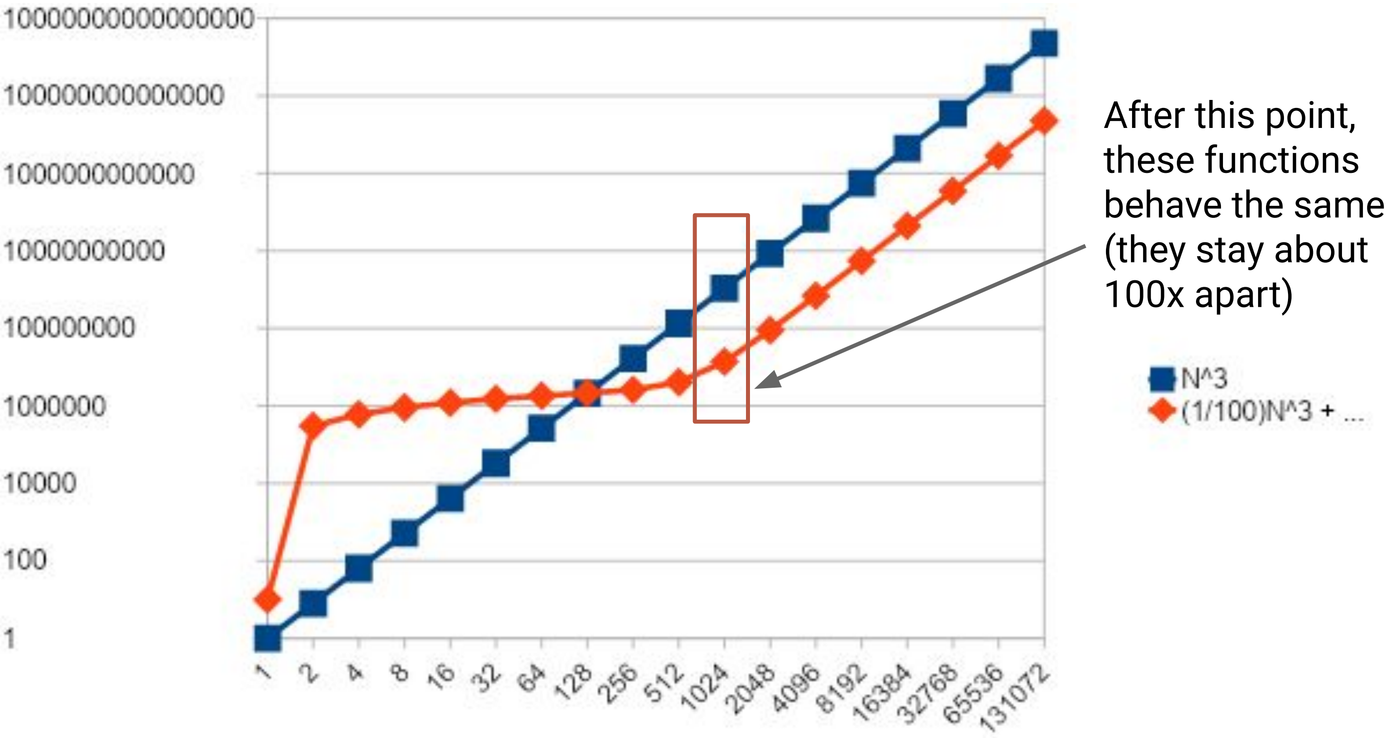

How Do We Capture Behavior at Scale?

$$\lim_{n \to \infty} \frac{\frac{1}{100} n^3 + 10n + 1000000 \log(n)}{n^3}$$

How Do We Capture Behavior at Scale?

$$\lim_{n\to\infty} \frac{\frac{1}{100}n^3 + 10n + 1000000\log(n)}{n^3}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{100}n^3}{n^3} + \frac{10n}{n^3} + \frac{1000000\log(n)}{n^3}$$

How Do We Capture Behavior at Scale?

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$$= \lim_{n\to\infty} \frac{1}{100} + \boxed{\lim_{n\to\infty} \frac{10}{n^2}} + \boxed{\lim_{n\to\infty} \frac{1000000\log(n)}{n^3}}$$

These terms go to 0

How Do We Capture Behavior at Scale?

$$\lim_{n\to\infty} \frac{\frac{1}{100}n^3 + 10n + 1000000\log(n)}{n^3}$$

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$$= \lim_{n\to\infty} \frac{1}{100} + \lim_{n\to\infty} \frac{10}{n^2} + \lim_{n\to\infty} \frac{1000000\log(n)}{n^3}$$

$$= \frac{1}{100}$$

Attempt #3: Asymptotic Analysis

Consider two functions, f(n) and g(n)

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$$

In this particular case, f grows w.r.t. n faster than g

So…if f(n) and g(n) are the number of steps two different algorithms take on a problem of size n, which is better?

Attempt #3: Asymptotic Analysis

Case 1: $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$ *(f grows faster; g is better)*

Case 2: $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$ *(g grows faster; f is better)*

Case 3: $\lim_{n \to \infty} \frac{f(n)}{g(n)} = some\ \ constant$ *(f and g "behave" the same)*

Goal of “Asymptotic Analysis”

We want to organize runtimes (growth functions) into different ***Complexity Classes***

Within the same complexity class, runtimes “behave the same”

Goal of “Asymptotic Analysis”

“Strategic Optimization” focuses on improving the complexity class of your code!

Back to Our Previous Example...

$$\frac{1}{100}n^3 + 10n + 1000000\log(n)$$

The 10n and 1000000 log(n) “don’t matter”

The 1/100 “does not matter”

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$$\frac{1}{100}n^3 + 10n + 1000000\log(n)$$

The 10n and 1000000 log(n) “don’t matter”

The 1/100 “does not matter”

n^3 is the dominant term, and that determines the “behavior”

Why Focus on Dominating Terms?

$f(n)$	**10**	**20**	**50**	**100**	**1000**
$log(log(n))$	0.43 ns	0.52 ns	0.62 ns	0.68 ns	0.82 ns
$log(n)$	0.83 ns	1.01 ns	1.41 ns	1.66 ns	2.49 ns
n	2.5 ns	5 ns	12.5 ns	25 ns	0.25 μs
$nlog(n)$	8.3 ns	22 ns	71 ns	0.17 μs	2.49 μs
n^2	25 ns	0.1 μs	0.63 μs	2.5 μs	0.25 ms
n^5	25 μs	0.8 ms	78 ms	2.5 s	**2.9 days**
2^n	0.25 μs	0.26 ms	**3.26 days**	**10^{13} years**	**10^{284} years**
$n!$	0.91 ms	**19 years**	**10^{47} years**	**10^{141} years**	🤯

Why Focus on Dominating Terms?

$$2^n \gg n^c \gg n \gg log(n) \gg c$$