CSE 250
Data Structures

Dr. Eric Mikida
epmikida@buffalo.edu
208 Capen Hall

Lec 11: Recursion

Announcements

- PA1 Implementation due Sunday, 9/24 @ 11:59PM
  - Continue with the same repo you've been using
  - Special weekend office hours (see Piazza)
- WA2 will be released after the PA1 deadline, due 9/31 @ 11:59PM
- Midterm #1 will be Monday 10/2 in class
  - **Covers:** Summations, Asymptotics, Sequences/Lists, Arrays, Linked Lists, Recursion, Bounds (Tight Upper/Lower, **Unqualified** vs Expected vs **Amortized**)

`Seq` Summary So Far

	ArrayList	Linked List (by index)	Linked List (by reference)
`get(...)`	$\Theta(1)$	$\Theta(\text{idx})$ or $\boldsymbol{O}(n)$	$\Theta(1)$
`set(...)`	$\Theta(1)$	$\Theta(\text{idx})$ or $\boldsymbol{O}(n)$	$\Theta(1)$
`size()`	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
`add(...)`	$\boldsymbol{O}(\boldsymbol{n})$, Amortized $\boldsymbol{\Theta(1)}$	$\Theta(\text{idx})$ or $\boldsymbol{O}(n)$	$\Theta(1)$
`remove(...)`	$\boldsymbol{O}(n)$	$\Theta(\text{idx})$ or $\boldsymbol{O}(n)$	$\Theta(1)$

What Data Structure is Best?

Scenario #1: You need to read in the lines of a CSV file, store them in a List, and later be able to access individual records based on index.

What Data Structure is Best?

Scenario #1: You need to read in the lines of a CSV file, store them in a List, and later be able to access individual records based on index.

`ArrayList`

Since the amortized runtime of add for **`ArrayList`** and **`LinkedList`**, adding the ***n*** lines of the CSV file will take $\boldsymbol{O(n)}$ time for both…

But **`ArrayLists`** will then have an advantage because looking up records by index will be $\boldsymbol{O(1)}$ whereas **`LinkedLists`** will be $\boldsymbol{O(n)}$

What Data Structure is Best?

Scenario #2: Users logging onto an online game need to be efficiently added to a List in the order they log on. From time to time you must be able to iterate through the list from beginning to end.

What Data Structure is Best?

Scenario #2: Users logging onto an online game need to be efficiently added to a List in the order they log on. From time to time you must be able to iterate through the list from beginning to end.

`LinkedList`

The enumeration will cost a total of $\boldsymbol{O(n)}$ for both types of List

But some users will experience longer waits being added to the List if implemented as an **`ArrayList`** due to the need for it to occasionally resize

Recursion

Factorial

factorial(n) = n * (n-1) * (n-2) * … * 2 * 1

Factorial

$$\text{factorial}(n) = n * \underbrace{(n-1) * (n-2) * \ldots * 2 * 1}_{\text{factorial}(n-1)}$$

Factorial

$$\text{factorial}(n) = n * \underbrace{(n-1) * \underbrace{(n-2) * \ldots * 2 * 1}_{\text{factorial}(n-2)}}_{\text{factorial}(n-1)}$$

Factorial

$$\text{factorial}(n) = n * \underbrace{(n-1) * \underbrace{(n-2) * \ldots * \underbrace{2 * 1}_{\text{factorial}(2)}}_{\text{factorial}(n-2)}}_{\text{factorial}(n-1)}$$

Factorial

$$\text{factorial}(n) = n * \underbrace{(n-1) * \underbrace{(n-2) * \ldots * \underbrace{2 * \overbrace{1}^{\text{factorial}(1)}}_{\text{factorial}(2)}}_{\text{factorial}(n-2)}}_{\text{factorial}(n-1)}$$

Factorial

```
public int factorial(int n) {
    if(n <= 1) { return 1; }
    else { return n * factorial(n - 1); }
}
```

Factorial

```
public int factorial(int n) {
    if(n <= 1) { return 1; }                    ← Base Case
    else { return n * factorial(n - 1); }
}
```

Factorial

```
public int factorial(int n) {
    if(n <= 1) { return 1; }                    ← Base Case
    else { return n * factorial(n - 1); }  ← Recursive Case
}
```

Fibonacci

fib(n) = 1, 1

Fibonacci

$$fib(n) = \underbrace{1, 1}_{+}, \boxed{2}$$

Fibonacci

$$\text{fib}(n) = 1,\ \underbrace{1,\ 2}_{+},\ \boxed{3}$$

Fibonacci

$$fib(n) = 1, 1, \underbrace{2, 3}_{+}, \boxed{5}$$

Fibonacci

fib(n) = 1, 1, 2, 3, 5, 8, 13, 21, 34, …

Fibonacci

fibb(n) = 1, 1, 2, 3, 5, 8, 13, 21, 34, …

fib(n) = fib(n-1) + fib(n-2)

Fibonacci

```java
public int fib(int n) {
    if(n < 2) { return 1; }
    else { return fib(n-1) + fib(n - 2); }
}
```

Fibonacci

```
public int fib(int n) {
    if(n < 2) { return 1; }                       ← Base Case
    else { return fib(n-1) + fib(n - 2); }
}
```

Fibonacci

```
public int fib(int n) {
    if(n < 2) { return 1; }                      ← Base Case
    else { return fib(n-1) + fib(n - 2); }       ← Recursive Case
}
```

Towers of Hanoi

Live demo!

But What is the Complexity?

```
public int factorial(int n) {
    if(n <= 1) { return 1; }
    else { return n * factorial(n - 1); }
}
```

But What is the Complexity?

```
public int factorial(int n) {
    if(n <= 1) { return 1; }                      ← Θ(1)
    else { return n * factorial(n - 1); }
}
```

But What is the Complexity?

```
public int factorial(int n) {
    if(n <= 1) { return 1; }                        ← Θ(1)
    else { return n * factorial(n - 1); }  ← Θ(1) + Θ(???)
}
```

But What is the Complexity?

```
public int factorial(int n) {
    if(n <= 1) { return 1; }                    ← Θ(1)
    else { return n * factorial(n - 1); }  ← Θ(1) + Θ(???)
}
```

How do we figure out complexity of a function, when part of the runtime of the function is calling itself?

To know the complexity of **`factorial`**, *we need to…know the complexity of* **`factorial`**?

Complexity of `factorial`

$$T(n) = \begin{cases} \Theta(1) & \textbf{if } n \leq 1 \\ T(n-1) + \Theta(1) & \textbf{otherwise} \end{cases}$$

Solve for *T*(*n*)

Complexity of `factorial`

Solve for $T(n)$

Approach:

1. Generate a hypothesis
2. Prove your hypothesis for the base case
3. Prove the hypothesis for the recursive case *inductively*

Step 1 - Generate a Hypothesis

Let's start by looking at the runtime for increasing values of *n*

Step 1 - Generate a Hypothesis

Let's start by looking at the runtime for increasing values of *n*

$\Theta(1)$

Step 1 - Generate a Hypothesis

Let's start by looking at the runtime for increasing values of *n*

$\Theta(1), 2\Theta(1)$

Step 1 - Generate a Hypothesis

Let's start by looking at the runtime for increasing values of *n*

$\Theta(1), 2\Theta(1), 3\Theta(1)$

Step 1 - Generate a Hypothesis

Let's start by looking at the runtime for increasing values of *n*

$\Theta(1), 2\Theta(1), 3\Theta(1), 4\Theta(1), 5\Theta(1), 6\Theta(1), 7\Theta(1)$

Step 1 - Generate a Hypothesis

Let's start by looking at the runtime for increasing values of *n*

$\Theta(1), 2\Theta(1), 3\Theta(1), 4\Theta(1), 5\Theta(1), 6\Theta(1), 7\Theta(1)$

What is the pattern?

Step 1 - Generate a Hypothesis

Let's start by looking at the runtime for increasing values of n

$\Theta(1), 2\Theta(1), 3\Theta(1), 4\Theta(1), 5\Theta(1), 6\Theta(1), 7\Theta(1)$

What is the pattern?

Hypothesis: $T(n) \in O(n)$

Step 1 - Generate a Hypothesis

Let's start by looking at the runtime for increasing values of n

$\Theta(1)$, $2\Theta(1)$, $3\Theta(1)$, $4\Theta(1)$, $5\Theta(1)$, $6\Theta(1)$, $7\Theta(1)$

What is the pattern?

Hypothesis: $T(n) \in O(n)$

(there is some $c > 0$ such that $T(n) \leq c \cdot n$)

Prove for the Base Case

First, lets make our constants explicit

$$T(n) = \begin{cases} c_0 & \textbf{if } n \leq 1 \\ T(n-1) + c_1 & \textbf{otherwise} \end{cases}$$

Prove $T(n) \in O(n)$ for the Base Case

Prove: $T(n) \in O(n)$ (ie: there exists a constant, c, such that $T(n) \leq c \cdot n$)

Base Case: n = 1

$$T(1) \leq c \cdot 1$$

Prove $T(n) \in O(n)$ for the Base Case

Prove: $T(n) \in O(n)$ (ie: there exists a constant, c, such that $T(n) \leq c \cdot n$)

Base Case: n = 1

$$T(1) \leq c \cdot 1$$

$$T(1) \leq c$$

Prove $T(n) \in O(n)$ for the Base Case

Prove: $T(n) \in O(n)$ (ie: there exists a constant, c, such that $T(n) \leq c \cdot n$)

Base Case: n = 1

$$T(1) \leq c \cdot 1$$

$$T(1) \leq c$$

$$c_0 \leq c$$

Prove $T(n) \in O(n)$ for the Base Case

Prove: $T(n) \in O(n)$ (ie: there exists a constant, c, such that $T(n) \leq c \cdot n$)

Base Case: n = 1

$T(1) \leq c \cdot 1$

$T(1) \leq c$

$c_0 \leq c$

True for any $c \geq c_0$

Prove $T(n) \in O(n)$ for the Base Case + 1

Prove: $T(n) \in O(n)$ (ie: there exists a constant, c, such that $T(n) \leq c \cdot n$)

Base Case + 1: n = 2

$$T(2) \leq c \cdot 2$$

Prove $T(n) \in O(n)$ for the Base Case + 1

Prove: $T(n) \in O(n)$ (ie: there exists a constant, c, such that $T(n) \leq c \cdot n$)

Base Case + 1: n = 2

$$T(2) \leq c \cdot 2$$

$$T(1) + c_1 \leq 2c$$

Prove $T(n) \in O(n)$ for the Base Case + 1

Prove: $T(n) \in O(n)$ (ie: there exists a constant, c, such that $T(n) \leq c \cdot n$)
Base Case + 1: n = 2

$$T(2) \leq c \cdot 2$$

$$T(1) + c_1 \leq 2c$$

$$c_0 + c_1 \leq 2c$$

Prove $T(n) \in O(n)$ for the Base Case + 1

Prove: $T(n) \in O(n)$ (ie: there exists a constant, c, such that $T(n) \leq c \cdot n$)

Base Case + 1: n = 2

$$T(2) \leq c \cdot 2$$

$$T(1) + c_1 \leq 2c$$

$$c_0 + c_1 \leq 2c$$

We already know there's a $c \geq c_0$, so…

Prove $T(n) \in O(n)$ for the Base Case + 1

Prove: $T(n) \in O(n)$ (ie: there exists a constant, c, such that $T(n) \leq c \cdot n$)

Base Case + 1: n = 2

$$T(2) \leq c \cdot 2$$

$$T(1) + c_1 \leq 2c$$

$$c_0 + c_1 \leq 2c$$

We already know there's a $c \geq c_0$, so…

True for any $c \geq c_1$

Prove $T(n) \in O(n)$ for the Base Case + 2

Prove: $T(n) \in O(n)$ (ie: there exists a constant, c, such that $T(n) \leq c \cdot n$)

Base Case + 2: n = 3

$$T(3) \leq c \cdot 3$$

Prove $T(n) \in O(n)$ for the Base Case + 2

Prove: $T(n) \in O(n)$ (ie: there exists a constant, c, such that $T(n) \leq c \cdot n$)

Base Case + 2: n = 3

$$T(3) \leq c \cdot 3$$

$$T(2) + c_1 \leq 3c$$

Prove $T(n) \in O(n)$ for the Base Case + 2

Prove: $T(n) \in O(n)$ (ie: there exists a constant, c, such that $T(n) \leq c \cdot n$)

Base Case + 2: n = 3

$$T(3) \leq c \cdot 3$$

$$T(2) + c_1 \leq 3c$$

We know there's a c s.t. $T(2) \leq 2c$,

Prove $T(n) \in O(n)$ for the Base Case + 2

Prove: $T(n) \in O(n)$ (ie: there exists a constant, c, such that $T(n) \leq c \cdot n$)

Base Case + 2: n = 3

$$T(3) \leq c \cdot 3$$

$$T(2) + c_1 \leq 3c$$

We know there's a c s.t. $T(2) \leq 2c$,

So if we show that $2c + c_1 \leq 3c$, then $T(2) + c_1 \leq 2c + c_1 \leq 3c$

Prove $T(n) \in O(n)$ for the Base Case + 2

Prove: $T(n) \in O(n)$ (ie: there exists a constant, c, such that $T(n) \leq c \cdot n$)
Base Case + 2: n = 3

$T(3) \leq c \cdot 3$

$T(2) + c_1 \leq 3c$

We know there's a c s.t. $T(2) \leq 2c$,

So if we show that $2c + c_1 \leq 3c$, then $T(2) + c_1 \leq 2c + c_1 \leq 3c$

True for any $c \geq c_1$

Prove $T(n) \in O(n)$ for the Base Case + 3

Prove: $T(n) \in O(n)$ (ie: there exists a constant, c, such that $T(n) \leq c \cdot n$)

Base Case + 2: n = 4

$$T(4) \leq c \cdot 4$$

$$T(3) + c_1 \leq 4c$$

We know there's a c s.t. $T(3) \leq 3c$,

So if we show that $3c + c_1 \leq 4c$, then $T(3) + c_1 \leq 3c + c_1 \leq 4c$

True for any $c \geq c_1$

Proving the Hypothesis Inductively

We're starting to see a pattern…

Proving the Hypothesis Inductively

We can prove our hypothesis for specific values of n…

n=1 n=2 n=3 n=4 n=5 n=6 n=7 n=8 …

Proving the Hypothesis Inductively

We can prove our hypothesis for specific values of n…

n=1	n=2	n=3	n=4	n=5	n=6	n=7	n=8	…

Proving the Hypothesis Inductively

We can prove our hypothesis for specific values of n…

n=1	n=2	n=3	n=4	n=5	n=6	n=7	n=8	…

Proving the Hypothesis Inductively

We can prove our hypothesis for specific values of n…

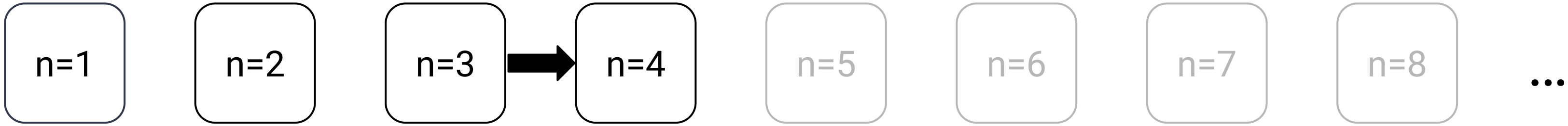
…but there are infinitely many possible values of n

n=1 n=2 n=3 n=4 n=5 n=6 n=7 n=8 …

Proving the Hypothesis Inductively

We can prove our hypothesis for specific values of n…

…but there are infinitely many possible values of n



Instead, let's prove that we can derive an unproven case from a proven one!

Proving the Hypothesis Inductively

Approach: Assume our hypothesis is true for any $n' < n$;
Now prove it must also hold true for n.

Proving the Hypothesis Inductively

Assume: There is a $c > 0$ s.t. $T(n - 1) \leq c \cdot (n - 1)$

Prove: There is a $c > 0$ s.t. $T(n) \leq c \cdot n$

$$T(n) \leq c \cdot n$$

Proving the Hypothesis Inductively

Assume: There is a $c > 0$ s.t. $T(n - 1) \leq c \cdot (n - 1)$

Prove: There is a $c > 0$ s.t. $T(n) \leq c \cdot n$

$$T(n) \leq c \cdot n$$

$$T(n - 1) + c_1 \leq c \cdot n$$

Proving the Hypothesis Inductively

Assume: There is a $c > 0$ s.t. $T(n - 1) \leq c \cdot (n - 1)$

Prove: There is a $c > 0$ s.t. $T(n) \leq c \cdot n$

$$T(n) \leq c \cdot n$$

$$T(n - 1) + c_1 \leq c \cdot n$$

By the inductive assumption, there is a c s.t. $T(n - 1) \leq (n - 1)c$

Proving the Hypothesis Inductively

Assume: There is a $c > 0$ s.t. $T(n - 1) \leq c \cdot (n - 1)$

Prove: There is a $c > 0$ s.t. $T(n) \leq c \cdot n$

$$T(n) \leq c \cdot n$$

$$T(n - 1) + c_1 \leq c \cdot n$$

By the inductive assumption, there is a c s.t. $T(n - 1) \leq (n - 1)c$

So if we show that $(n - 1)c + c_1 \leq nc$, then…

Proving the Hypothesis Inductively

Assume: There is a $c > 0$ s.t. $T(n - 1) \leq c \cdot (n - 1)$

Prove: There is a $c > 0$ s.t. $T(n) \leq c \cdot n$

$$T(n) \leq c \cdot n$$

$$T(n - 1) + c_1 \leq c \cdot n$$

By the inductive assumption, there is a c s.t. $T(n - 1) \leq (n - 1)c$

So if we show that $(n - 1)c + c_1 \leq nc$, then…

$$T(n - 1) + c_1 \leq (n - 1)c + c_1 \leq nc$$

Proving the Hypothesis Inductively

Assume: There is a $c > 0$ s.t. $T(n - 1) \le c \cdot (n - 1)$

Prove: There is a $c > 0$ s.t. $T(n) \le c \cdot n$

$$T(n) \le c \cdot n$$

$$T(n - 1) + c_1 \le c \cdot n$$

By the inductive assumption, there is a c s.t. $T(n - 1) \le (n - 1)c$

So if we show that $(n - 1)c + c_1 \le nc$, then…

$$T(n - 1) + c_1 \le (n - 1)c + c_1 \le nc$$

True for any $c \ge c_1$

Proving the Hypothesis Inductively

Assume: There is a $c > 0$ s.t. $T(n - 1) \le c \cdot (n - 1)$

Prove: There is a $c > 0$ s.t. $T(n) \le c \cdot n$

$$T(n) \le c \cdot n$$

$$T(n - 1) + c_1 \le c \cdot n$$

By the inductive assumption, there is a c s.t. $T(n - 1) \le (n - 1)c$

So if we show that $(n - 1)c + c_1 \le nc$, then...

$$T(n - 1) + c_1 \le (n - 1)c + c_1 \le nc$$

True for any $c \ge c_1$

Therefore, we've proven our hypothesis for the Base Case, and inductively for the Recursive Case.
Therefore, the complexity of factorial is $\Theta(n)$

How much space is used?

```
factorial(n)
```

How much space is used?

`factorial(n-1)`
`factorial(n)`

How much space is used?

`factorial(n-2)`
`factorial(n-1)`
`factorial(n)`

How much space is used?

`factorial(n-3)`
`factorial(n-2)`
`factorial(n-1)`
`factorial(n)`

How much space is used?

⋮

`factorial(n-4)`
`factorial(n-3)`
`factorial(n-2)`
`factorial(n-1)`
`factorial(n)`

Tail Recursion

If the last thing we do in the function is a single recursive call, we shouldn't need to create an entire stack of all the function calls…

```
public int factorial(int n) {
    if(n <= 1) { return 1; }
    else { return n * factorial(n - 1); }
}
```

…smart compilers can often automatically convert to a loop…

```
public int factorial(int n) {
    int total = 1;
    for (int i = 0; i < n; i++) { total *= i; }
    return total;
}
```

Fibonacci

What about a function without tail recursion, or with multiple recursive calls?

What is the complexity of `fib(n)`?

```
public int fib(int n) {
    if(n < 2) { return 1; }
    else { return fib(n-1) + fib(n - 2); }
}
```

Next time...

Divide and Conquer

Recursion Trees