# CSE 250
## Data Structures

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# Lec 06: Asymptotic Analysis (cont)

# Announcements and Feedback

- Recitations start this week
- Office hours schedule updated
- PA1 released
  - Testing phase due Sunday 9/17 @ 11:59PM
    - Recitation this week will go over tips for testing
  - Implementation phase due Sunday 9/24 @ 11:59PM

# "Shape" of a Function

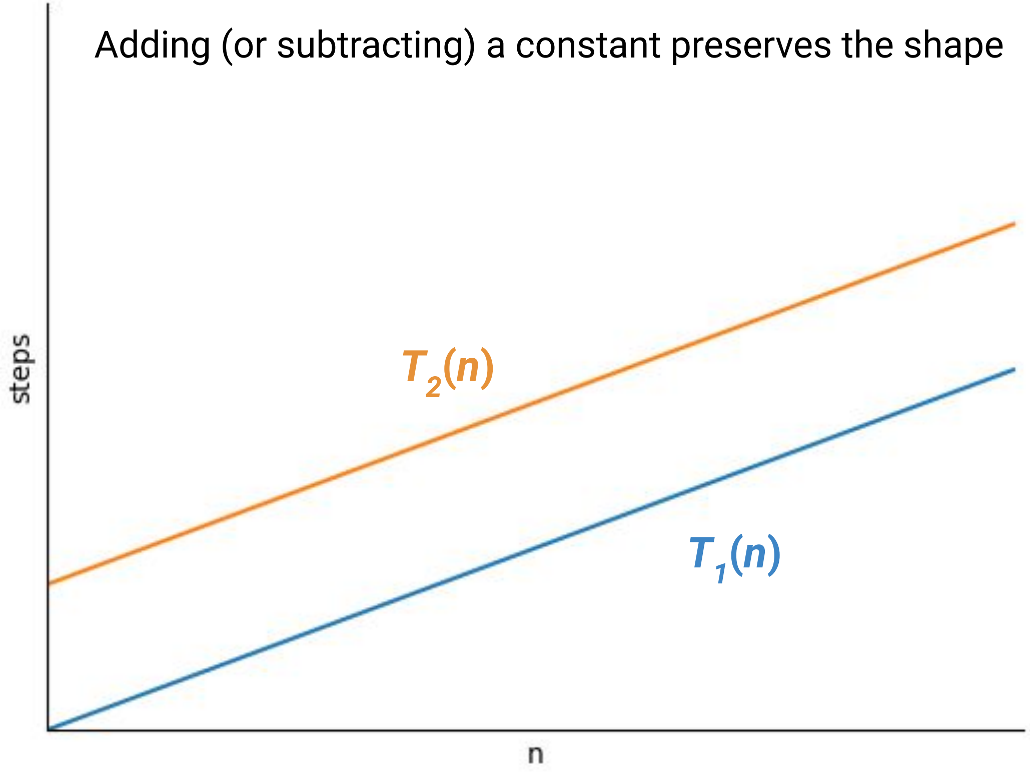
When do we consider two functions to have the same shape?

# Additive Factors

Consider two growth functions:

$T_1(n) = 3n$

$T_2(n) = 3n + 3$

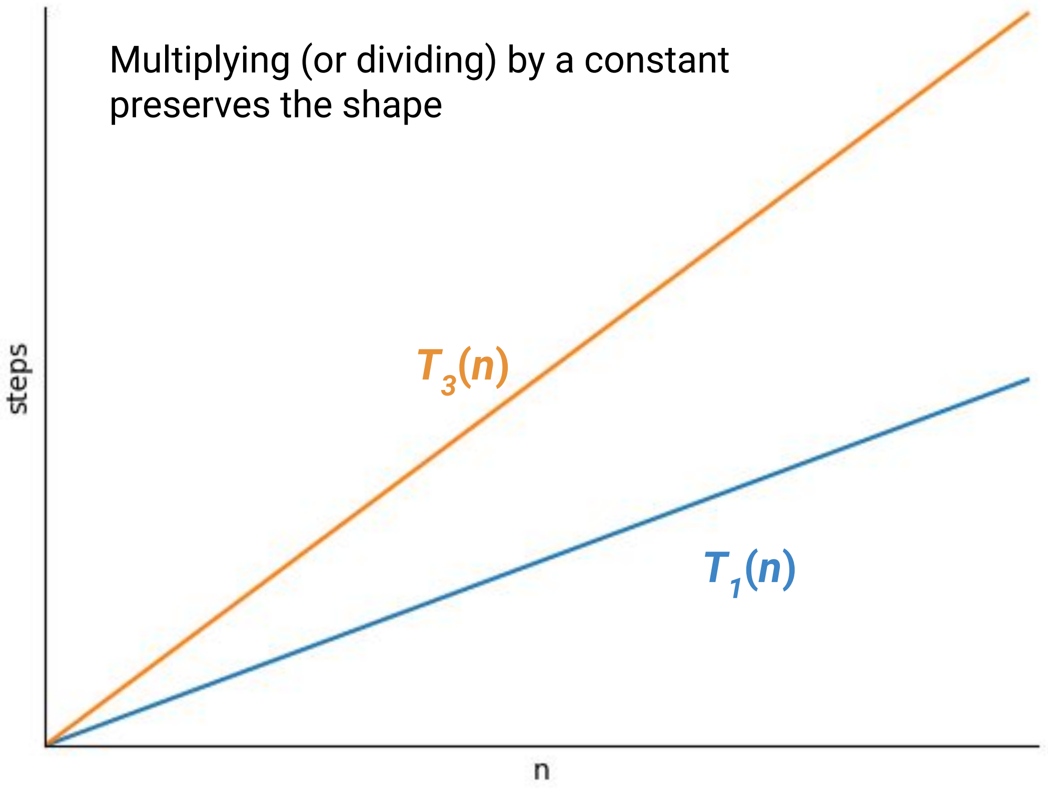


# Multiplicative Factors

Consider two growth functions:

$T_1(n) = 3n$

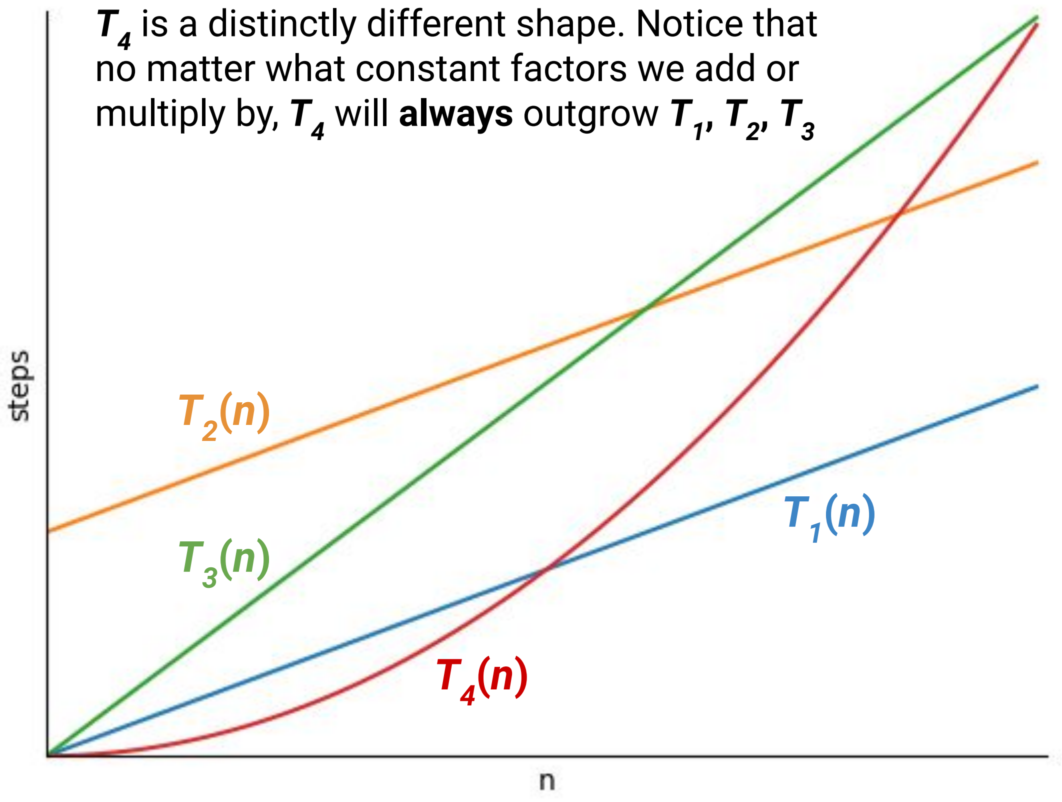
$T_3(n) = 6n$



# A Counter Example

Now consider:

$T_4(n) = n^2$



# Complexity Class

***f*** and ***g*** are in the same complexity class, denoted $\boldsymbol{g(n) \in \Theta(f(n))}$, iff:

> ***g*** is bounded from above by something ***f***-shaped
>
> ***and***
>
> ***g*** is bounded from below by something ***f***-shaped

# Bounded from Above: Big O

$g(n)$ is bounded from above by $f(n)$ if:

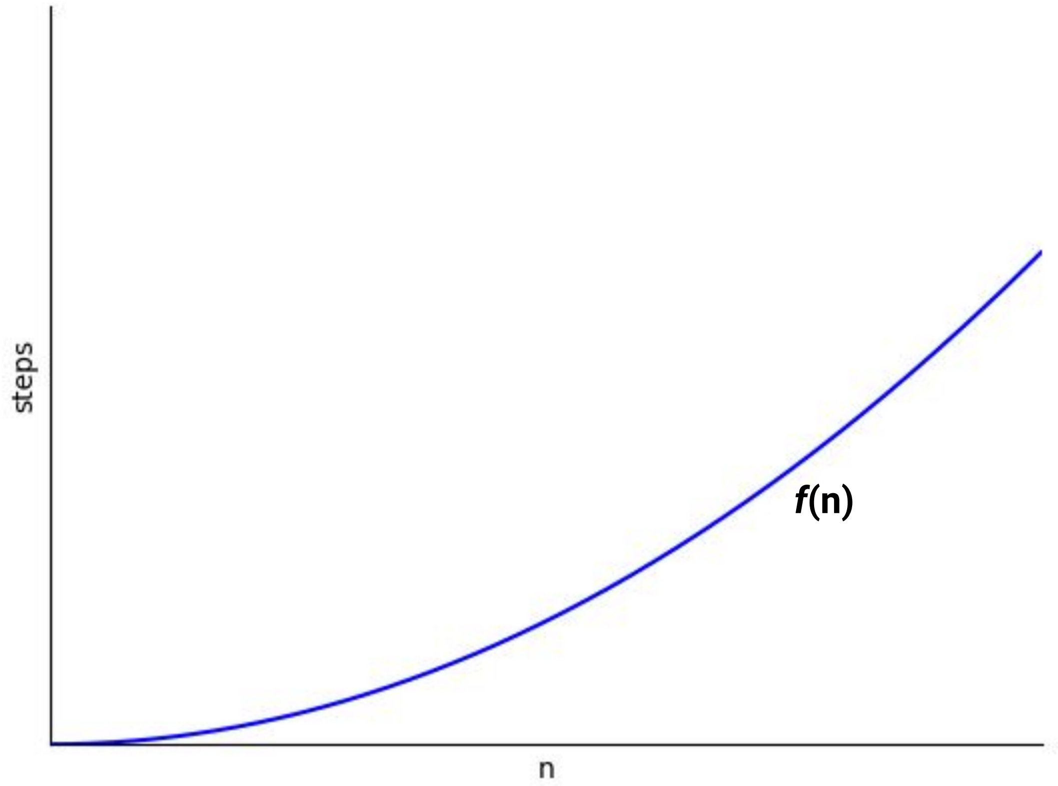
There exists a constant $n_0$ **> 0** and a constant $c$ **> 0** such that:

$$\text{For all } n > n_0,\ g(n) \leq c \cdot f(n)$$

In this case, we say that $g(n) \in O(f(n))$

$O(f(n))$ is the set of all functions bounded from above by $f(n)$

# Bounded from Above: Big O



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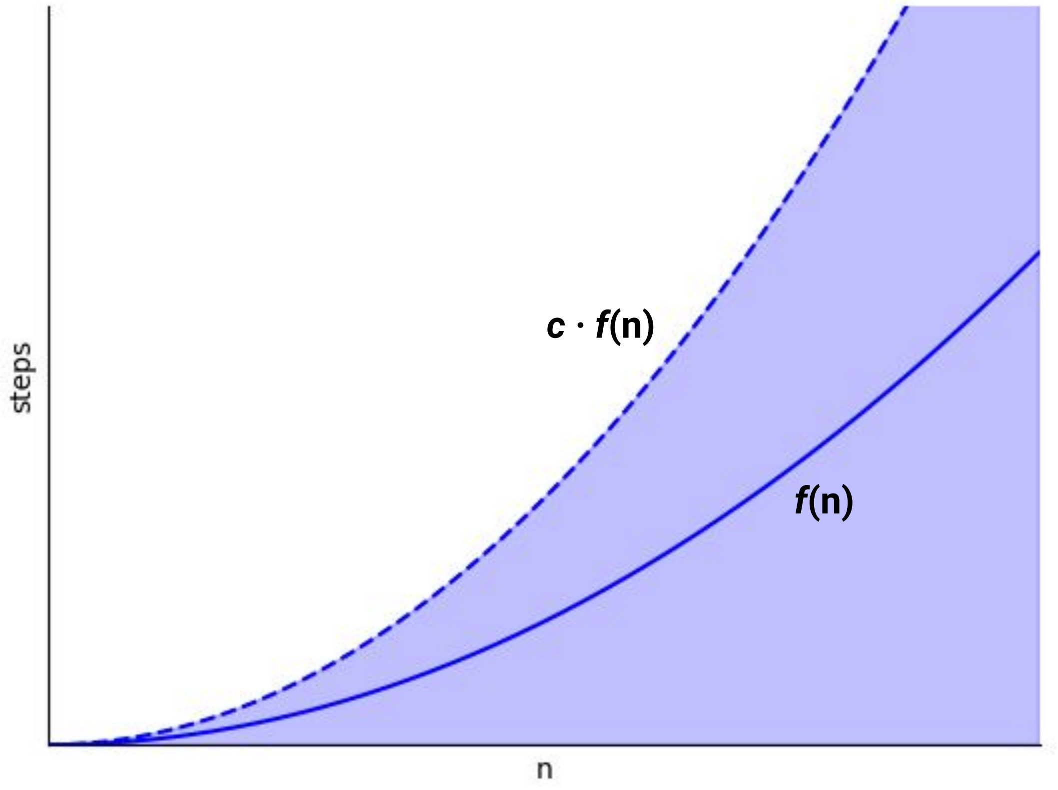
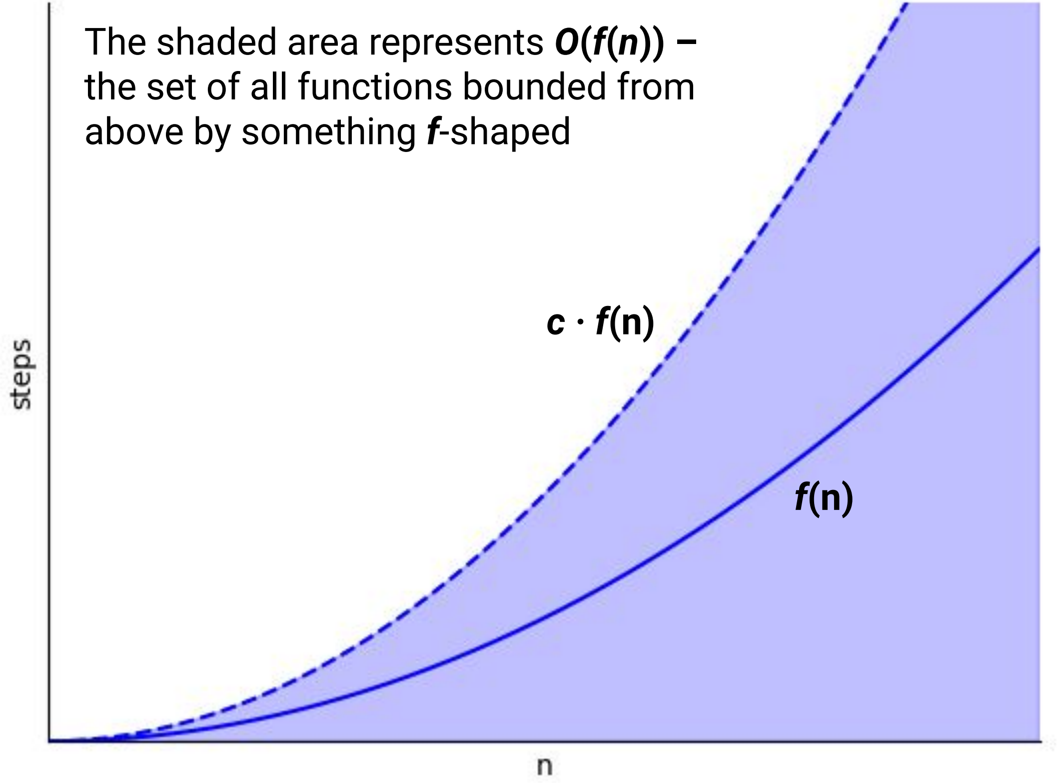


# Bounded from Above: Big O



# Bounded from Above: Big O

$$g(n) = \frac{n^2}{2} + 4n + 7$$

Prove that $\boldsymbol{g(n) \in O(n^2)}$

$$\frac{n^2}{2} + 4n + 7 \leq c \cdot n^2$$

# Inequality Tricks

$f(n) \geq g(n)$ is true if $f(n)/a \geq g(n)/a$ (for any $a > 0$)

$f(n) \geq g(n)$ is true if $f(n)*a \geq g(n)*a$ (for any $a > 0$)

$x + a \geq y + b$ is true if $x \geq y$ and $a \geq b$ (for any $a$, $b$)

$x \geq y$ is true if $x \geq a$ and $a \geq y$ (for any $a$)

$1 \leq \log(n) \leq n \leq n^2 \leq n^k \text{ (for } k \geq 2) \leq 2^n$

# Bounded from Above: Big O

$$\frac{n^2}{2} + 4n + 7 \leq c \cdot n^2$$

First show:

$$\frac{n^2}{2} \leq c_1 \cdot n^2 \qquad 4n \leq c_2 \cdot n^2 \qquad 7 \leq c_3 \cdot n^2$$

# Bounded from Above: Big O

$$\frac{n^2}{2} \leq c_1 \cdot n^2$$

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$$\frac{n^2}{2} \leq c_1 \cdot n^2$$

This is true for all ***n* ≥ 0** if we set $c_1$ to **1/2**

# Bounded from Above: Big O

$$4n \leq c_2 \cdot n^2$$

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$$4n \leq c_2 \cdot n^2$$

This is true for all $\boldsymbol{n} \geq \mathbf{0}$ if we set $\boldsymbol{c_2}$ to **4**

# Bounded from Above: Big O

$$7 \leq c_3 \cdot n^2$$

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$$7 \leq c_3 \cdot n^2$$

This is true for all $\boldsymbol{n} \geq \mathbf{1}$ if we set $\boldsymbol{c_3}$ to **7**

# Bounded from Above: Big O

$$\frac{n^2}{2} + 4n + 7 \leq c \cdot n^2$$

First show:

$$\frac{n^2}{2} \leq c_1 \cdot n^2 \qquad 4n \leq c_2 \cdot n^2 \qquad 7 \leq c_3 \cdot n^2$$

# Bounded from Above: Big O

$$\boxed{\frac{n^2}{2}} + \boxed{4n} + \boxed{7} \leq c \cdot n^2$$

First show:

$$\frac{n^2}{2} \leq \frac{1}{2} \cdot n^2 \qquad 4n \leq 4 \cdot n^2 \qquad 7 \leq 7 \cdot n^2$$

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$$\frac{n^2}{2} + 4n + 7 \leq \frac{1}{2} \cdot n^2 + 4 \cdot n^2 + 7 \cdot n^2$$

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$$\frac{n^2}{2} + 4n + 7 \leq \frac{1}{2} \cdot n^2 + 4 \cdot n^2 + 7 \cdot n^2$$

# Bounded from Above: Big O

$$\boxed{\frac{n^2}{2}} + \boxed{4n} + \boxed{7} \leq c \cdot n^2$$

First show:

$$\frac{n^2}{2} \leq \frac{1}{2} \cdot n^2 \qquad 4n \leq 4 \cdot n^2 \qquad 7 \leq 7 \cdot n^2$$

$$\frac{n^2}{2} + 4n + 7 \leq \frac{1}{2} \cdot n^2 + 4 \cdot n^2 + 7 \cdot n^2 = (\frac{1}{2} + 4 + 7) \cdot n^2$$

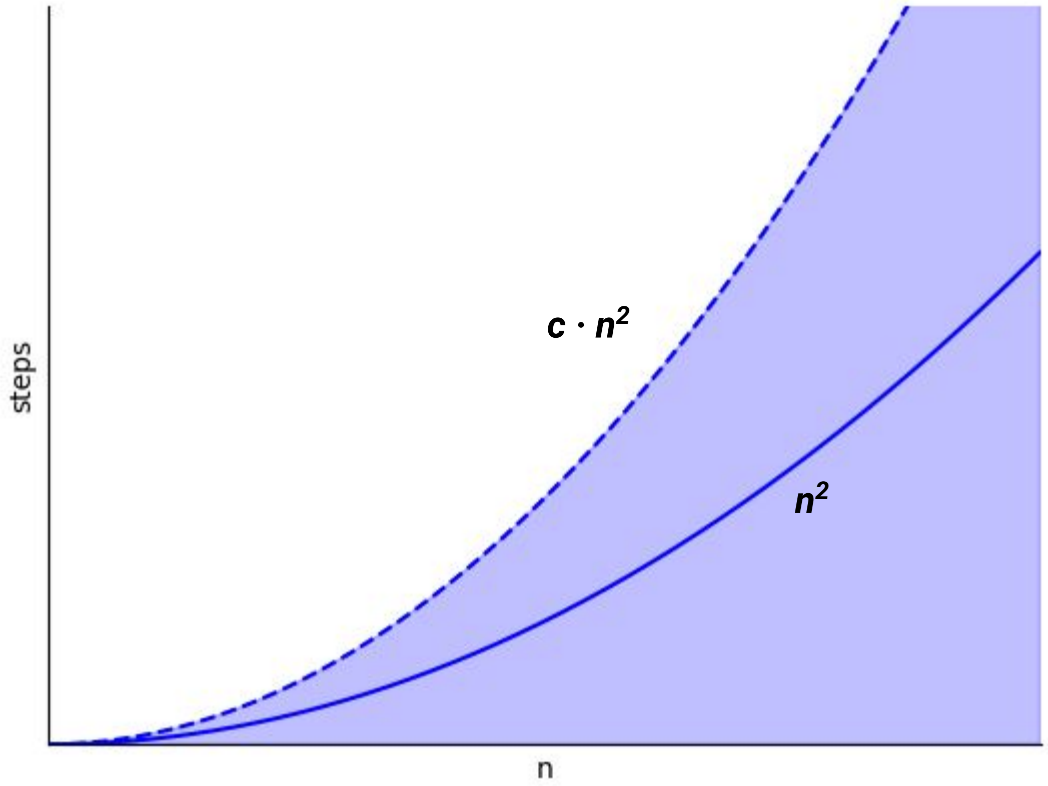
# Bounded from Above: Big O

$$\frac{n^2}{2} + 4n + 7 \leq c \cdot n^2$$

Therefore if we let ***c* = 11.5**, then for all ***n* ≥ 1** the above holds true

Therfore $\boldsymbol{g(n) \in O(n^2)}$

# Bounded from Above: Big O



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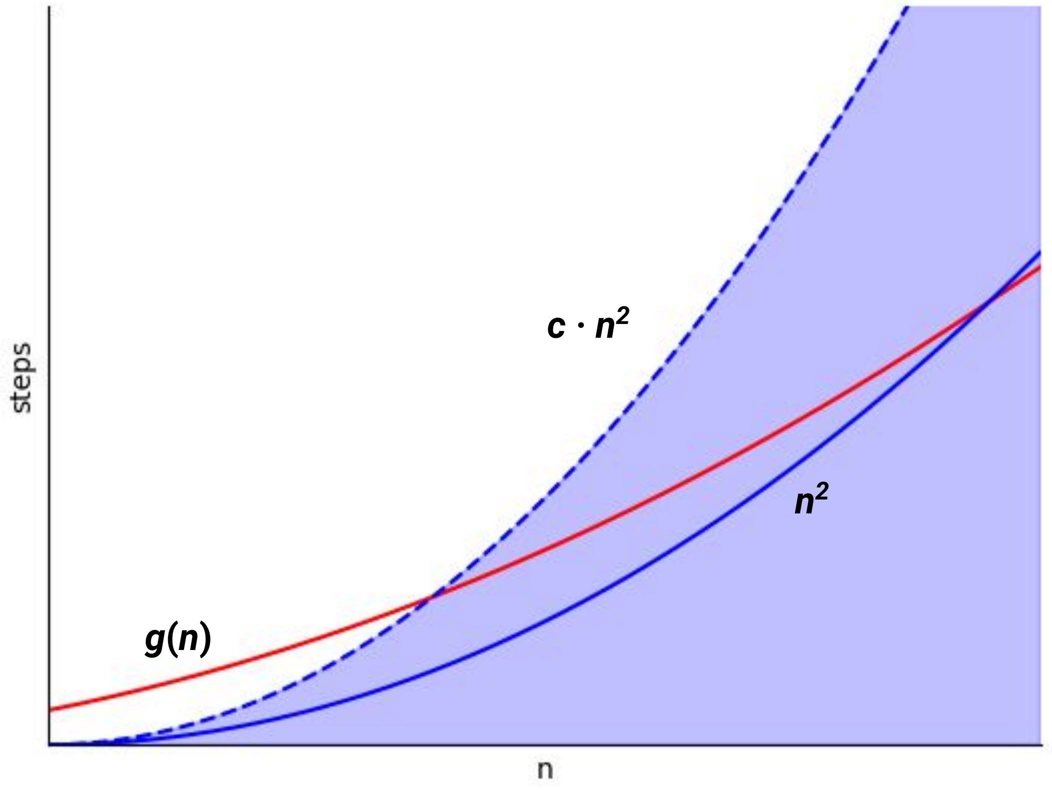

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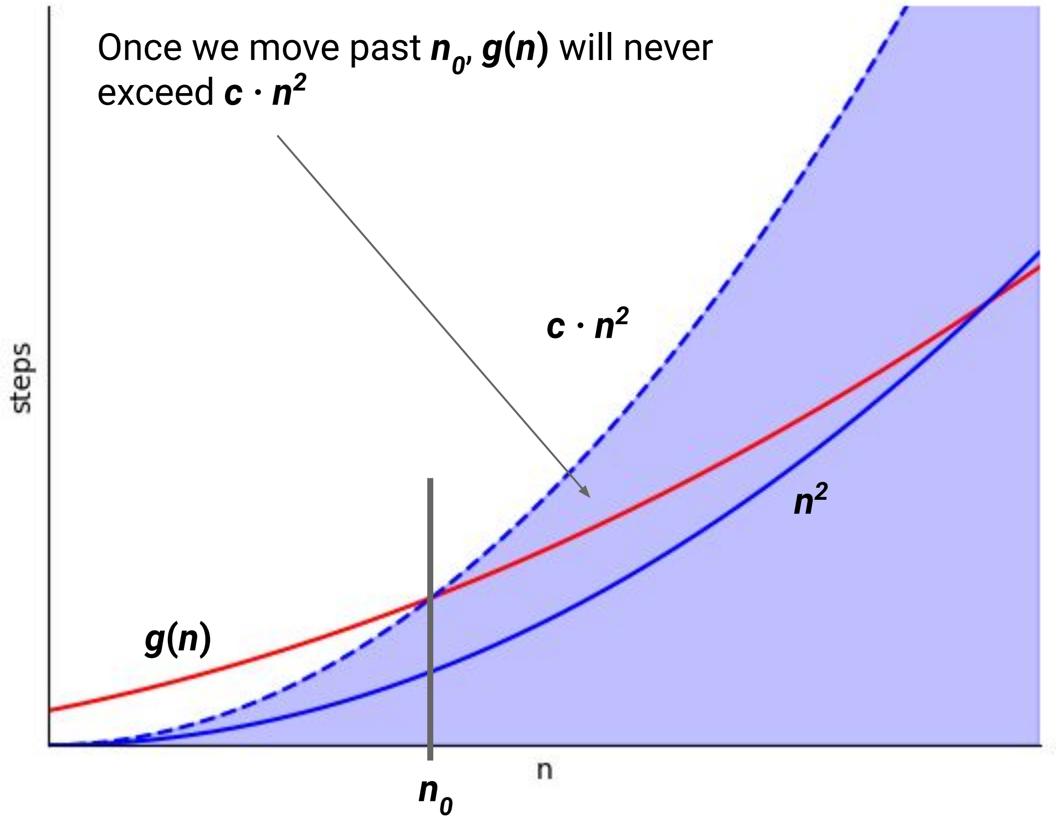


# Bounded from Below: Big Ω

***g*(*n*)** is bounded from below by ***f*(*n*)** if:

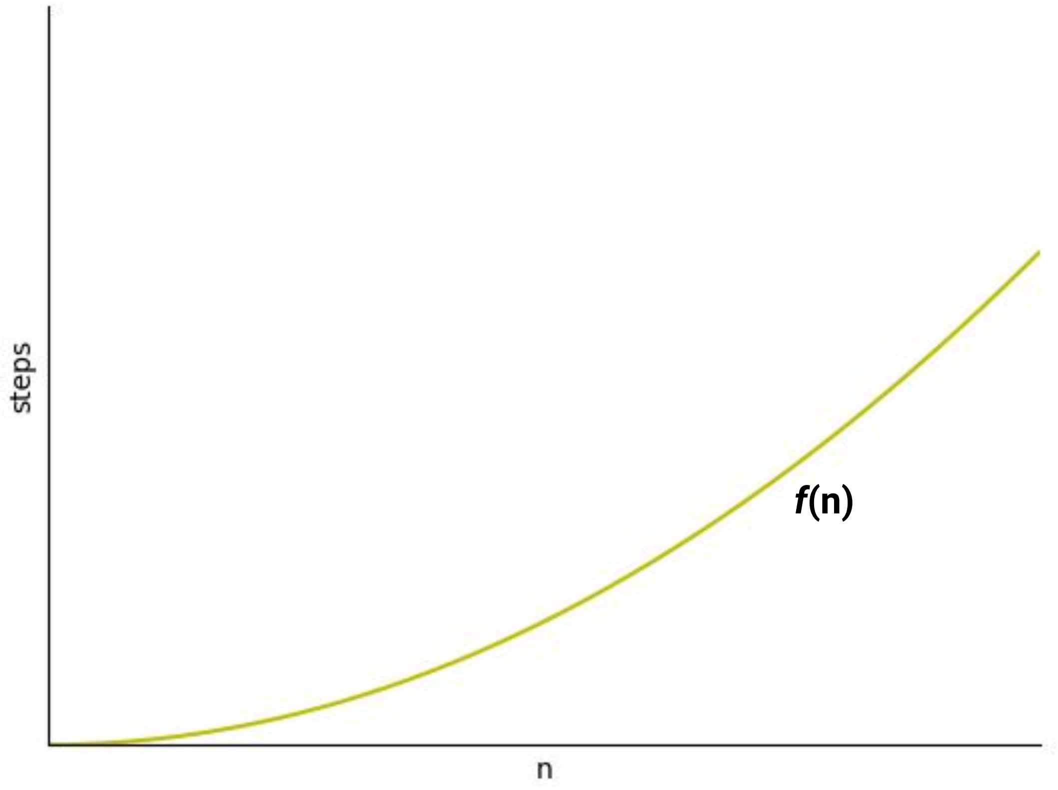
There exists a constant $n_0 > 0$ and a constant $c > 0$ such that:

$$\text{For all } n > n_0,\ g(n) \geq c \cdot f(n)$$

In this case, we say that $g(n) \in \Omega(f(n))$

$\Omega(f(n))$ is the set of all functions bounded from below by ***f*(*n*)**

# Bounded from Below: Big Ω



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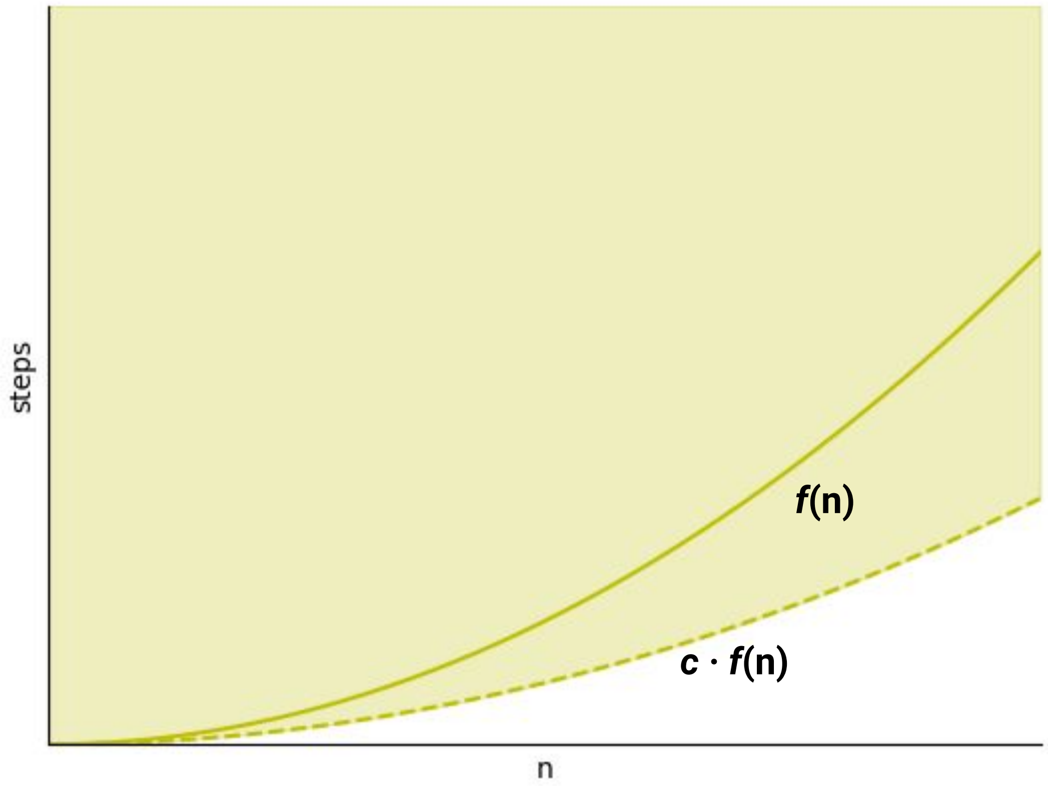
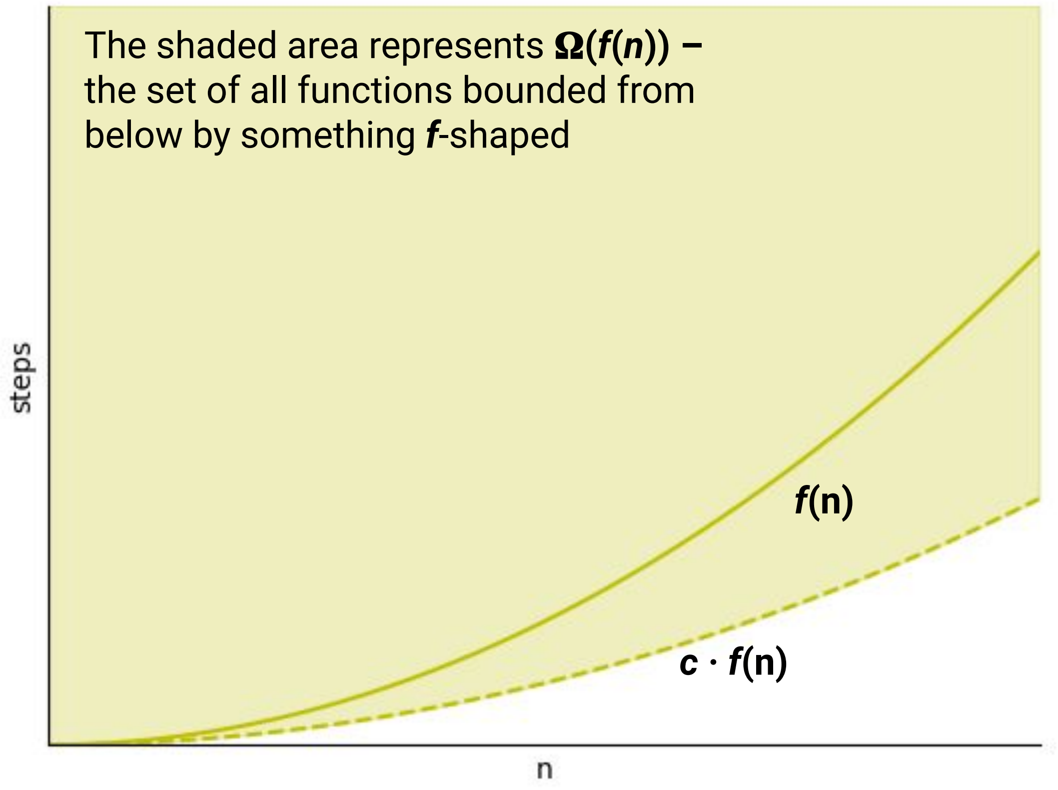


# Bounded from Below: Big Ω



# Complexity Class: Big Θ

***f*** and ***g*** are in the same complexity class, denoted $\boldsymbol{g(n) \in \Theta(f(n))}$, iff:

***g*** is bounded from above by something ***f***-shaped

***and***

***g*** is bounded from below by something ***f***-shaped

# Complexity Class: Big $\Theta$

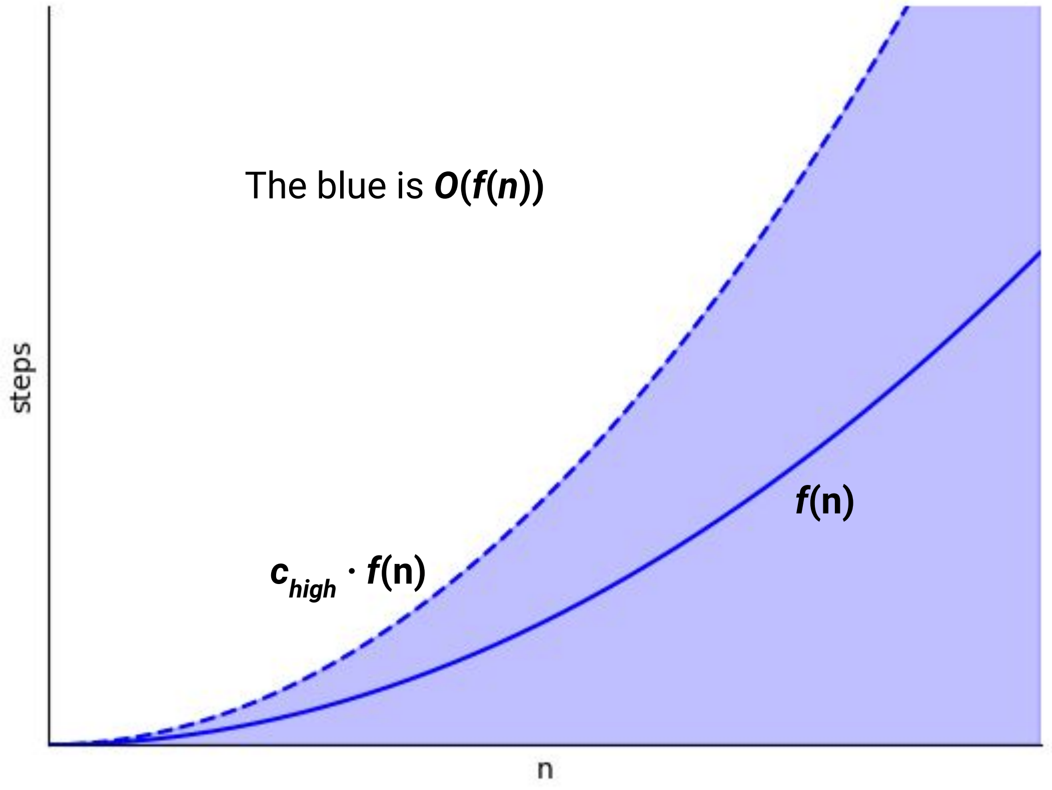
***f*** and ***g*** are in the same complexity class, denoted $g(n) \in \Theta(f(n))$, iff:

$g(n) \in O(f(n))$

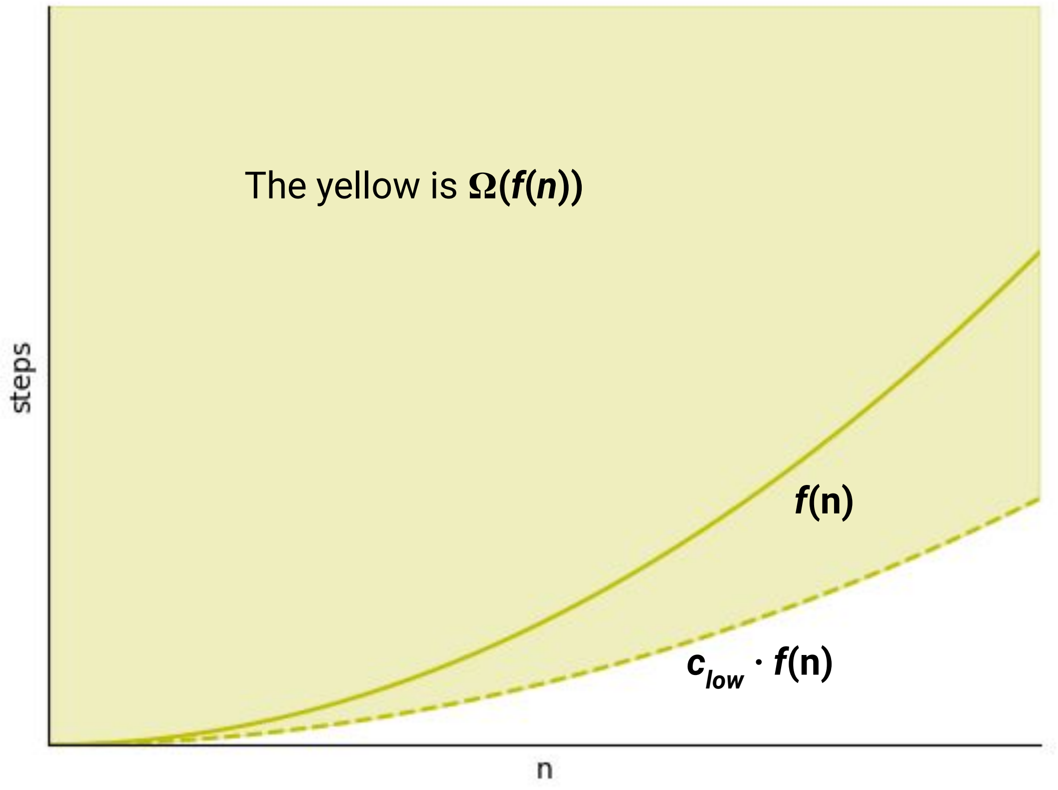
***and***

$g(n) \in \Omega(f(n))$

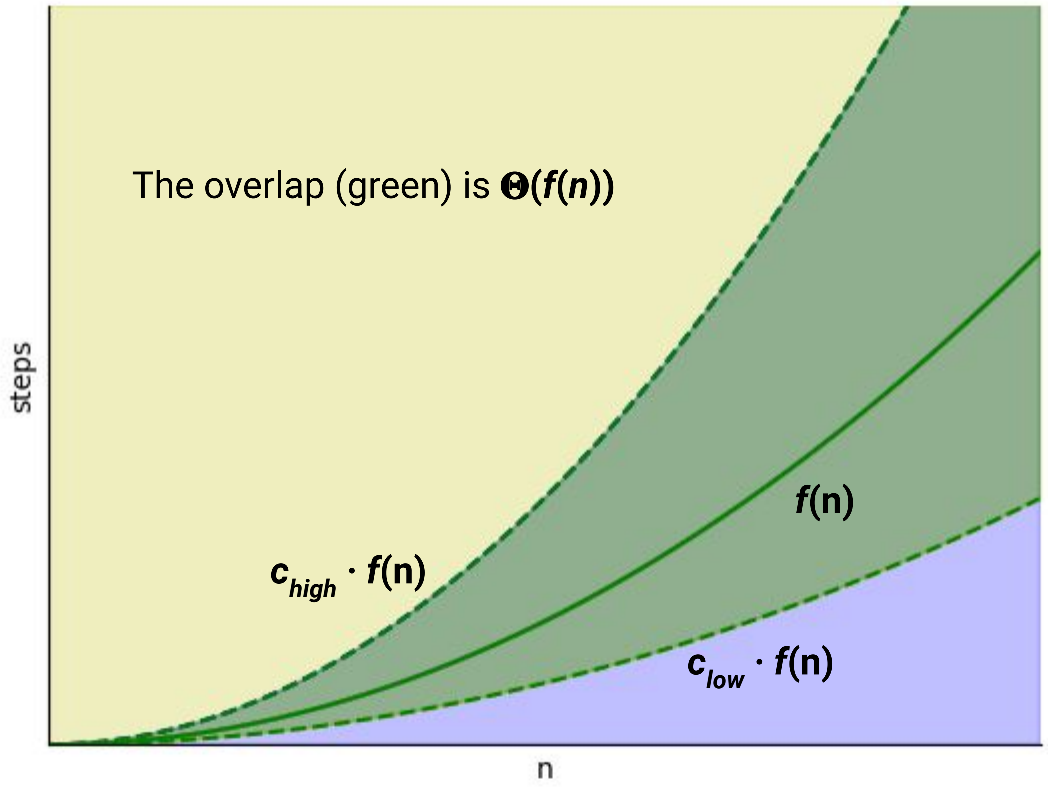
# Complexity Class: Big Θ



# Complexity Class: Big Θ

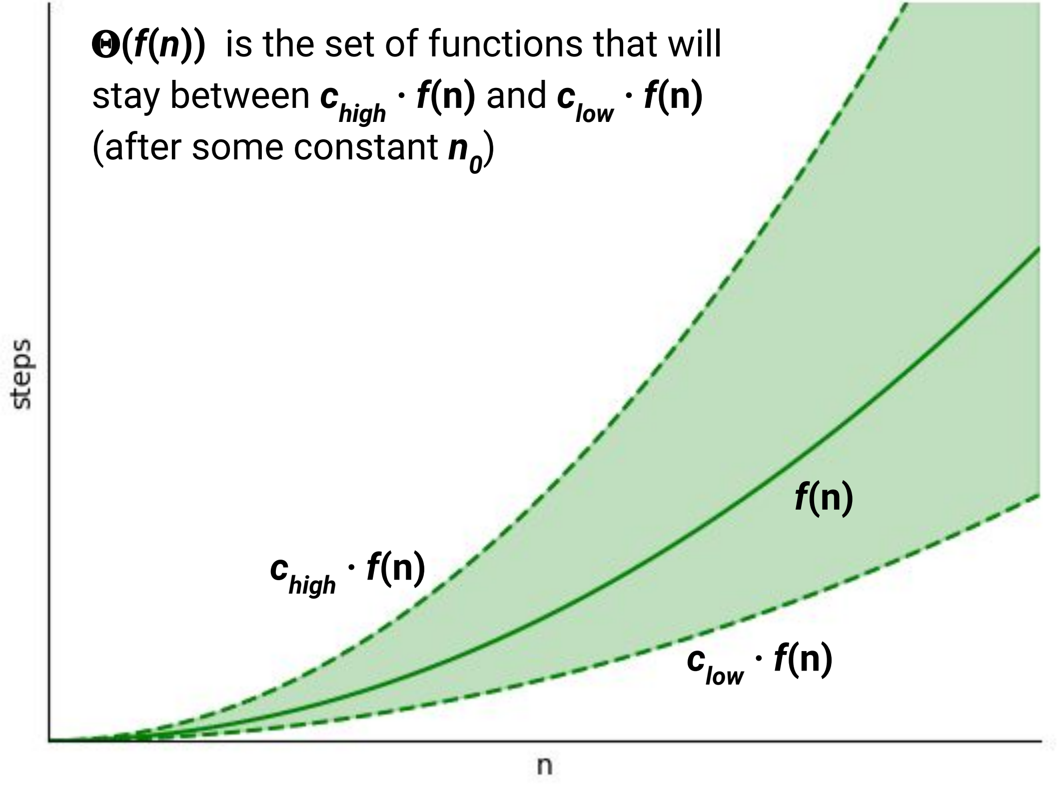


# Complexity Class: Big Θ

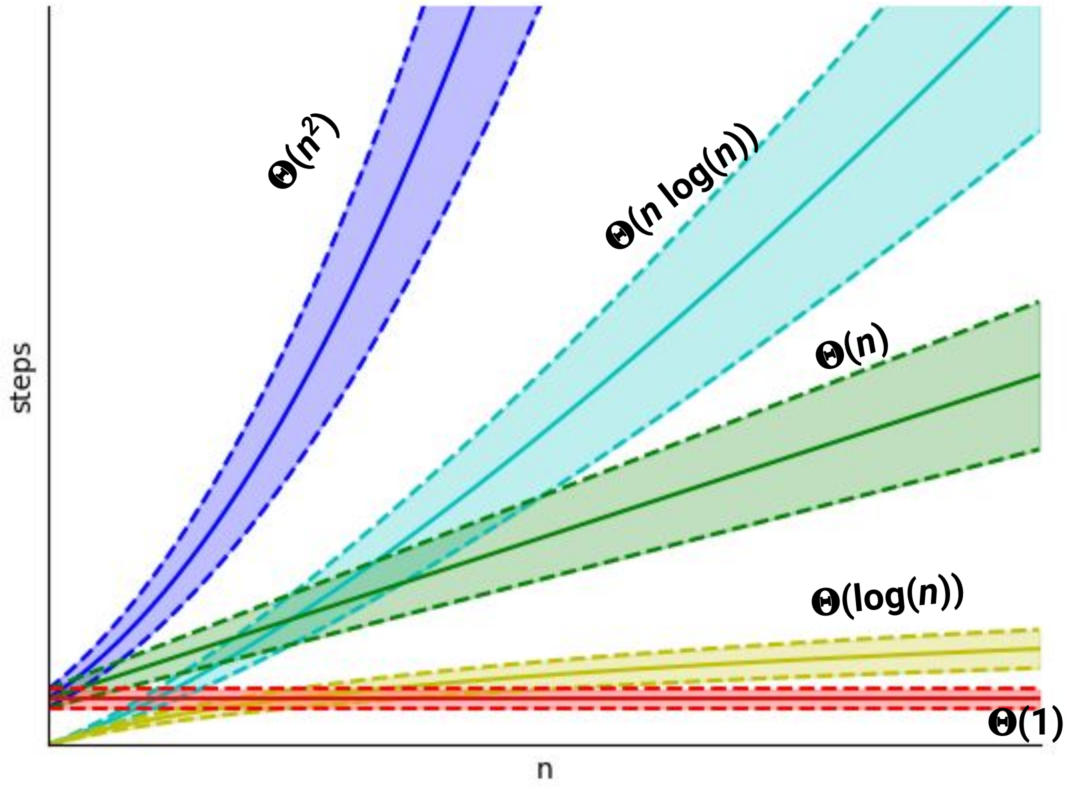


# Complexity Class: Big Θ

**Θ**(***f***(***n***)) is the set of functions that will stay between $\boldsymbol{c_{high}} \cdot \boldsymbol{f}(\mathbf{n})$ and $\boldsymbol{c_{low}} \cdot \boldsymbol{f}(\mathbf{n})$ (after some constant $\boldsymbol{n_0}$)

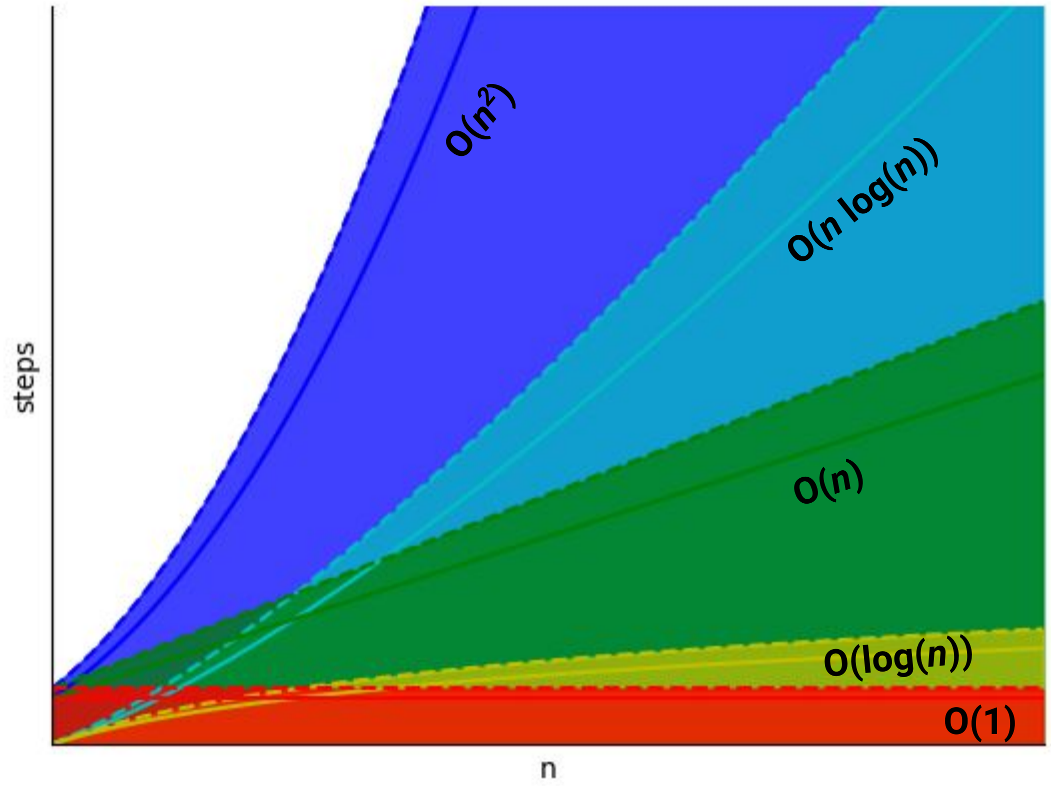


# Complexity Class Ranking




$\Theta(1) < \Theta(\log(n)) < \Theta(n) < \Theta(n \log(n)) < \Theta(n^2) < \Theta(n^3) < \Theta(2^n)$

# Big O Subsets




$O(1) \subset O(\log(n)) \subset O(n) \subset O(n \log(n)) \subset O(n^2) \subset O(n^3) \subset O(2^n)$

# Shortcut

What complexity class do each of the following belong to:

$f(n) = 4n + n^2$

$g(n) = 2^n + 4n$

$h(n) = 100\, n \log(n) + 73n$

# Shortcut

What complexity class do each of the following belong to:

$f(n) = 4n + n^2 \in \Theta(n^2)$

$g(n) = 2^n + 4n \in \Theta(2^n)$

$h(n) = 100\, n \log(n) + 73n \in \Theta(n \log(n))$

# Shortcut

What complexity class do each of the following belong to:

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$g(n) = 2^n + 4n \in \Theta(2^n)$

$h(n) = 100\, n \log(n) + 73n \in \Theta(n \log(n))$

**Shortcut:** Just consider the complexity of the most dominant term

# Why Focus on Dominating Terms?

| $f(n)$ | 10 | 20 | 50 | 100 | 1000 |
|---|---|---|---|---|---|
| $log(log(n))$ | 0.43 ns | 0.52 ns | 0.62 ns | 0.68 ns | 0.82 ns |
| $log(n)$ | 0.83 ns | 1.01 ns | 1.41 ns | 1.66 ns | 2.49 ns |
| $n$ | 2.5 ns | 5 ns | 12.5 ns | 25 ns | 0.25 µs |
| $nlog(n)$ | 8.3 ns | 22 ns | 71 ns | 0.17 µs | 2.49 µs |
| $n^2$ | 25 ns | 0.1 µs | 0.63 µs | 2.5 µs | 0.25 ms |
| $n^5$ | 25 µs | 0.8 ms | 78 ms | 2.5 s | **2.9 days** |
| $2^n$ | 0.25 µs | 0.26 ms | **3.26 days** | **$10^{13}$ years** | **$10^{284}$ years** |
| $n!$ | 0.91 ms | **19 years** | **$10^{47}$ years** | **$10^{141}$ years** | 🤯 |

# Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in O(n^2)$ ✓

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Is $f(n)$ in $O(n^3)$?

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Is $f(n)$ in $O(n^3)$? ✓

# Tight Bounds

***f*(*n*) = 4*n* + *n*²** $\in \Theta(n^2)$, therefore ***f*(*n*) = 4*n* + *n*²** $\in O(n^2)$ ✓

Is $f(n)$ in $O(n^3)$? ✓

Is $f(n)$ in $O(2^n)$?

# Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in O(n^2)$ ✓

Is $f(n)$ in $O(n^3)$? ✓

Is $f(n)$ in $O(2^n)$? ✓

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Is $f(n)$ in $O(n)$? ✗

# Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in O(n^2)$ ✓

Is $f(n)$ in $O(n^3)$? ✓

Is $f(n)$ in $O(2^n)$? ✓

Is $f(n)$ in $O(n)$? ✗

$n^2$, $n^3$, and $2^n$ all bound $f(n)$ from above

$n^2$ is a **tight** upper bound of $f(n)$

(there is no smaller upper bound for $f(n)$)

# Tight Bounds

$\boldsymbol{f(n) = 4n + n^2 \in \Theta(n^2)}$, therefore $\boldsymbol{f(n) = 4n + n^2 \in \Omega(n^2)}$ ✓

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$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in \Omega(n^2)$ ✓

Is $f(n)$ in $\Omega(n^3)$?

# Tight Bounds

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Is $f(n)$ in $\Omega(n^3)$? ✗

# Tight Bounds

$\boldsymbol{f(n) = 4n + n^2 \in \Theta(n^2)}$, therefore $\boldsymbol{f(n) = 4n + n^2 \in \Omega(n^2)}$ ✓

Is $\boldsymbol{f(n)}$ in $\boldsymbol{\Omega(n^3)}$? ✗

Is $\boldsymbol{f(n)}$ in $\boldsymbol{\Omega(\log(n))}$?

# Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in \Omega(n^2)$ ✓

Is $f(n)$ in $\Omega(n^3)$? ✗

Is $f(n)$ in $\Omega(\log(n))$? ✓

# Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in \Omega(n^2)$ ✓

Is $f(n)$ in $\Omega(n^3)$? ✗

Is $f(n)$ in $\Omega(\log(n))$? ✓

Is $f(n)$ in $\Omega(n)$?

# Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in \Omega(n^2)$ ✓

Is $f(n)$ in $\Omega(n^3)$? ✗

Is $f(n)$ in $\Omega(\log(n))$? ✓

Is $f(n)$ in $\Omega(n)$? ✓

# Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in \Omega(n^2)$ ✓

Is $f(n)$ in $\Omega(n^3)$? ✗

Is $f(n)$ in $\Omega(\log(n))$? ✓

Is $f(n)$ in $\Omega(n)$? ✓

$n^2$, $n$, and $\log(n)$ all bound $f(n)$ from below
$n^2$ is a **tight** lower bound of $f(n)$
(there is no larger lower bound for $f(n)$)

# Tight Bounds

If $g(n) \in \Theta(f(n))$, then:

- $g(n) \in O(f(n))$ is a tight upper bound
- $g(n) \in \Omega(f(n))$ is a tight lower bound

# Tight Bounds

If $g(n) \in \Theta(f(n))$, then:

- $g(n) \in O(f(n))$ is a tight upper bound
- $g(n) \in \Omega(f(n))$ is a tight lower bound

But what if the tight upper bound and tight lower bound are not the same?

# Multi-class Functions

$$T(n) = \begin{cases} n^2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

What is the tight upper bound of this function?

# Multi-class Functions

$$T(n) = \begin{cases} n^2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

What is the tight upper bound of this function? $\boldsymbol{T(n) \in O(n^2)}$

# Multi-class Functions

$$T(n) = \begin{cases} n^2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

What is the tight upper bound of this function? $\boldsymbol{T(n) \in O(n^2)}$

What is the tight lower bound of this function?

# Multi-class Functions

$$T(n) = \begin{cases} n^2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

What is the tight upper bound of this function? $\boldsymbol{T(n) \in O(n^2)}$

What is the tight lower bound of this function? $\boldsymbol{T(n) \in \Omega(n)}$

# Multi-class Functions

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What is the tight upper bound of this function? $T(n) \in O(n^2)$

What is the tight lower bound of this function? $T(n) \in \Omega(n)$

What is the complexity class of this function?

# Multi-class Functions

$$T(n) = \begin{cases} n^2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

What is the tight upper bound of this function? $T(n) \in O(n^2)$

What is the tight lower bound of this function? $T(n) \in \Omega(n)$

What is the complexity class of this function? It does not have one!

# Multi-class Functions

$$T(n) = \begin{cases} n^2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

It is not bounded from above by $n$, therefore it cannot be in $\Theta(n)$

It is not bounded from below by $n^2$, therefore it cannot be in $\Theta(n^2)$

What is the tight upper bound of this function? $T(n) \in O(n^2)$

What is the tight lower bound of this function? $T(n) \in \Omega(n)$

What is the complexity class of this function? It does not have one!