CSE 250
Data Structures

Dr. Eric Mikida
epmikida@buffalo.edu
208 Capen Hall

Lec 22: Priority Queues

Announcements

- PA2 Testing was due yesterday
- PA2 Implementation due Sunday 11/5 @ 11:59PM
  - Autolab coming soon
  - Test locally!

Examples

How might we order the following?

- (B,10), (D,3), (E,40)
- "A+", "C", "B-"
- Taco Tuesday, Fish Friday, Meatless Monday
- Buffalo Bills, Denver Broncos, Baltimore Ravens
- Halloween, Friday the 13th, The Babadook

Ordering

An **ordering (over type A)**, **(*A*, ≤)**:

- A set of things of type ***A***
- A "relation" or comparator, **≤**, that relates two things in the set

Examples

5 ≤ 30 ≤ 999
Numerical order

(E,40) ≤ (B,10) ≤ (D,3)
Reverse-numerical order on the 2nd field

C+ ≤ B- ≤ B ≤ B+ ≤ A- ≤ A
Letter grades

AA ≤ AM ≤ BZ ≤ CA ≤ CD
Compare 1st then 2nd, 3rd…(Lexical order)

Ordering Properties

Team A ≤ Team B

Team B won its match against Team A

Ordering Properties

Team A ≤ Team B

Team B won its match against Team A

Team B ≤ Team C

Team C won its match against Team B

Ordering Properties

Team A ≤ Team B

Team B won its match against Team A

Team B ≤ Team C

Team C won its match against Team B

Team C ≤ Team A

Team A won its match against Team C

Ordering Properties

Team A ≤ Team B

Team B won its match against Team A

Team B ≤ Team C

Team C won its match against Team B

Team C ≤ Team A

Team A won its match against Team C

Is this an ordering??

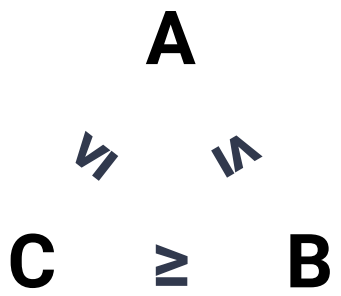
Ordering Properties

Team A ≤ Team B
Team B won its match against Team A

Team B ≤ Team C
Team C won its match against Team B

Team C ≤ Team A
Team A won its match against Team C

Is this an ordering??



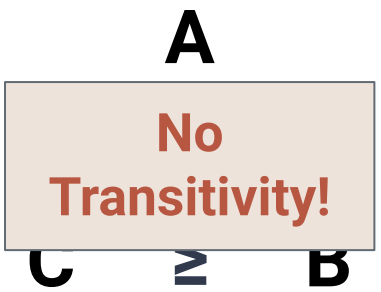
Ordering Properties

Team A ≤ Team B
Team B won its match against Team A

Team B ≤ Team C
Team C won its match against Team B

Team C ≤ Team A
Team A won its match against Team C

Is this an ordering?? **NO!**



Ordering Properties

An ordering must be…

Reflexive

$x \leq x$

Antisymmetric

If $x \leq y$ and $y \leq x$ then $x = y$

Transitive

If $x \leq y$ and $y \leq z$ then $x \leq z$

Another Example

Define an ordering over CSE Courses:

Course 1 ≤ Course 2 iff Course 1 is a prereq of Course 2

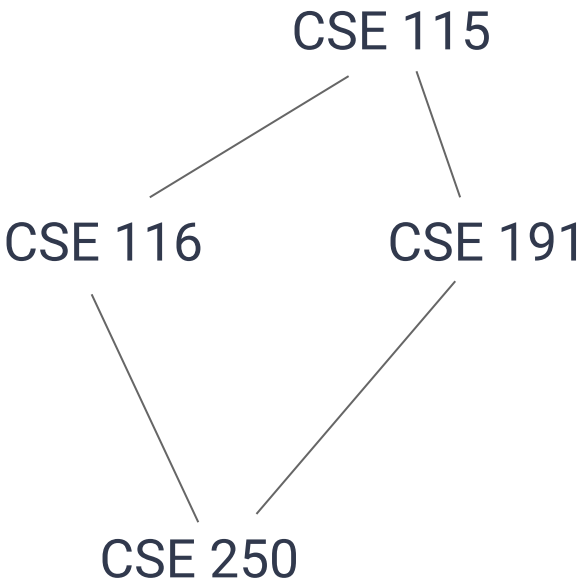
CSE 115 ≤ CSE 116

CSE 116 ≤ CSE 250

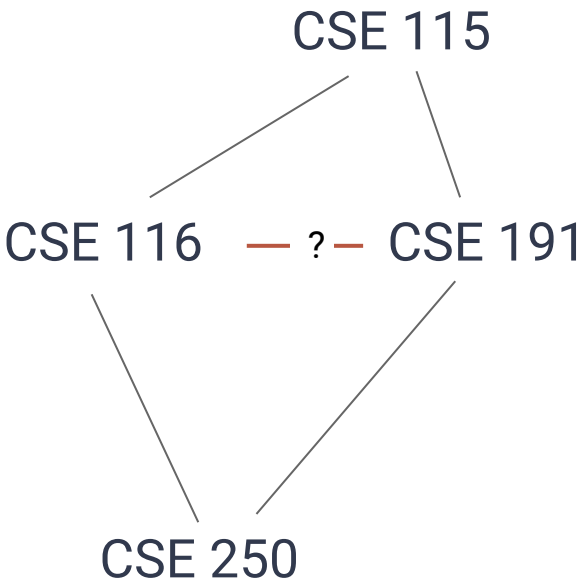
CSE 115 ≤ CSE 191

CSE 191 ≤ CSE 250

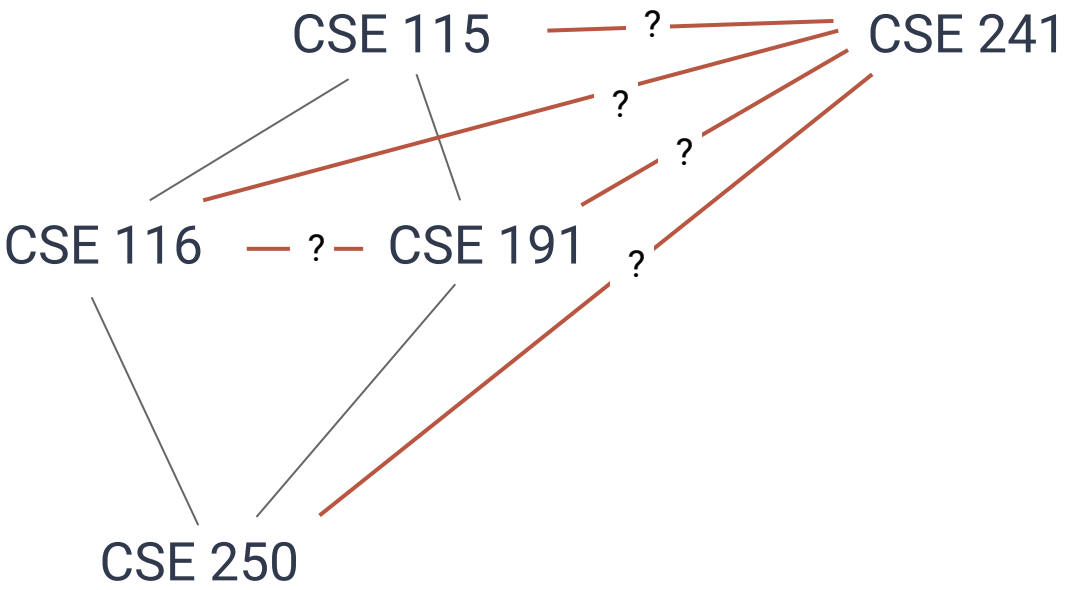
Ordering Properties



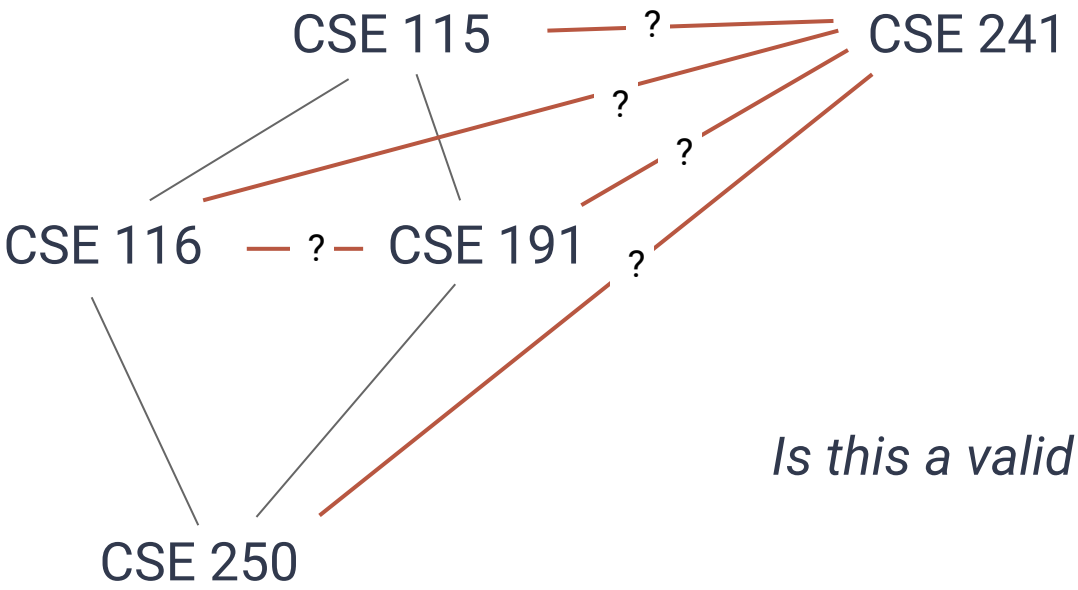
Ordering Properties



Ordering Properties

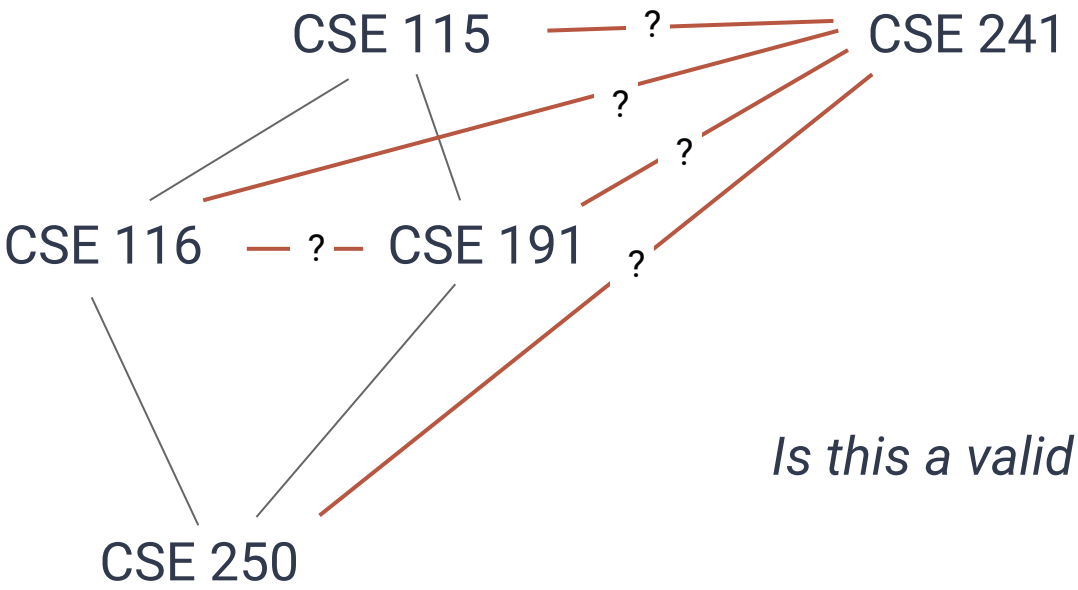


Ordering Properties



Is this a valid ordering?

Ordering Properties



Is this a valid ordering? **YES**

(Partial) Ordering Properties

A **partial ordering** must be…

Reflexive

$x \leq x$

Antisymmetric

If $x \leq y$ and $y \leq x$ then $x = y$

Transitive

If $x \leq y$ and $y \leq z$ then $x \leq z$

(Total) Ordering Properties

An **total ordering** must be…

Reflexive

$x \leq x$

Antisymmetric

If $x \leq y$ and $y \leq x$ then $x = y$

Transitive

If $x \leq y$ and $y \leq z$ then $x \leq z$

Complete

Either $x \leq y$ or $y \leq x$ for any $x,y \in A$

Example

Consider two different ways to "rank" movies:

Halloween, It, Hereditary, Get Out, Descent, Friday the 13th

Example

Consider two different ways to "rank" movies:

Halloween, It, Hereditary, Get Out, Descent, Friday the 13th

I could organize these movies in a tier list based on my preferences:

A-tier: Halloween, Get Out, Friday the 13th

B-tier: It, Descent

C-tier: Hereditary

Example

Consider two different ways to "rank" movies:

Halloween, It, Hereditary, Get Out, Descent, Friday the 13th

I could organize these movies in a tier list based on my preferences:

A-tier: Halloween, Get Out, Friday the 13th

B-tier: It, Descent

C-tier: Hereditary

This is a partial ordering
It is reflexive, antisymmetric and transitive
…but not all pairs are directly order
Consider Halloween and Friday the 13th

Example

Consider two different ways to "rank" movies:

Halloween, It, Hereditary, Get Out, Descent, Friday the 13th

I could also rank these movies based on my preferences:

1. Halloween
2. Get Out
3. Friday the 13th
4. Descent
5. It
6. Hereditary

Example

Consider two different ways to "rank" movies:

Halloween, It, Hereditary, Get Out, Descent, Friday the 13th

I could also rank these movies based on my preferences:

1. Halloween
2. Get Out
3. Friday the 13th
4. Descent
5. It
6. Hereditary

This is a total ordering
It is reflexive, antisymmetric and transitive
…and every pair can be directly compared

Some Other Definitions

For an ordering $(A, \leq)$

The **greatest** element is an element $x \in A$ s.t. there is no y **in** A, where $x \leq y$

The **least** element is an element $x \in A$ s.t. there is no y **in** A, where $y \leq x$

Some Other Definitions

For an ordering $(A, \leq)$

The **greatest** element is an element $x \in A$ s.t. there is no y **in** A, where $x \leq y$

The **least** element is an element $x \in A$ s.t. there is no y **in** A, where $y \leq x$

*A **partial** ordering may not have a **unique** greatest/least element*

Describing an Ordering

≤ can be described **explicitly**, by a set of tuples:

{(a,a),(a,b),(a,c),...,(b,b),...,(z,z)}

Describing an Ordering

≤ can be described **explicitly**, by a set of tuples:

{(a,a),(a,b),(a,c),...,(b,b),...,(z,z)}

If (x,y) is in the set, then x **≤** y

Describing an Ordering

≤ can be described by a **mathematical rule**:

$$\{(x,y) \mid x, y \in \mathbb{Z},\ \exists\ a \in \mathbb{Z}^{+} \cup \{0\} : x + a = y \}$$

Describing an Ordering

≤ can be described by a **mathematical rule**:

$$\{(x,y) \mid x, y \in \mathbb{Z},\ \exists\ a \in \mathbb{Z}^{+} \cup \{0\} : x + a = y \}$$

x* ≤ *y iff ***x***,***y*** are integers and there is a non-negative integer ***a*** s.t. ***x+a=y***

Multiple Orderings

Multiple Orderings can be defined for the same set

- RottenTomatoes vs Metacritic vs Box Office Gross
- "Best Movie" first vs "Worst Movie" first
- Rank by number of swear words, killcount, etc

Multiple Orderings

Multiple Orderings can be defined for the same set

- RottenTomatoes vs Metacritic vs Box Office Gross
- "Best Movie" first vs "Worst Movie" first
- Rank by number of swear words, killcount, etc

We use subscripts to separate orderings ($\leq_1$, $\leq_2$, $\leq_3$, ...)

Transformations

We can transform orderings:

Transformations

We can transform orderings:

Reverse: If $x \leq_1 y$ then define $y \leq_r x$

Transformations

We can transform orderings:

Reverse: If $x \leq_1 y$ then define $y \leq_r x$

Lexical: Given $\leq_1, \leq_2, \leq_3, \ldots$

- if $x \leq_1 y$ then $x \leq_L y$
- else if $x =_1 y$ and $x \leq_2 y$ then $x \leq_L y$
- else if $x =_2 y$ and $x \leq_3 y$ then $x \leq_L y$
- …

Examples of Lexical Ordering

Names: First letter, then second letter, then third…

Movies: Average of reviews, then number of reviews…

Tuples: First field, then second field, then third…

Sports Teams: Games won, points scored, speed of victory…

Ordering Over Keys

≤ can be described as an **ordering over a key derived from the element:**

$$x \leq_{\textbf{edge}} y \text{ iff } \text{weight}(x) \leq \text{weight}(y)$$

$$x \leq_{\textbf{student}} y \text{ iff } \text{name}(x) \leq_{\textbf{Lex}} \text{name}(y)$$

Ordering Over Keys

≤ can be described as an **ordering over a key derived from the element:**

$$x \leq_{\textbf{edge}} y \text{ iff } weight(x) \leq weight(y)$$

$$x \leq_{\textbf{student}} y \text{ iff } name(x) \leq_{\textbf{Lex}} name(y)$$

We say that weight/name are keys

Topological Sort

A **Topological Sort** of *partial* order $(A, \leq_1)$ is *any total* order $(A, \leq_2)$ that "agrees" with $(A, \leq_1)$:

For any two elements x,y in A:

if $x \leq_1 y$ then $x \leq_2 y$

if $y \leq_1 x$ then $y \leq_2 x$

Otherwise, either $x \leq_2 y$ or $y \leq_2 x$

Topological Sort

The following are all topological sorts over our partial order from earlier:

- CSE 115, CSE 116, CSE 191, CSE 241, CSE 250
- CSE 241, CSE 115, CSE 116, CSE 191, CSE 250
- CSE 115, CSE 191, CSE 116, CSE 250, CSE 241

Topological Sort

The following are all topological sorts over our partial order from earlier:

- CSE 115, CSE 116, CSE 191, CSE 241, CSE 250
- CSE 241, CSE 115, CSE 116, CSE 191, CSE 250
- CSE 115, CSE 191, CSE 116, CSE 250, CSE 241

(In this case, the partial ordering is a schedule requirement, and each topological sort is a possible schedule)

And now for an ordering-based ADT…

A New ADT...`PriorityQueue`

`PriorityQueue<T>`

`void add(T value)`

Insert **`value`** into the priority queue

`T poll()`

Remove the highest priority value in the priority queue

`T peek()`

Peek at the highest priority value in the priority queue

A New ADT...`PriorityQueue`

`PriorityQueue<T>`

`void add(T value)`

Insert `value` into the priority queue

In Java, by default the smallest element has the highest priority

`T poll()`

Remove the **highest priority** value in the priority queue

`T peek()`

Peek at the **highest priority** value in the priority queue

Sorted Lists

Note this is not the first time we've seen an ordered data structure

Our Linked List from PA1 was an ordered data structure as well…

How do we store the following→

How do we store the following→

```
add(5)
```

How do we store the following→

```
add(5)
add(9)
```

How do we store the following→

```
add(5)
add(9)
add(2)
```

How do we store the following→

```
add(5)
add(9)
add(2)
add(7)
```

How do we store the following→

```
add(5)
add(9)
add(2)
add(7)
peek()    // Should be 9
poll()    // should be 9
```

How do we store the following→

```
add(5)
add(9)
add(2)
add(7)
peek()    // Should be 9
poll()    // should be 9
size()    // should be 3
peek()    // should be 7
```

How do we store the following→

```
add(5)
add(9)
add(2)
add(7)
peek()    // Should be 9
poll()    // should be 9
size()    // should be 3
peek()    // should be 7
poll()    // 7
poll()    // 5
poll()    // 2
```

How do we store the following→

```
add(5)
add(9)
add(2)
add(7)
peek()     // Should be 9
poll()     // should be 9
size()     // should be 3
peek()     // should be 7
poll()     // 7
poll()     // 5
poll()     // 2
isEmpty() // should be true
```

How do we store the following→

Insertion Order?	5, 9, 7, 2
Sorted Order?	2, 5, 7, 9
Reverse Sorted Order?	9, 7, 5, 2

```
add(5)
add(9)
add(2)
add(7)
peek()    // Should be 9
poll()    // should be 9
size()    // should be 3
peek()    // should be 7
poll()    // 7
poll()    // 5
poll()    // 2
isEmpty() // should be true
```

Priority Queues

Two mentalities…

Lazy: Keep everything a mess

Proactive: Keep everything organized

Priority Queues

Two mentalities…

Lazy: Keep everything a mess ("Selection Sort")

Proactive: Keep everything organized

Priority Queues

Two mentalities…

Lazy: Keep everything a mess ("Selection Sort")

Proactive: Keep everything organized ("Insertion Sort")

Lazy Priority Queue

Base Data Structure: Linked List

`void add(T value)`

Append **`value`** to the end of the linked list.

`T peek()/T poll()`

Traverse the list to find the smallest value.

Lazy Priority Queue

Base Data Structure: Linked List

`void add(T value)`

Append `value` to the end of the linked list. $\Theta(1)$

`T peek()/T poll()`

Traverse the list to find the smallest value. $O(n)$

Sorting with Our Priority Queue

```
1  public List<T> PQueueSort(List<T> input) {
2    List<T> out = new ArrayList<>();
3    PriorityQueue<T> pq = new PriorityQueue<>();
4    for (T item : input) { pq.add(item); }
5    while (!pq.isEmpty()) { out.add(pq.poll()); }
6    return out;
7  }
```

Sorting with Our Priority Queue

```
1  public List<T> PQueueSort(List<T> input) {
2    List<T> out = new ArrayList<>();
3    PriorityQueue<T> pq = new PriorityQueue<>();
4    for (T item : input) { pq.add(item); }   ← Add everything to a priority queue
5    while (!pq.isEmpty()) { out.add(pq.poll()); }
6    return out;
7  }
```

Sorting with Our Priority Queue

```
1  public List<T> PQueueSort(List<T> input) {
2    List<T> out = new ArrayList<>();
3    PriorityQueue<T> pq = new PriorityQueue<>();
4    for (T item : input) { pq.add(item); }
5    while (!pq.isEmpty()) { out.add(pq.poll()); }  ← Remove it all (and add to out)
6    return out;
7  }
```

Selection Sort
(w/"Lazy" PriorityQueue)

	List	PriorityQueue
Input	(7,4,8,2,5,3,9)	()

Selection Sort
(w/"Lazy" PriorityQueue)

	List	PriorityQueue
Input	(7,4,8,2,5,3,9)	()
Step 1	(4,8,2,5,3,9)	(7)

Selection Sort
(w/"Lazy" PriorityQueue)

	List	PriorityQueue
Input	(7,4,8,2,5,3,9)	()
Step 1	(4,8,2,5,3,9)	(7)
Step 2	(8,2,5,3,9)	(7,4)

Selection Sort
(w/"Lazy" PriorityQueue)

	List	PriorityQueue
Input	(7,4,8,2,5,3,9)	()
Step 1	(4,8,2,5,3,9)	(7)
Step 2	(8,2,5,3,9)	(7,4)
	…	…
Step *n*	()	(7,4,8,2,5,3,9)

Selection Sort
(w/"Lazy" PriorityQueue)

	List	PriorityQueue
Input	(7,4,8,2,5,3,9)	()
Step 1	(4,8,2,5,3,9)	(7)
Step 2	(8,2,5,3,9)	(7,4)
	…	…
Step *n*	()	(7,4,8,2,5,3,9)
Step *n* + 1	[2,_,_,_,_,_,_]	(7,4,8,5,3,9)

Selection Sort
(w/"Lazy" PriorityQueue)

	List	PriorityQueue
Input	(7,4,8,2,5,3,9)	()
Step 1	(4,8,2,5,3,9)	(7)
Step 2	(8,2,5,3,9)	(7,4)
	…	…
Step *n*	()	(7,4,8,2,5,3,9)
Step *n* + 1	[2,_,_,_,_,_,_]	(7,4,8,5,3,9)
Step *n* + *2*	[2,3,_,_,_,_,_]	(7,4,8,5,9)

Selection Sort
(w/"Lazy" PriorityQueue)

	List	PriorityQueue
Input	(7,4,8,2,5,3,9)	()
Step 1	(4,8,2,5,3,9)	(7)
Step 2	(8,2,5,3,9)	(7,4)
	…	…
Step *n*	()	(7,4,8,2,5,3,9)
Step *n* + 1	[2,_,_,_,_,_,_]	(7,4,8,5,3,9)
Step *n* + *2*	[2,3,_,_,_,_,_]	(7,4,8,5,9)
Step *n* + *3*	[2,3,4,_,_,_,_]	(7,8,5,9)

Selection Sort
(w/"Lazy" PriorityQueue)

	List	PriorityQueue
Input	(7,4,8,2,5,3,9)	()
Step 1	(4,8,2,5,3,9)	(7)
Step 2	(8,2,5,3,9)	(7,4)
	…	…
Step *n*	()	(7,4,8,2,5,3,9)
Step *n* + 1	[2,_,_,_,_,_,_]	(7,4,8,5,3,9)
Step *n* + *2*	[2,3,_,_,_,_,_]	(7,4,8,5,9)
Step *n* + *3*	[2,3,4,_,_,_,_]	(7,8,5,9)
Step *n* + *4*	[2,3,4,5,_,_,_]	(7,8,9)

Selection Sort
(w/"Lazy" PriorityQueue)

	List	PriorityQueue
Input	(7,4,8,2,5,3,9)	()
Step 1	(4,8,2,5,3,9)	(7)
Step 2	(8,2,5,3,9)	(7,4)
	…	…
Step *n*	()	(7,4,8,2,5,3,9)
Step *n* + 1	[2,_,_,_,_,_,_]	(7,4,8,5,3,9)
Step *n* + *2*	[2,3,_,_,_,_,_]	(7,4,8,5,9)
Step *n* + *3*	[2,3,4,_,_,_,_]	(7,8,5,9)
Step *n* + *4*	[2,3,4,5,_,_,_]	(7,8,9)
	…	…
Step 2*n*	[2,3,4,5,7,8,9]	()

Selection Sort (w/"Lazy" PriorityQueue)

```
1  public List<T> PQueueSort(List<T> input) {
2    List<T> out = new ArrayList<>();
3    PriorityQueue<T> pq = new PriorityQueue<>();
4    for (T item : input) { pq.add(item); }
5    while (!pq.isEmpty()) { out.add(pq.poll()); }
6    return out;
7  }
```

What is the complexity?

Selection Sort (w/"Lazy" PriorityQueue)

```
1  public List<T> PQueueSort(List<T> input) {
2    List<T> out = new ArrayList<>();
3    PriorityQueue<T> pq = new PriorityQueue<>();
4    for (T item : input) { pq.add(item); }
5    while (!pq.isEmpty()) { out.add(pq.poll()); }
6    return out;
7  }
```

← **`poll()`** is an ***O*(*n*)** operation

What is the complexity? ***O*(n^2)**

Proactive Priority Queue

Base Data Structure: Linked List

`void add(T value)`

Insert `value` in ascending sorted order.

`T peek()/T poll()`

Get the first value in the list.

Proactive Priority Queue

Base Data Structure: Linked List

`void add(T value)`

Insert `value` in ascending sorted order. $O(n)$

`T peek()/T poll()`

Get the first value in the list. $\Theta(1)$

Insertion Sort
(w/"Proactive" PriorityQueue)

	List	PriorityQueue
Input	(7,4,8,2,5,3,9)	()

Insertion Sort
(w/"Proactive" PriorityQueue)

	List	PriorityQueue
Input	(7,4,8,2,5,3,9)	()
Step 1	(4,8,2,5,3,9)	(7)

Insertion Sort
(w/"Proactive" PriorityQueue)

	List	PriorityQueue
Input	(7,4,8,2,5,3,9)	()
Step 1	(4,8,2,5,3,9)	(7)
Step 2	(8,2,5,3,9)	(4,7)

Insertion Sort
(w/"Proactive" PriorityQueue)

	List	PriorityQueue
Input	(7,4,8,2,5,3,9)	()
Step 1	(4,8,2,5,3,9)	(7)
Step 2	(8,2,5,3,9)	(4,7)
Step 3	(2,5,3,9)	(4,7,8)

Insertion Sort
(w/"Proactive" PriorityQueue)

	List	PriorityQueue
Input	(7,4,8,2,5,3,9)	()
Step 1	(4,8,2,5,3,9)	(7)
Step 2	(8,2,5,3,9)	(4,7)
Step 3	(2,5,3,9)	(4,7,8)
Step 4	(5,3,9)	(2,4,7,8)
	…	…
Step *n*	[_,_,_,_,_,_,_]	(2,3,4,5,7,8,9)

Insertion Sort
(w/"Proactive" PriorityQueue)

	List	PriorityQueue
Input	(7,4,8,2,5,3,9)	()
Step 1	(4,8,2,5,3,9)	(7)
Step 2	(8,2,5,3,9)	(4,7)
Step 3	(2,5,3,9)	(4,7,8)
Step 4	(5,3,9)	(2,4,7,8)
	…	…
Step *n*	[_,_,_,_,_,_,_]	(2,3,4,5,7,8,9)
Step *n* + *2*	[2,_,_,_,_,_,_]	(3,4,5,7,8,9)

Insertion Sort
(w/"Proactive" PriorityQueue)

	List	PriorityQueue
Input	(7,4,8,2,5,3,9)	()
Step 1	(4,8,2,5,3,9)	(7)
Step 2	(8,2,5,3,9)	(4,7)
Step 3	(2,5,3,9)	(4,7,8)
Step 4	(5,3,9)	(2,4,7,8)
	…	…
Step *n*	[_,_,_,_,_,_,_]	(2,3,4,5,7,8,9)
Step *n* + *2*	[2,_,_,_,_,_,_]	(3,4,5,7,8,9)
Step *n* + *3*	[2,3,_,_,_,_,_]	(4,5,7,8,9)

Insertion Sort
(w/"Proactive" PriorityQueue)

	List	PriorityQueue
Input	(7,4,8,2,5,3,9)	()
Step 1	(4,8,2,5,3,9)	(7)
Step 2	(8,2,5,3,9)	(4,7)
Step 3	(2,5,3,9)	(4,7,8)
Step 4	(5,3,9)	(2,4,7,8)
	…	…
Step *n*	[_,_,_,_,_,_,_]	(2,3,4,5,7,8,9)
Step *n* + *2*	[2,_,_,_,_,_,_]	(3,4,5,7,8,9)
Step *n* + *3*	[2,3,_,_,_,_,_]	(4,5,7,8,9)
	…	…
Step 2*n*	[2,3,4,5,7,8,9]	()

Insertion Sort (w/"Proactive" PriorityQueue)

```
1  public List<T> PQueueSort(List<T> input) {
2    List<T> out = new ArrayList<>();
3    PriorityQueue<T> pq = new PriorityQueue<>();
4    for (T item : input) { pq.add(item); }
5    while (!pq.isEmpty()) { out.add(pq.poll()); }
6    return out;
7  }
```

What is the complexity?

Insertion Sort (w/"Proactive" PriorityQueue)

```
1  public List<T> PQueueSort(List<T> input) {
2    List<T> out = new ArrayList<>();
3    PriorityQueue<T> pq = new PriorityQueue<>();
4    for (T item : input) { pq.add(item); }  ← add() is an O(n) operation
5    while (!pq.isEmpty()) { out.add(pq.poll()); }
6    return out;
7  }
```

What is the complexity? $\boldsymbol{O(n^2)}$

Priority Queues

Operation	Lazy	Proactive
enqueue	$O(1)$	$O(n)$
dequeue	$O(n)$	$O(1)$
head	$O(n)$	$O(1)$

Priority Queues

Operation	Lazy	Proactive
enqueue	$O(1)$	$O(n)$
dequeue	$O(n)$	$O(1)$
head	$O(n)$	$O(1)$

Can we do better?

Priority Queues

Lazy - Fast Enqueue, Slow Dequeue

Proactive - Slow Enqueue, Fast Dequeue

Priority Queues

Lazy - Fast Enqueue, Slow Dequeue

Proactive - Slow Enqueue, Fast Dequeue

??? - Fast(-ish) Enqueue, Fast(-ish) Dequeue

Priority Queues

Idea: Keep the priority queue "kinda" sorted.

Hopefully "kinda" sorted is cheaper to maintain than a full sort,

but still gives us some of the benefits.