CSE 250
Data Structures

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Lec 23: Heaps

Announcements

- PA2 autolab is live

`PriorityQueue` ADT

`PriorityQueue<T>`

`void add(T value)`

Insert **`value`** into the priority queue

`T poll()`

Remove the highest priority value in the priority queue

`T peek()`

Peek at the highest priority value in the priority queue

Priority Queues

Two mentalities…

Lazy: Keep everything a mess ("Selection Sort")

Proactive: Keep everything organized ("Insertion Sort")

Priority Queues

Operation	**Lazy**	**Proactive**
add	$O(1)$	$O(n)$
poll	$O(n)$	$O(1)$
peek	$O(n)$	$O(1)$

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Can we do better?

Priority Queues

Lazy - Fast add, Slow removal

Proactive - Slow add, Fast removal

Priority Queues

Lazy - Fast add, Slow removal

Proactive - Slow add, Fast removal

??? - Fast(-ish) add, Fast(-ish) removal

Priority Queues

Idea: Keep the priority queue "kinda" sorted.

Hopefully "kinda" sorted is cheaper to maintain than a full sort,

but still gives us some of the benefits.

Priority Queues

Idea: Keep the priority queue "kinda" sorted.

Keep higher priority towards the front of the list,

and keep the front of the list more sorted than the back…

Binary Heaps

Challenge: If we are only "kinda" sorting, how do we know which elements are actually sorted?

Binary Heaps

Idea: Organize the priority queue as a *directed* tree!

A directed edge from ***a*** to ***b*** means that $\boldsymbol{a \geq b}$

More Tree Terminology

Child - An adjacent node connected by an out-edge

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Depth (of a tree) - The maximum depth of any node in the tree

Level (of a node) - depth + 1

More Tree Terminology

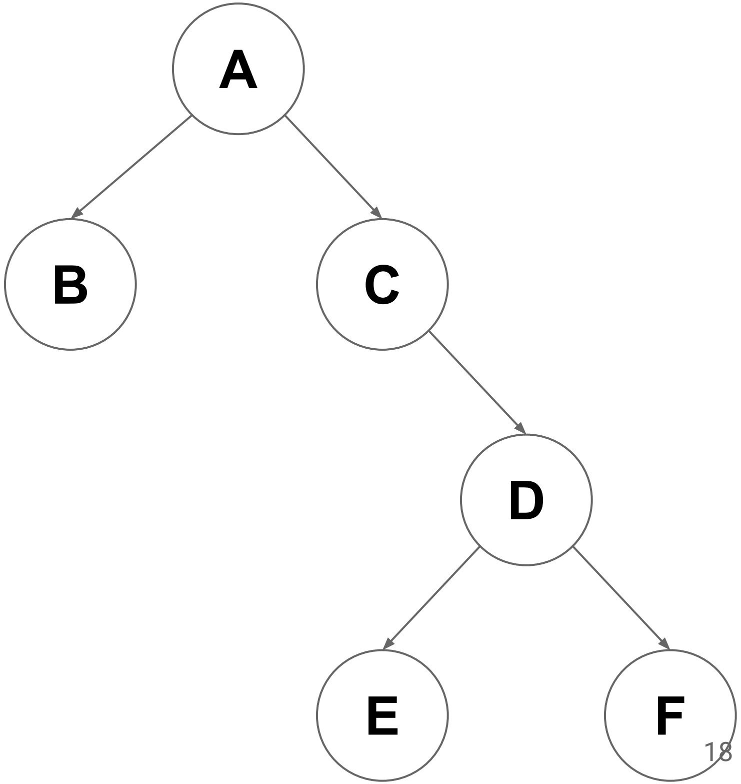
A is the root

B and **C** are children of **A**
D is a child of **C**
E and **F** are children of **D**

B, **E** and **F** are leaves

The depth of **A** is 0, **B** and **C:** 1, **D**: 2, **E** and **F**: 3

The depth of the tree is 3



Binary Min Heaps

Organize our priority queue as a directed tree

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This makes it easy to encode into an array (later today)

Binary Min Heaps

Organize our priority queue as a directed tree

Directed: A directed edge from ***a*** to ***b*** means that $\boxed{a \leq b}$

A max heap would reverse this ordering

Binary: Max out-degree of 2 (easy to reason about)

Complete: Every "level" except the last is full (from left to right)

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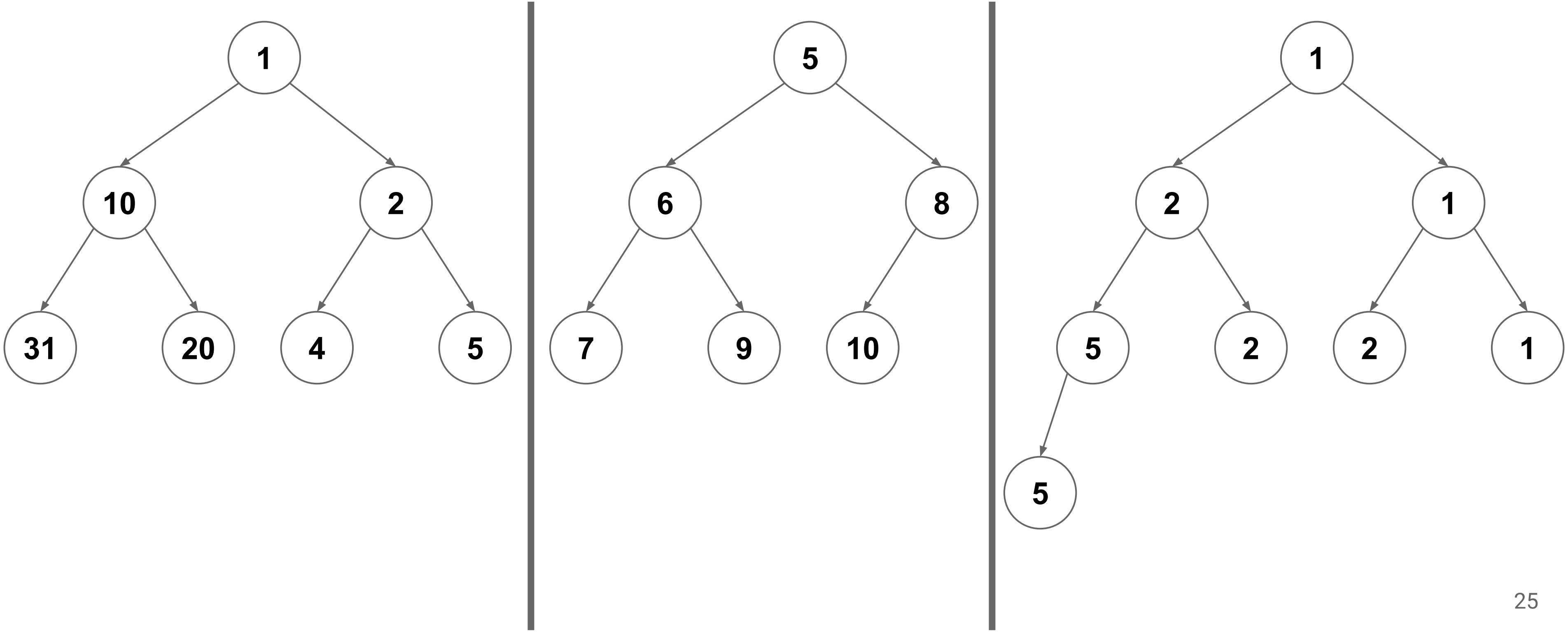
Valid Min Heaps



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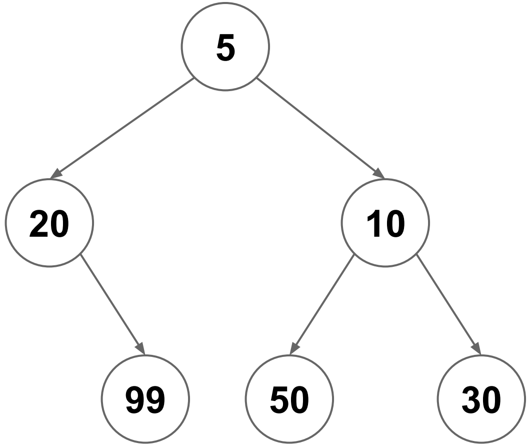

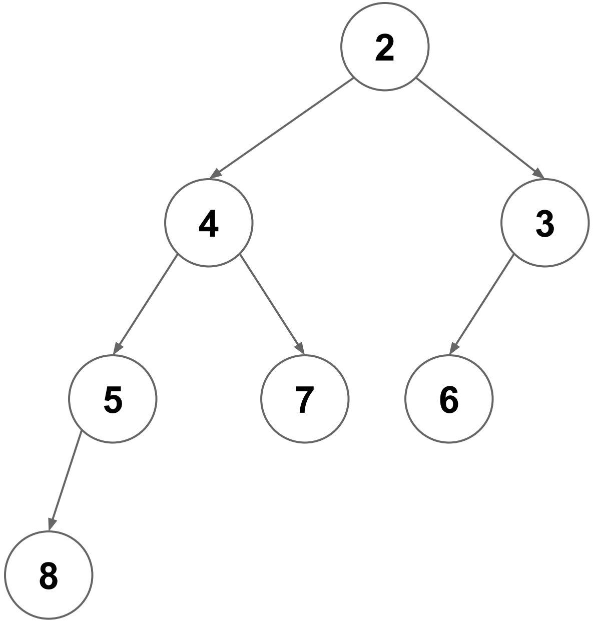

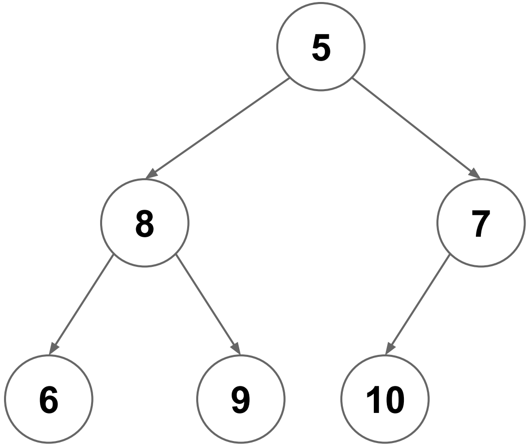

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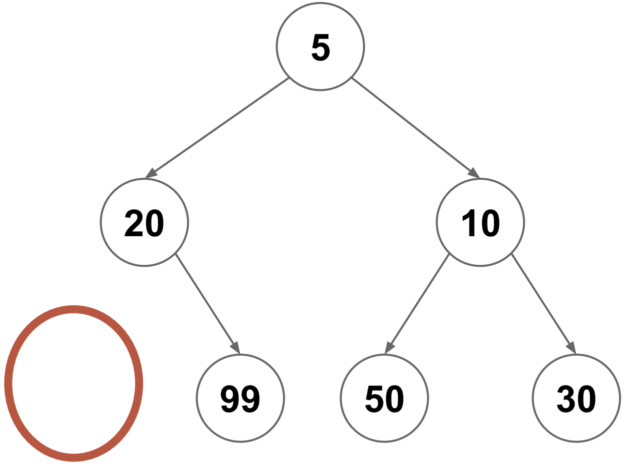
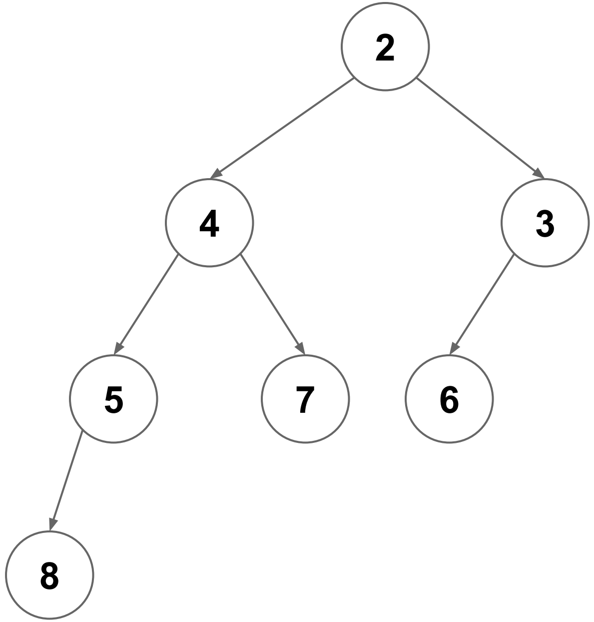
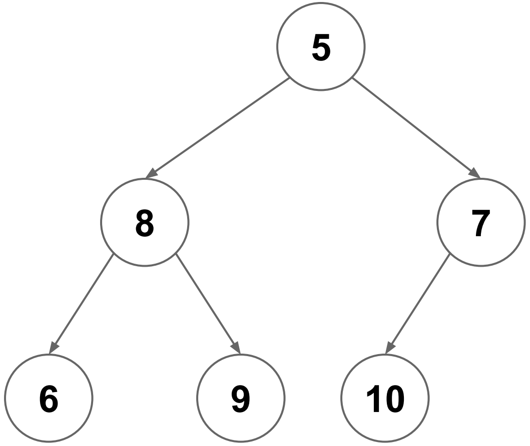


Need to fill from left to right





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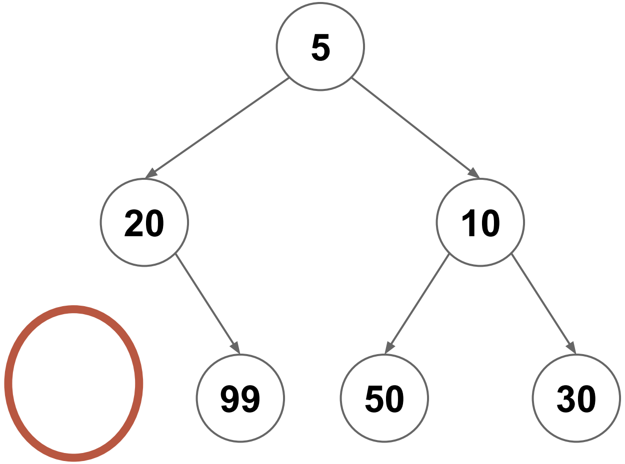

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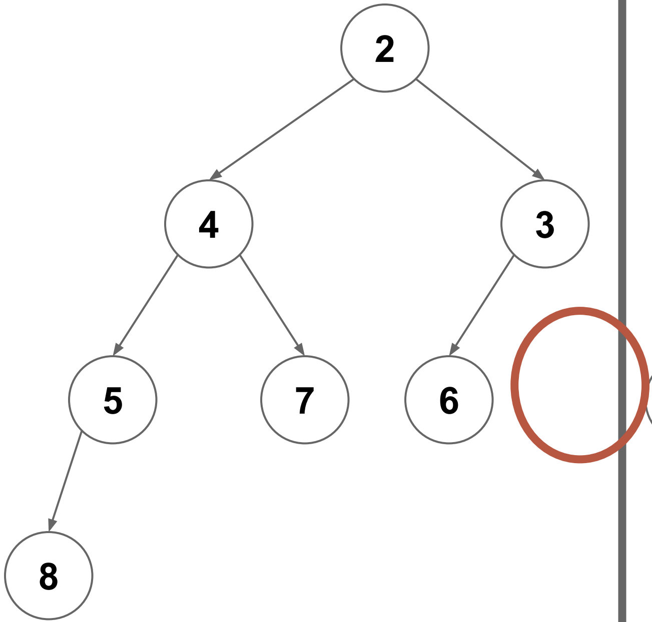
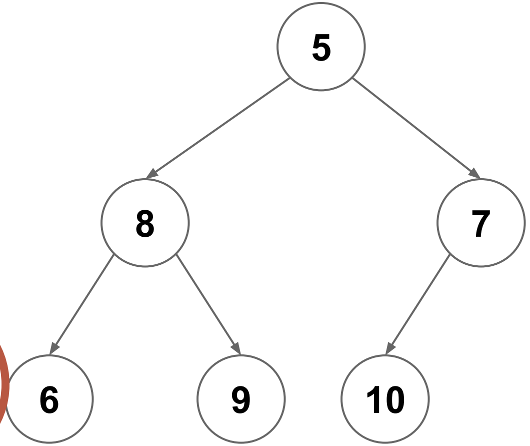


Need complete levels



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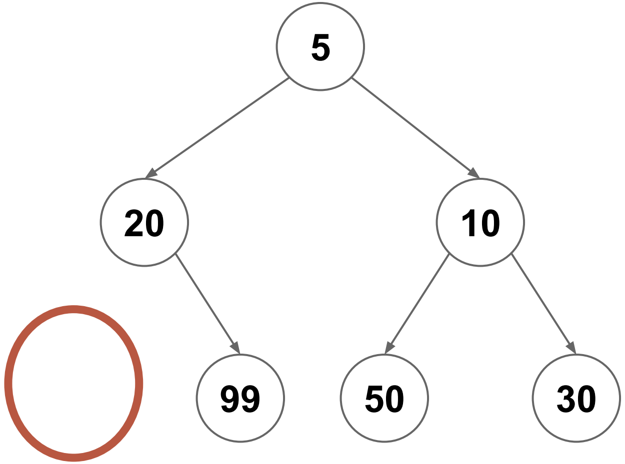

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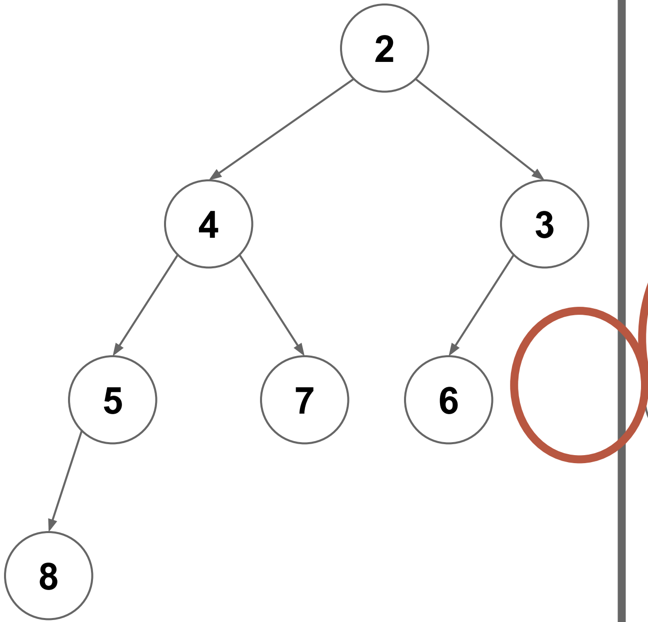

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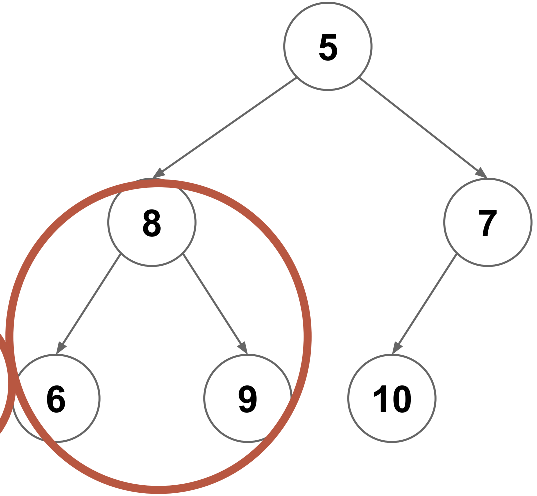

Parents must be less than or equal to children

Heaps

*What is the depth of a binary heap containing **n** items?*

Level 1: holds up to 1 item

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Level 4: holds up to 8 items

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Level 1: holds up to 1 item

Level 2: holds up to 2 items

Level 3: holds up to 4 items

Level 4: holds up to 8 items

…

Level *i*: holds up to 2^{i-1} items

Heaps

What is the depth of a binary heap containing ***n*** *items?*

$$n = O\left(\sum_{i=1}^{\ell_{max}} 2^i\right) = O\left(2^{\ell_{max}}\right)$$

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$$n = O\left(\sum_{i=1}^{\ell_{max}} 2^i\right) = O\left(2^{\ell_{max}}\right)$$

$$\ell_{max} = O\left(\log(n)\right)$$

The `MinHeap` ADT

`void pushHeap(T value)`

Place an item into the heap

`T popHeap()`

Remove and return the minimal element from the heap

`T peek()`

Peek at the minimal element in the heap

`int size()`

The number of elements in the heap

pushHeap

Idea: Insert the element at the next available spot, then fix the heap.

`pushHeap`

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1. Call the insertion point **`current`**
2. While **`current != root`** and **`current < parent`**
   a. Swap **`current`** with **`parent`**
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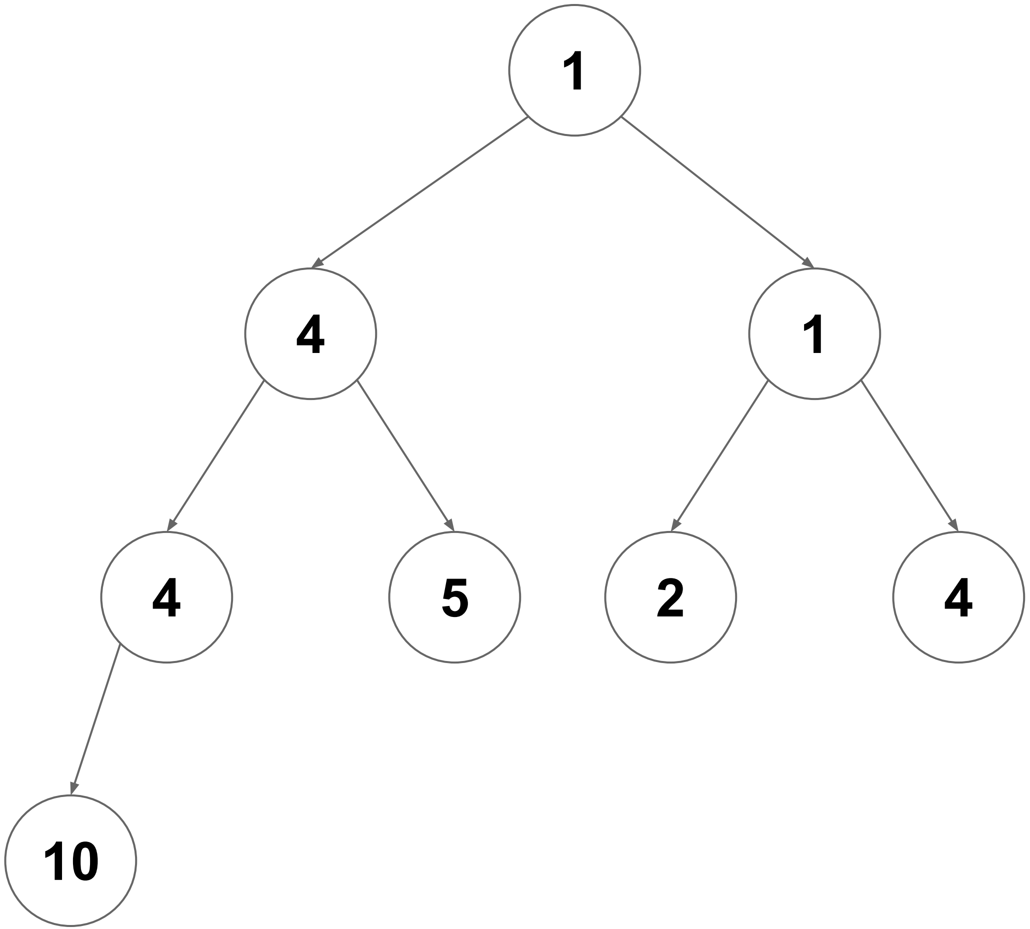
pushHeap

What if we add 3?



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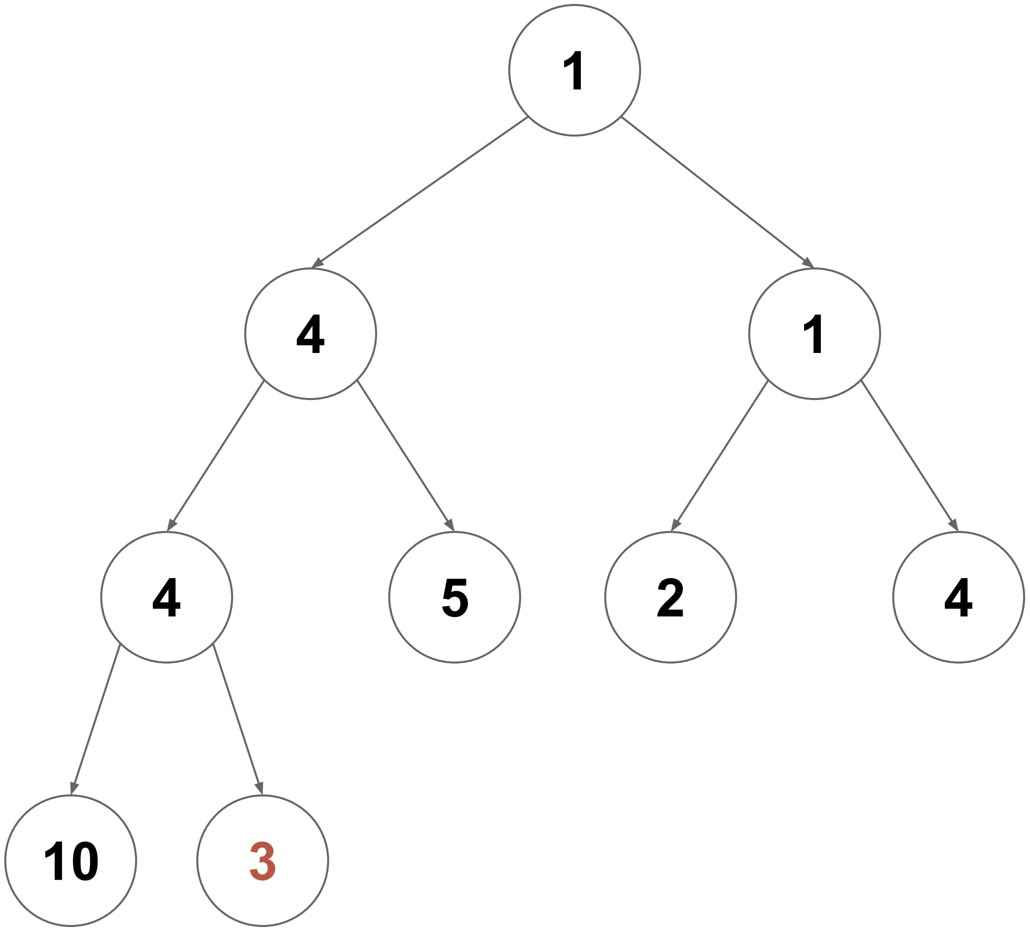
What if we add 3?

Place in the next available spot



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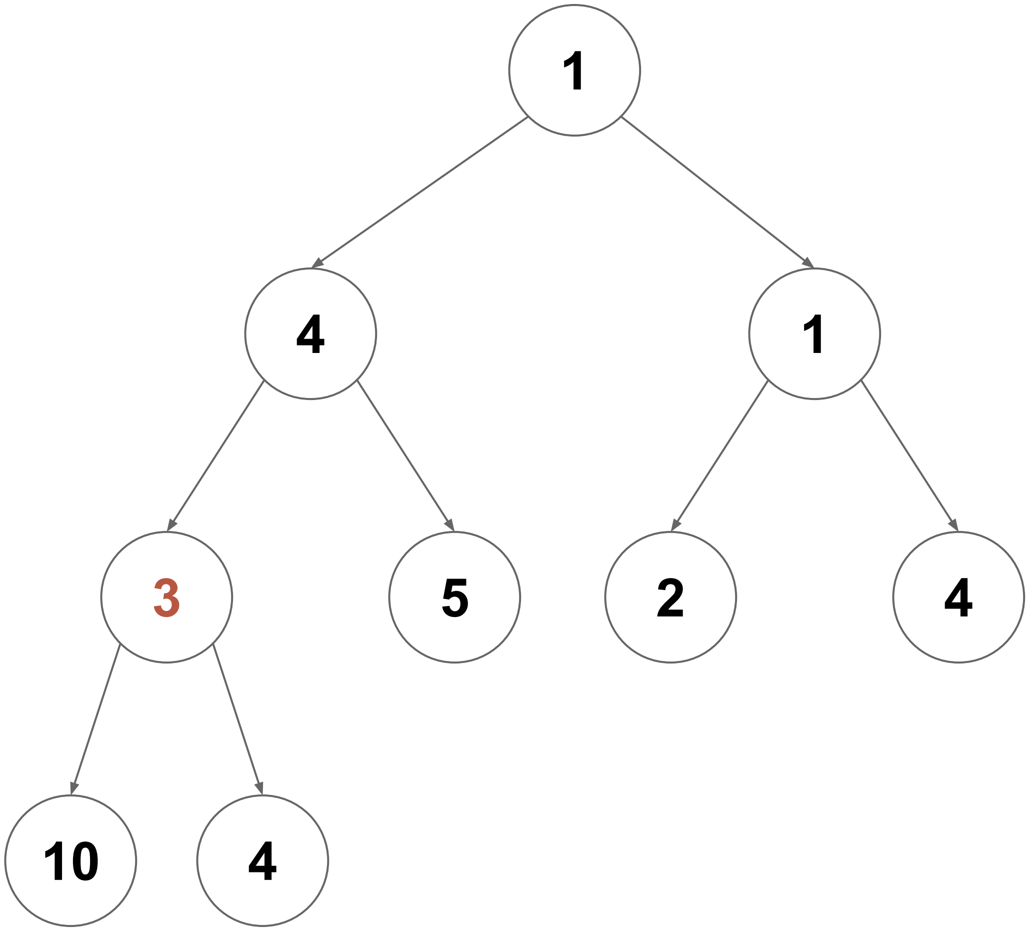
What if we enqueue 3?

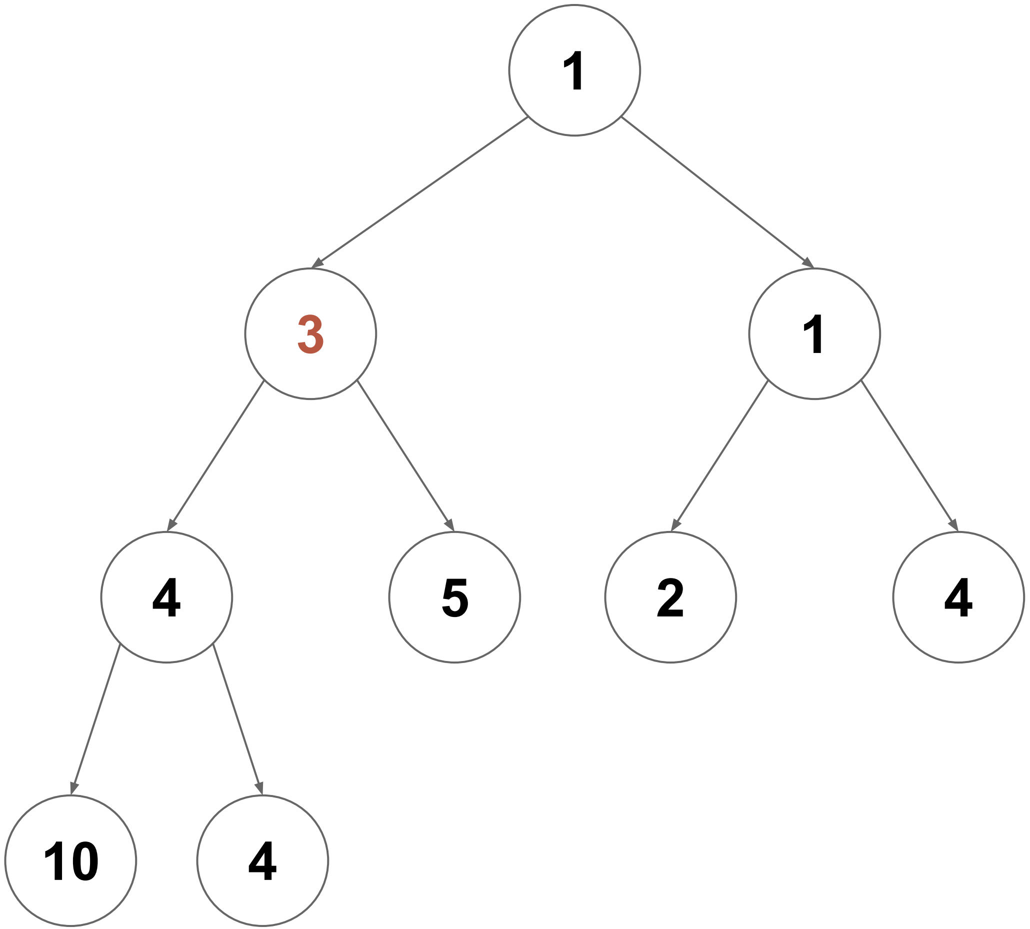
Swap with parent if it is smaller than the parent



pushHeap

What if we enqueue 3?

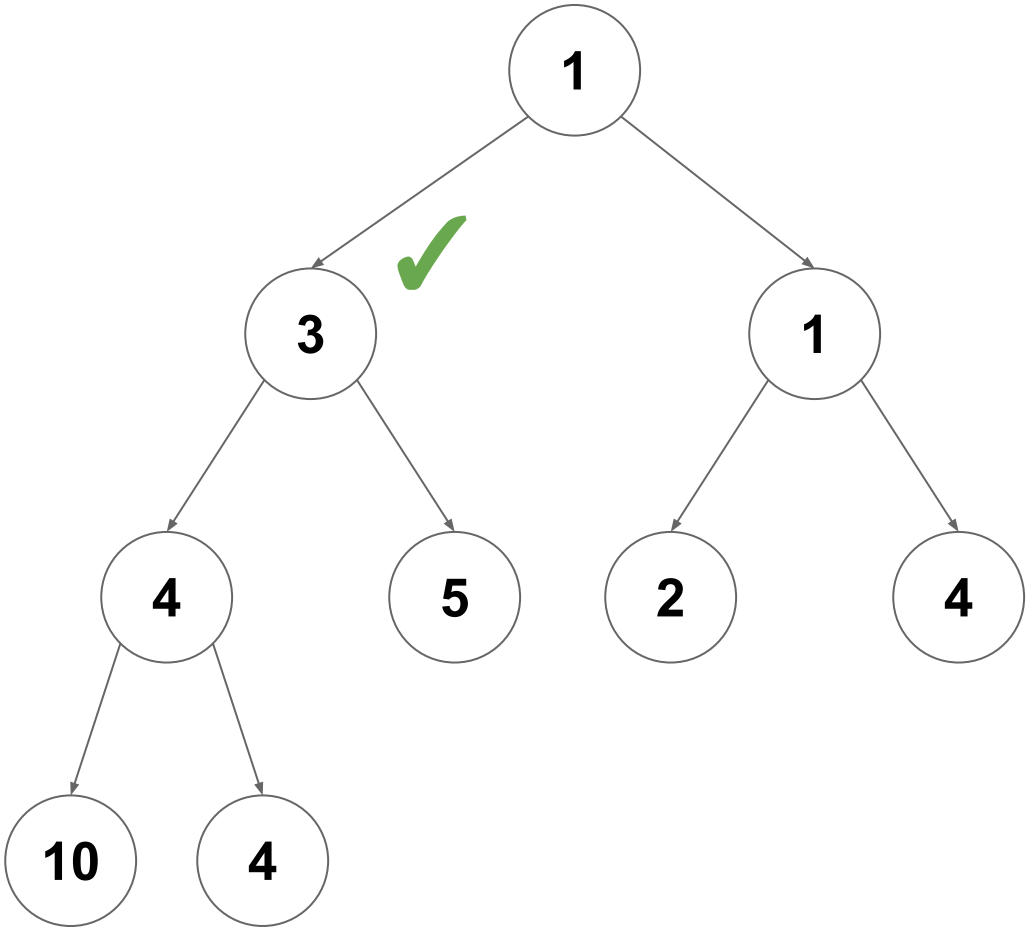
Continue swapping upwards…



pushHeap

What if we enqueue 3?

Stop swapping when we are no longer smaller than our parent



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What is the complexity (or how many swaps occur)?

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What is the complexity (or how many swaps occur)? ***O*(log(*n*))**

popHeap

Idea: Replace root with the last element then fix the heap

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Idea: Replace root with the last element then fix the heap

1. Start with **`current = root`**
2. While **`current`** has a **`child < current`**
   a. Swap **`current`** with its smallest **`child`**
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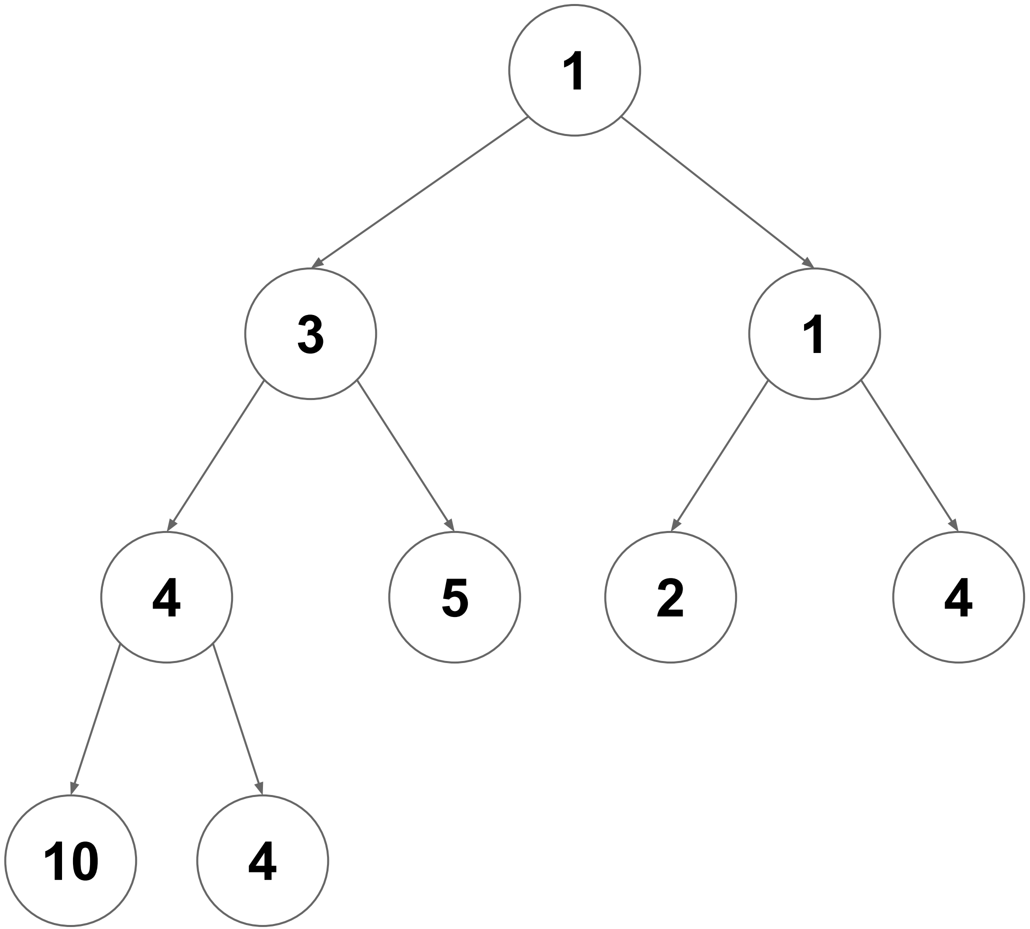
popHeap

What if we call popHeap?



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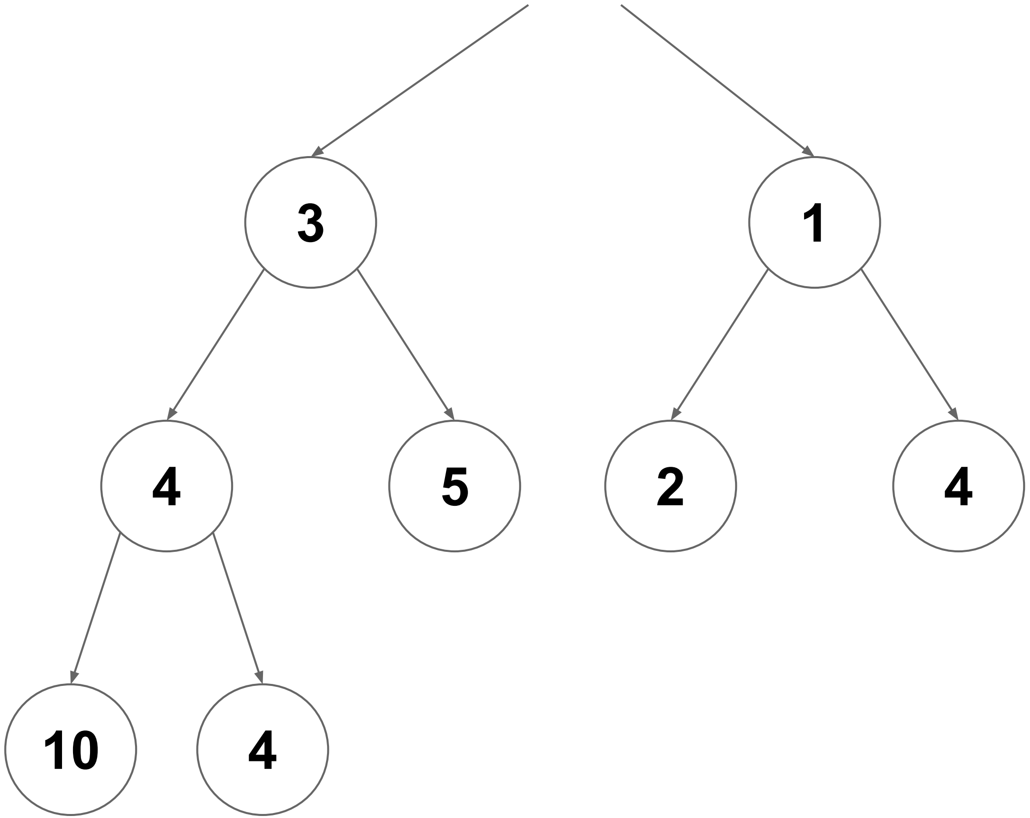
Remove and return the root



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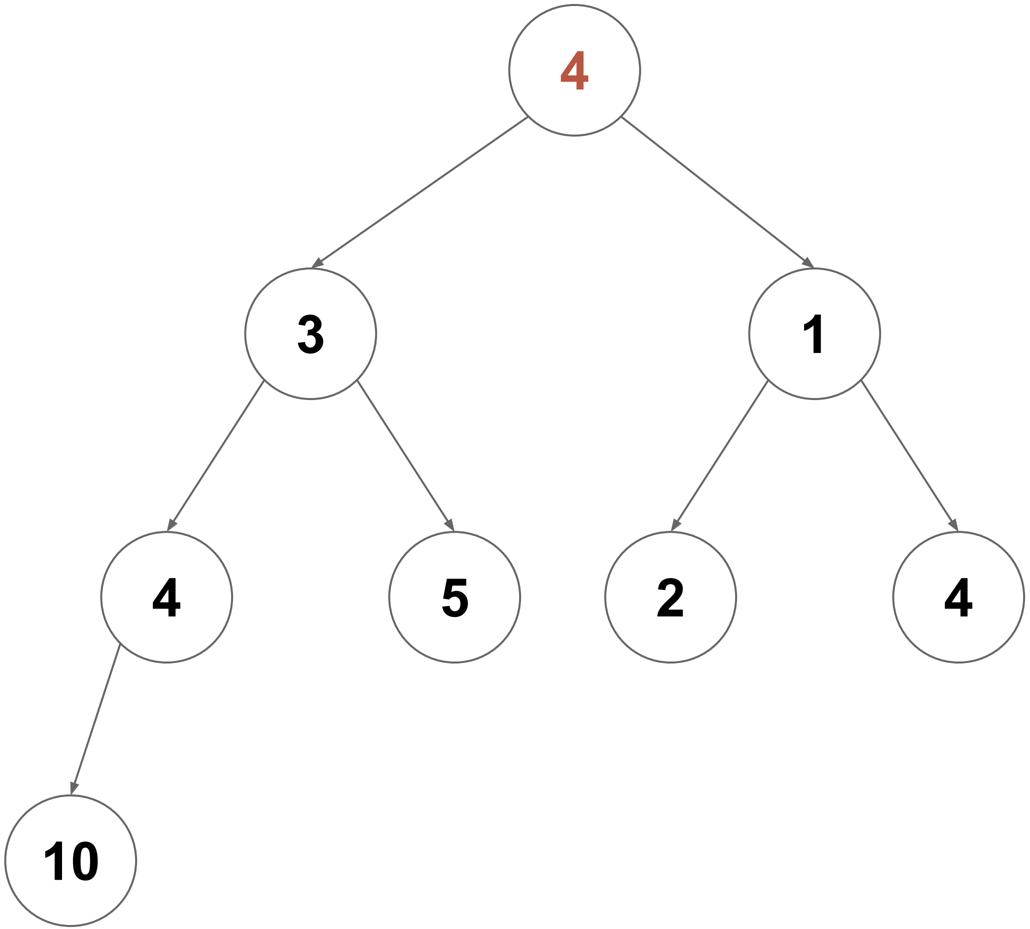
Make the last item the new root



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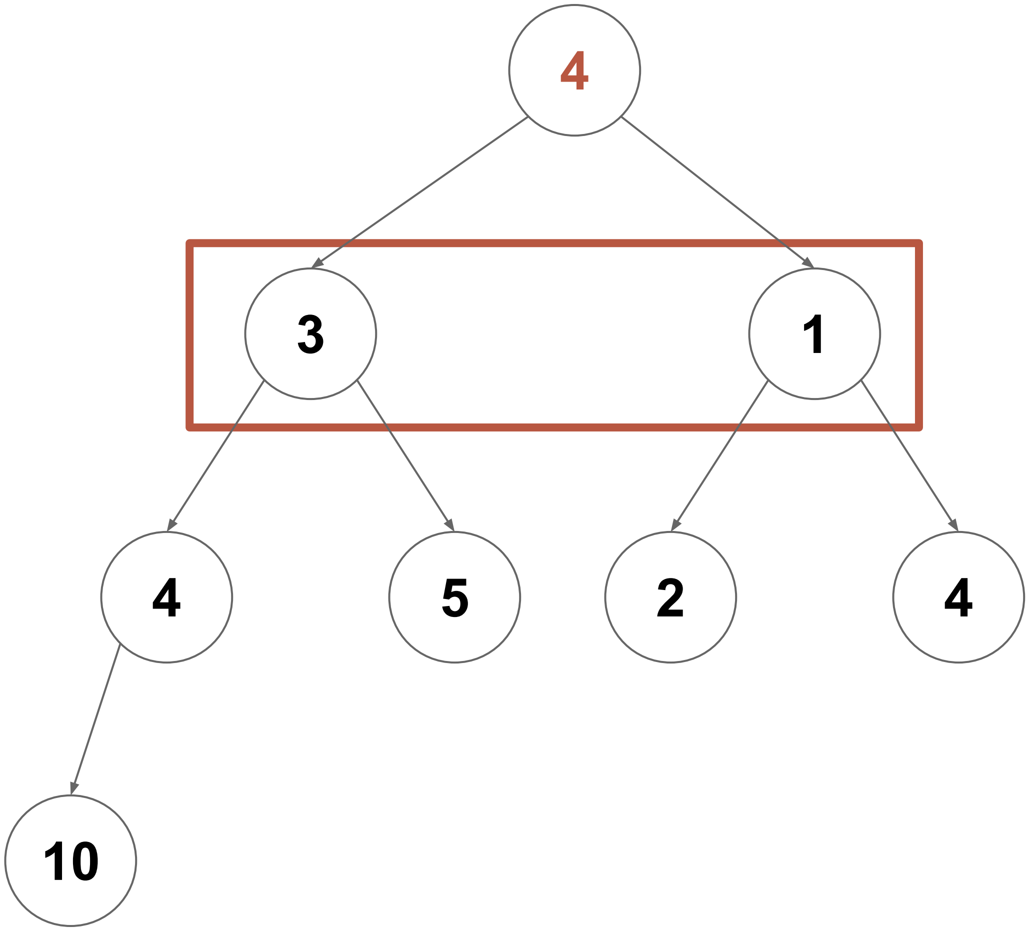
Check for our smallest child



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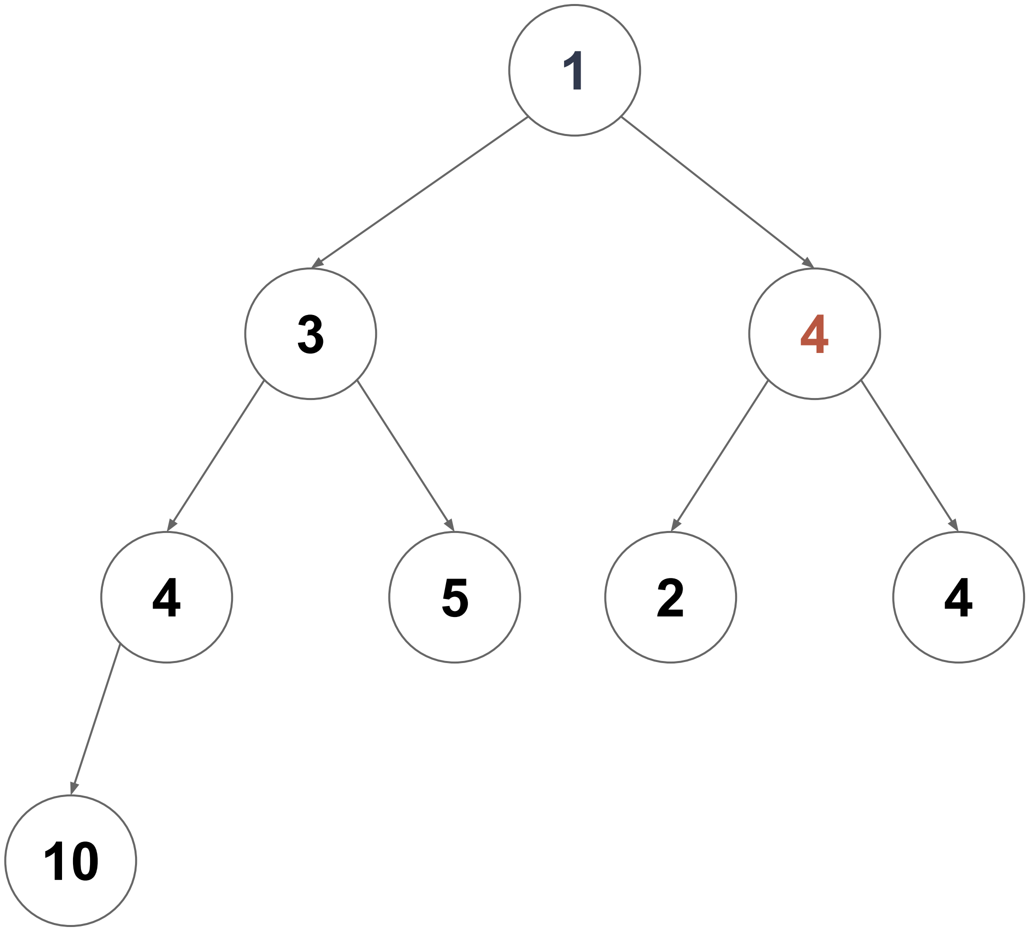
If the smallest child is smaller than us, swap



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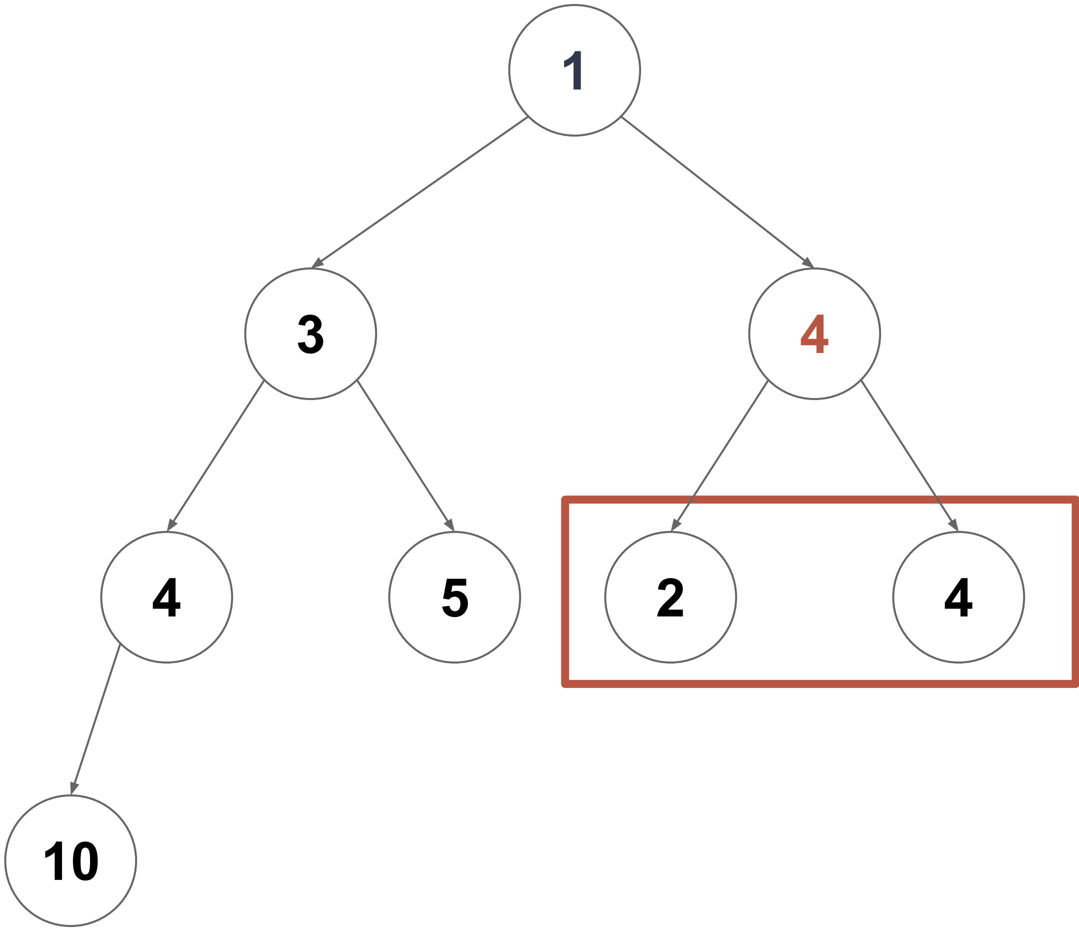
What if we call popHeap?

Continue swapping down the tree as necessary…



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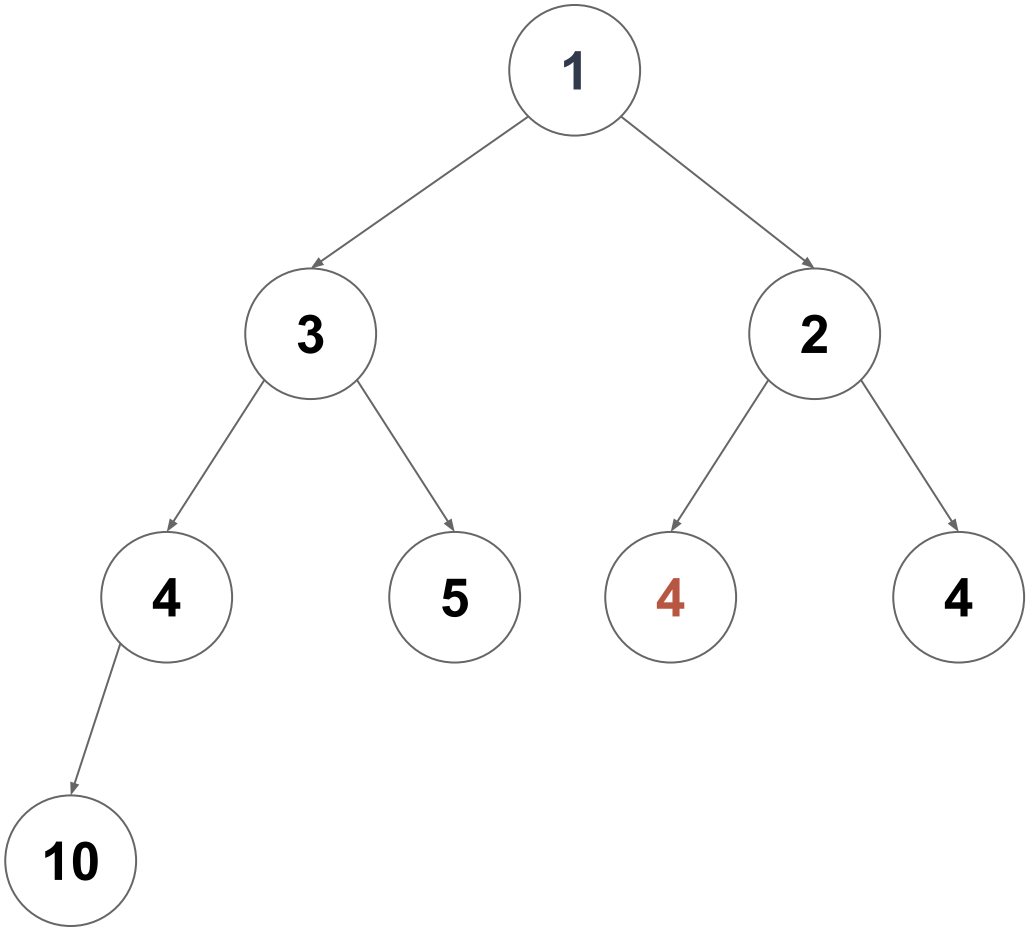

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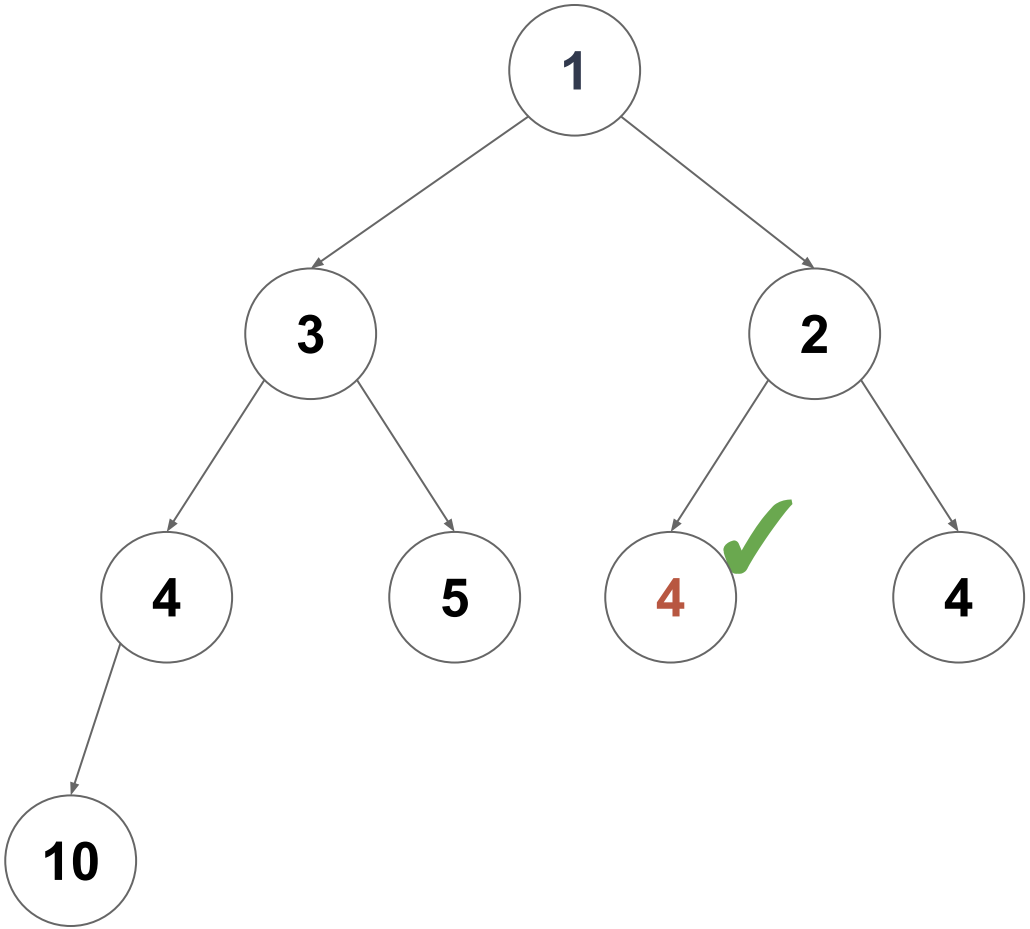
Stop swapping when our children are no longer bigger



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Priority Queues

Operation	Lazy	Proactive	Heap
add	$O(1)$	$O(n)$	$O(\log(n))$
poll	$O(n)$	$O(1)$	$O(\log(n))$
peek	$O(n)$	$O(1)$	$O(1)$

Storing heaps

Notice that:

1. Each level has a maximum size
2. Each level grows left-to-right
3. Only the last layer grows

How can we compactly store a heap?

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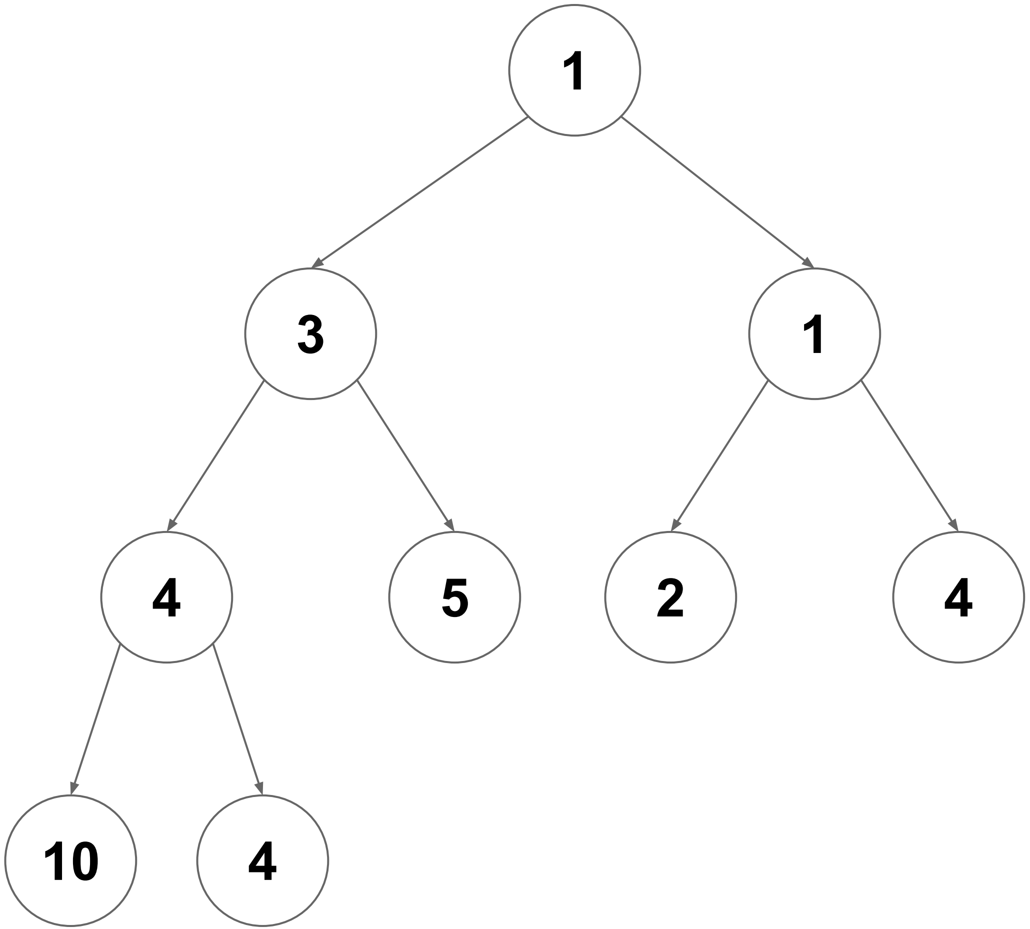
How can we compactly store a heap?

Idea: Use an **`ArrayList`**

Storing Heaps

How can we store this heap in an array buffer?



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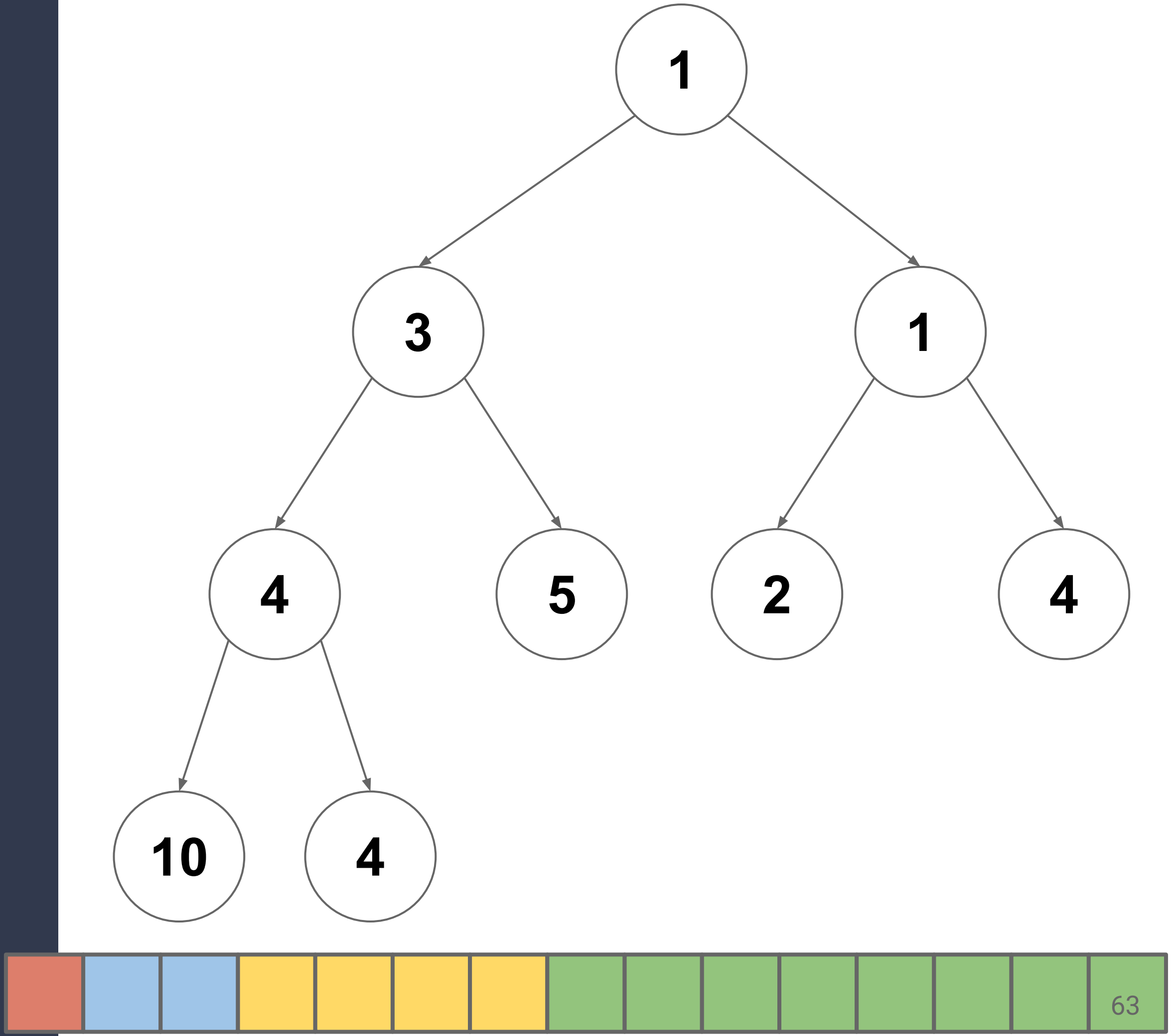

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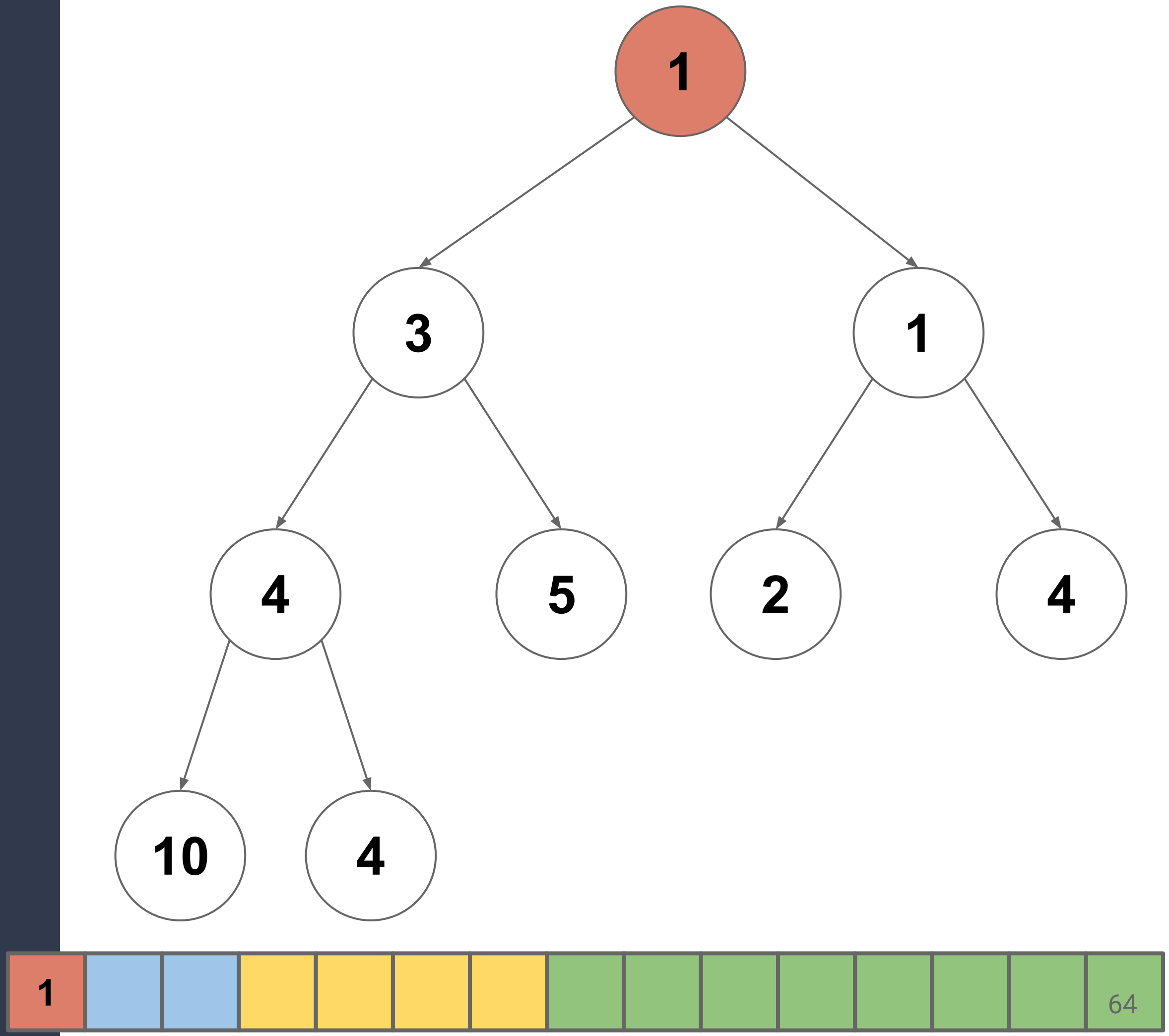

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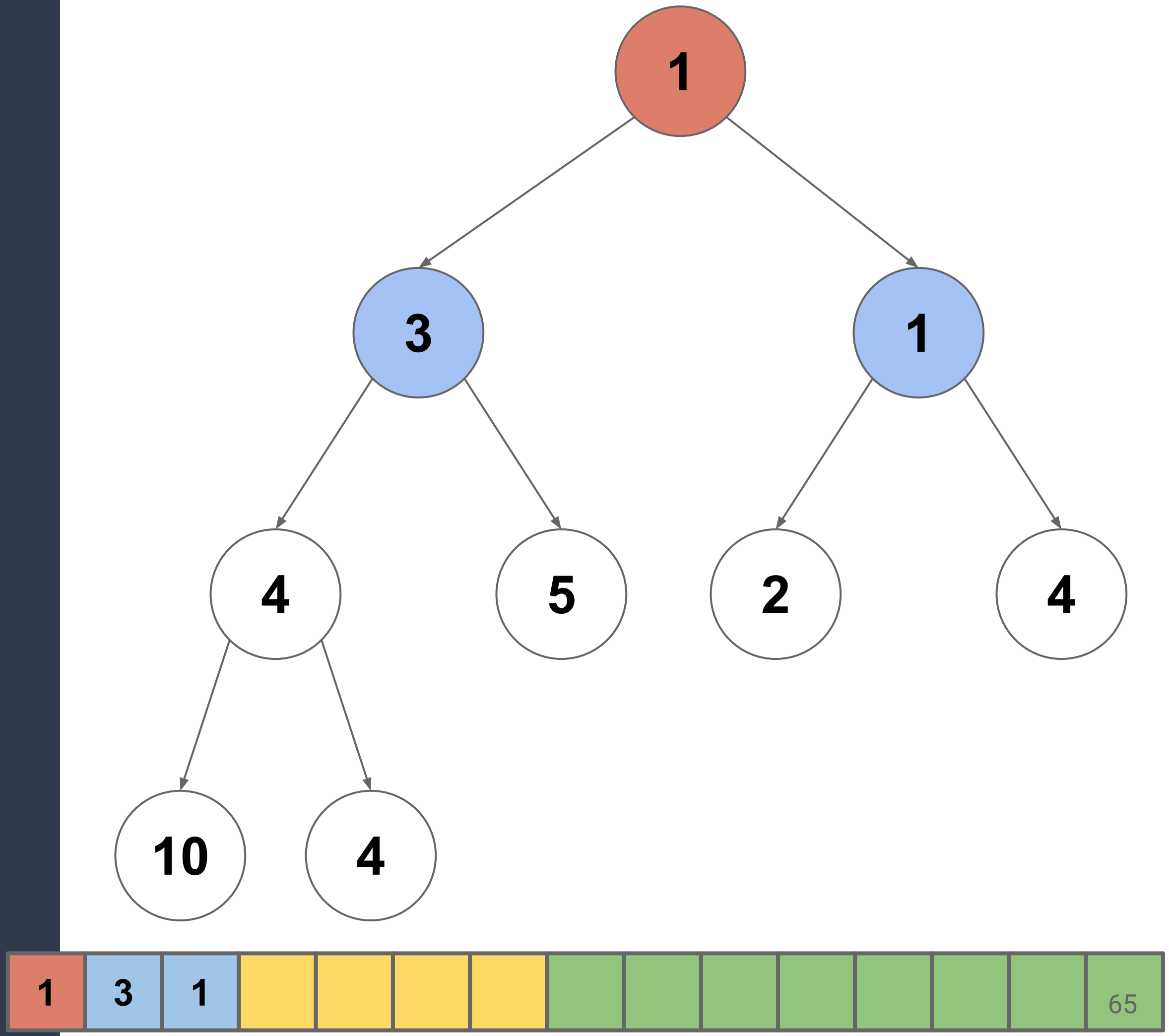

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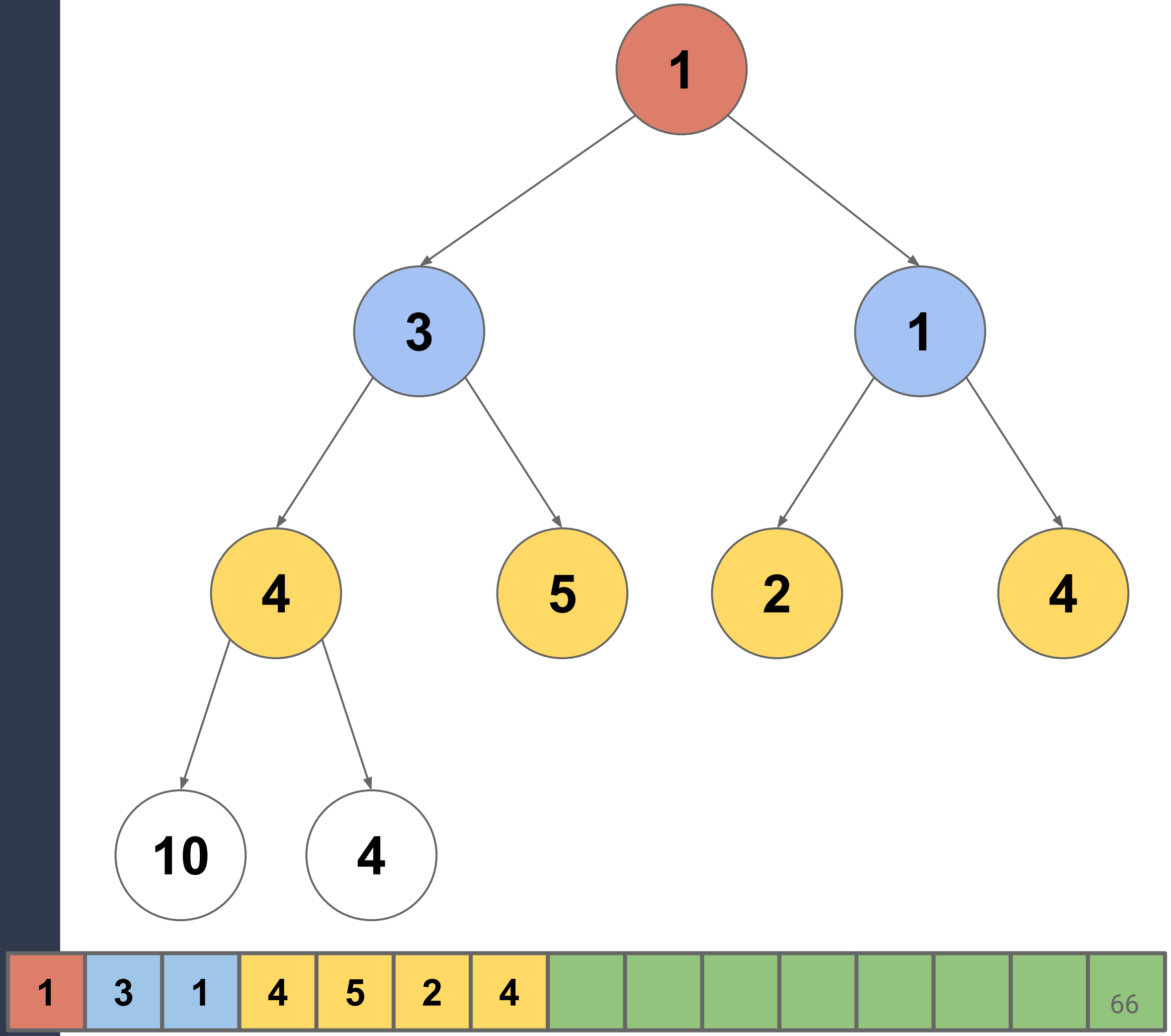

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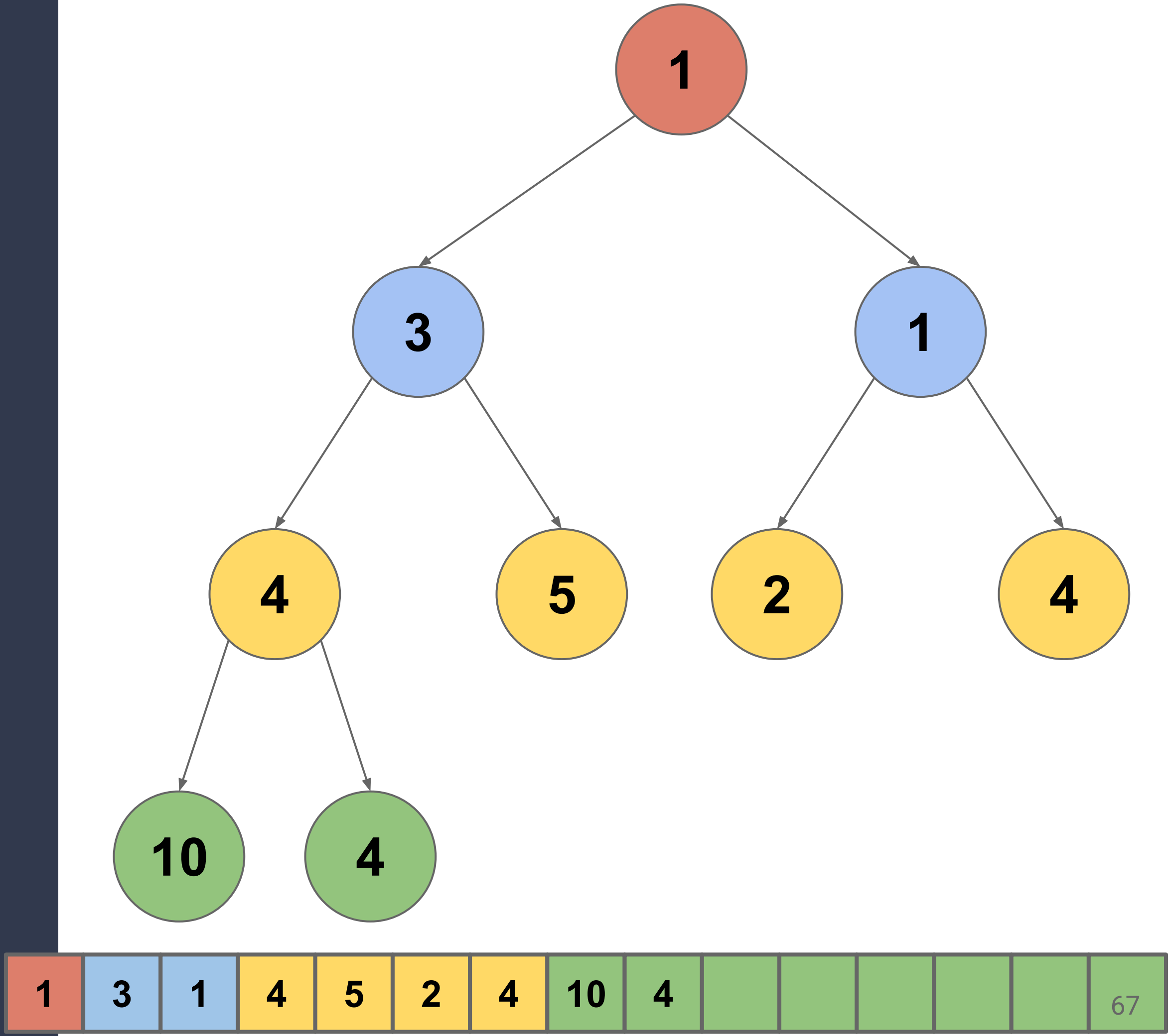

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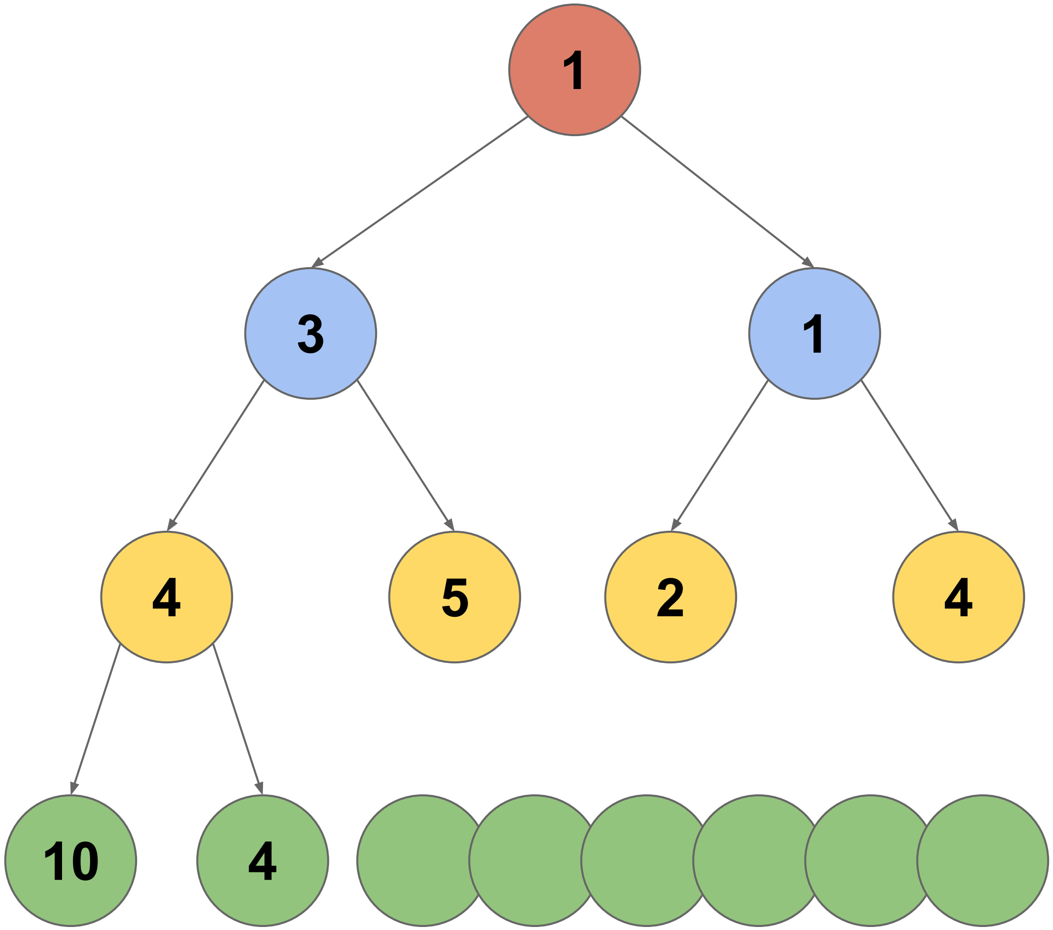

1	3	1	4	5	2	4	10	4						

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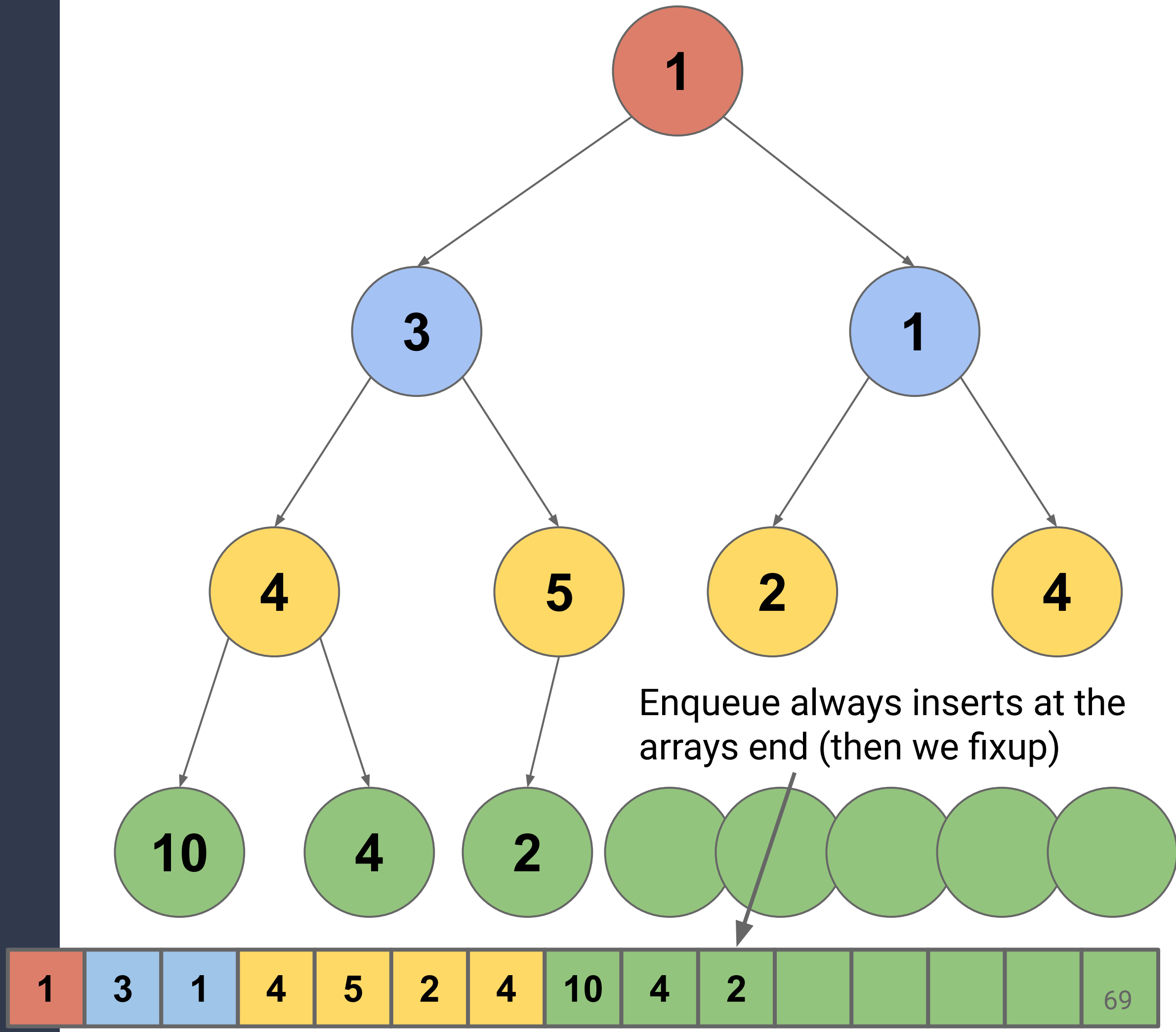

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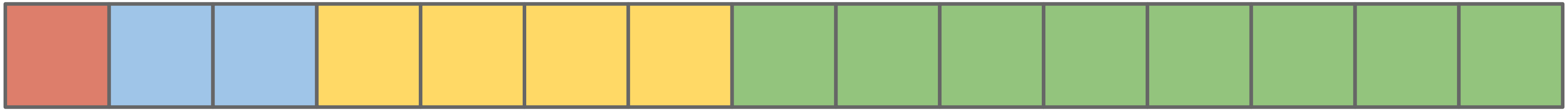
Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

7, 4, 8, 2, 5, 3, 9



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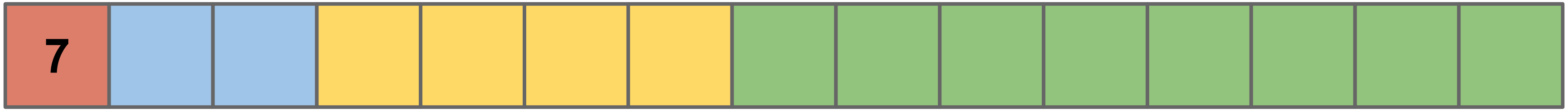

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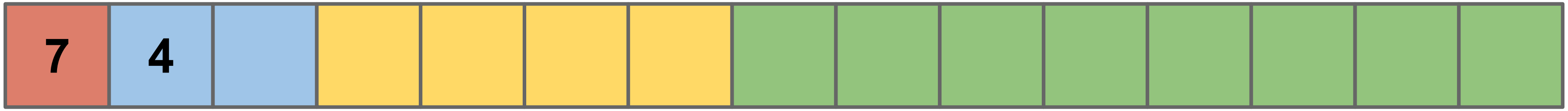

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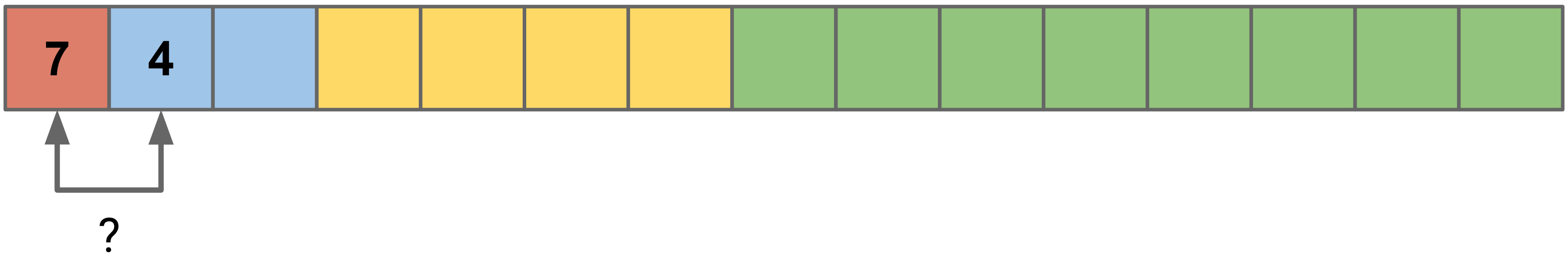
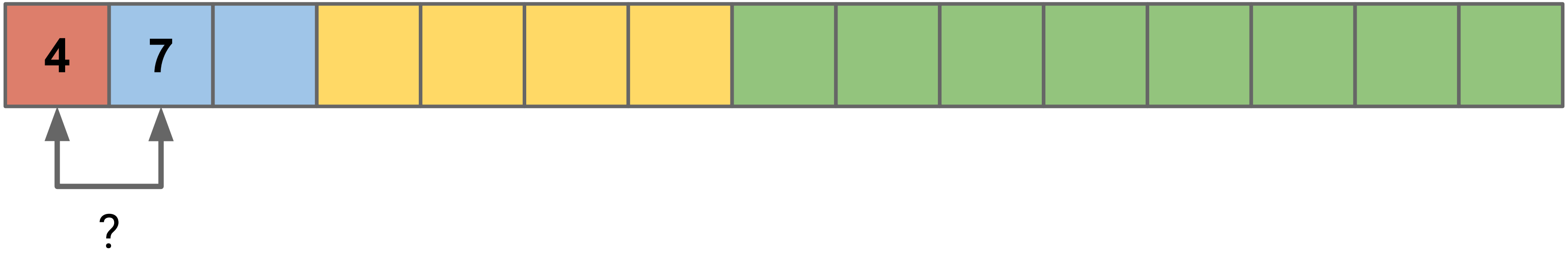


Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

7, **4**, 8, 2, 5, 3, 9



Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

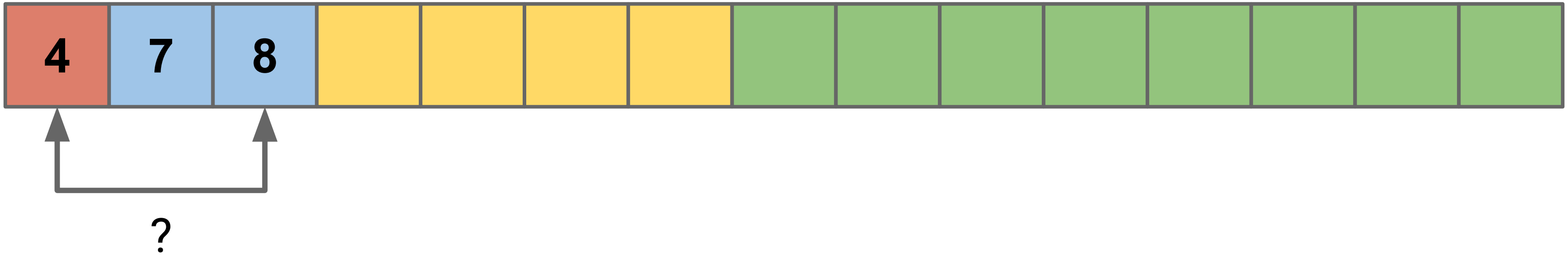
7, 4, **8**, 2, 5, 3, 9

4	**7**	**8**												

Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

7, 4, **8**, 2, 5, 3, 9



Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

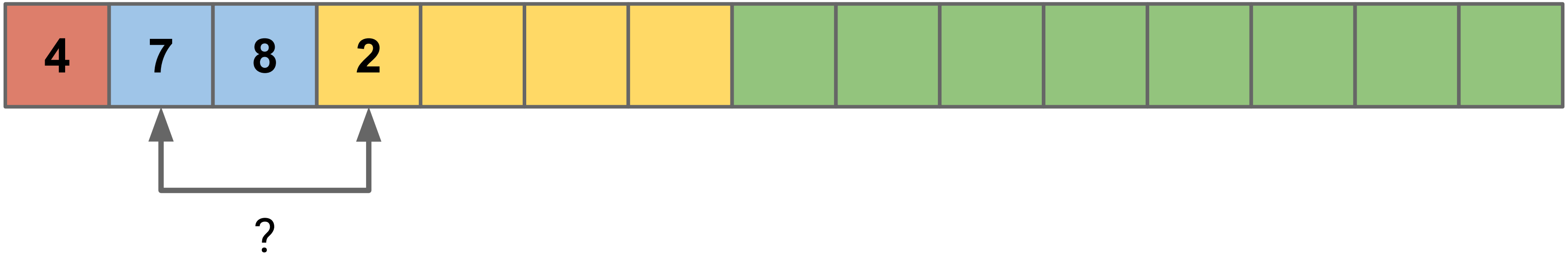
7, 4, 8, **2**, 5, 3, 9

4	**7**	**8**	**2**											

Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

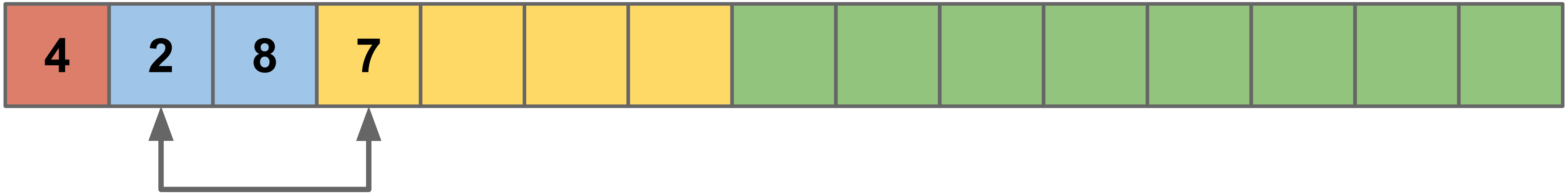
7, 4, 8, **2**, 5, 3, 9



Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

7, 4, 8, **2**, 5, 3, 9

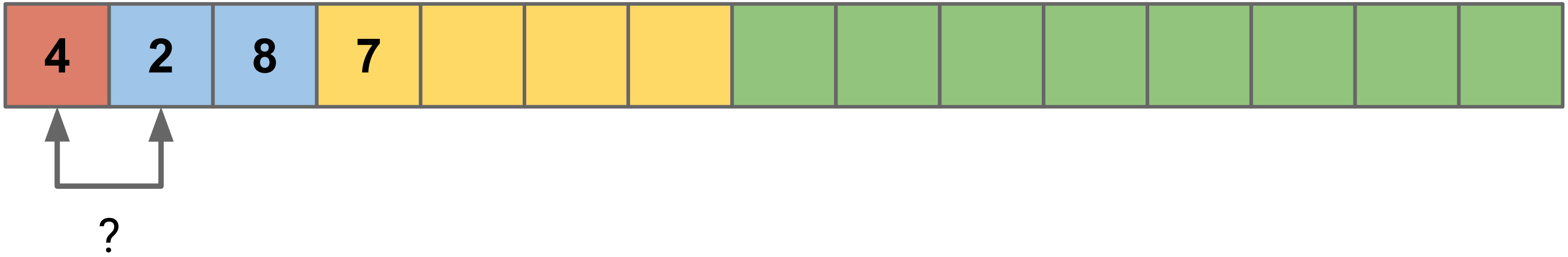


?

Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

7, 4, 8, **2**, 5, 3, 9



Heap Sort

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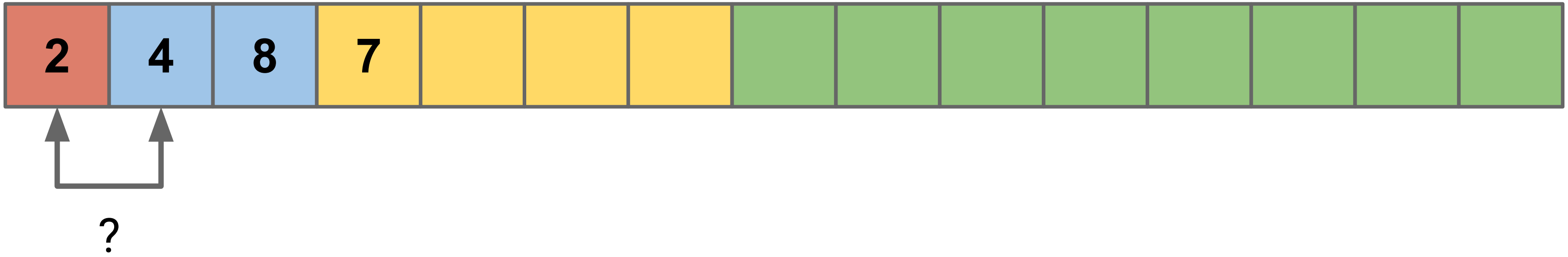
2. Reconstruct sequence with dequeue

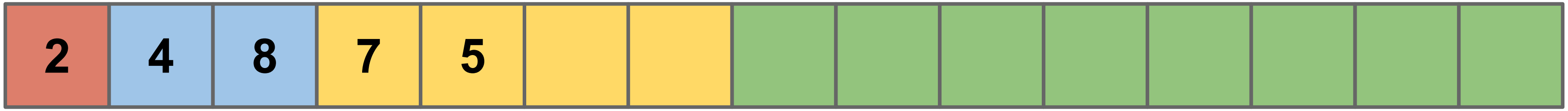
7, 4, 8, **2**, 5, 3, 9



Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

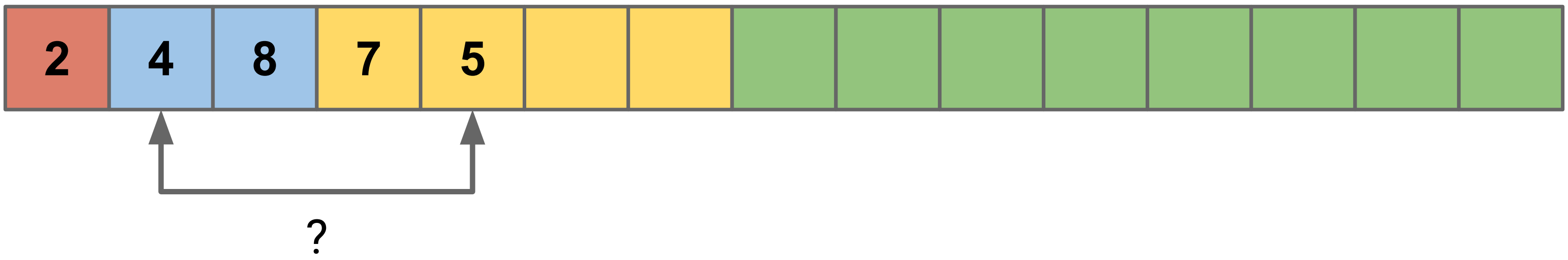
7, 4, 8, 2, **<u>5</u>**, 3, 9



Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

7, 4, 8, 2, **5**, 3, 9



Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

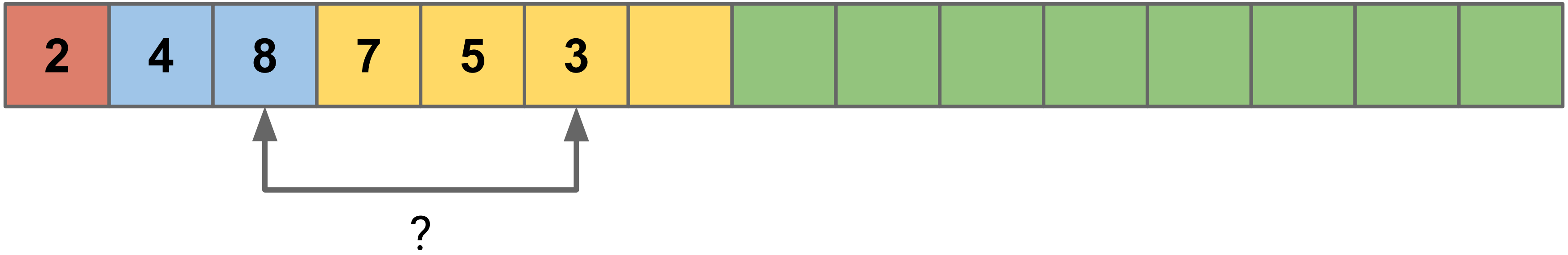
7, 4, 8, 2, 5, <u>**3**</u>, 9

2	**4**	**8**	**7**	**5**	**3**									

Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

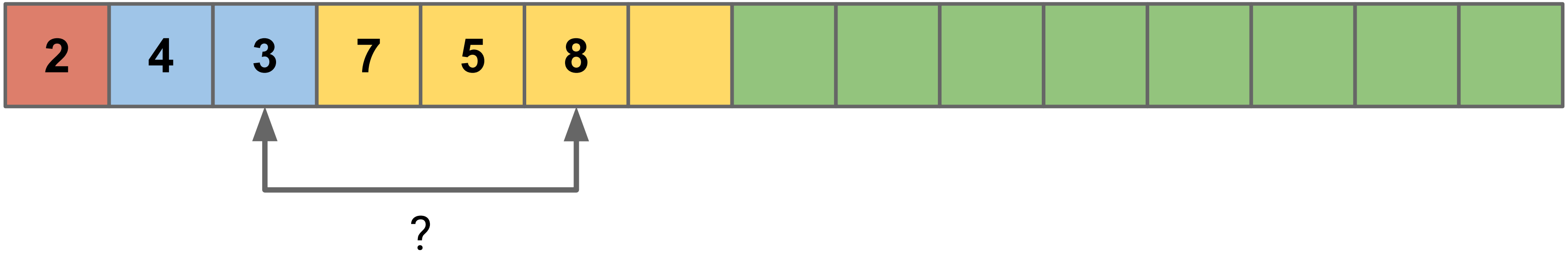
7, 4, 8, 2, 5, **3**, 9



Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

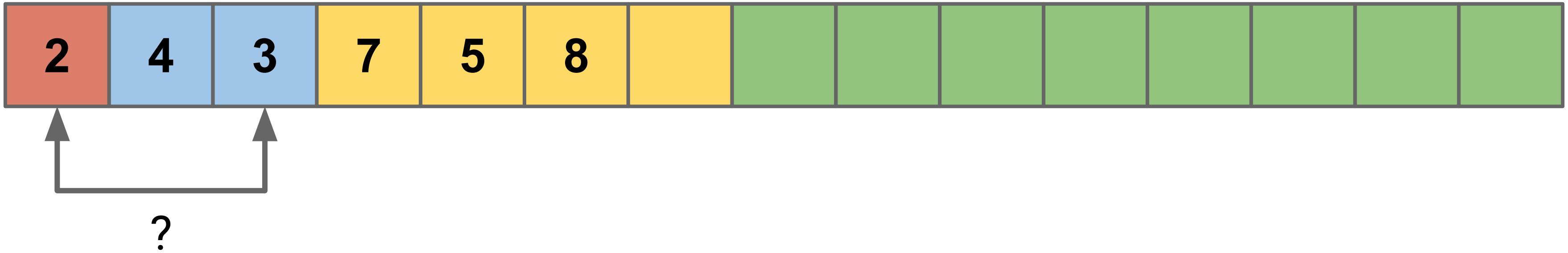
7, 4, 8, 2, 5, **<u>3</u>**, 9



Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

7, 4, 8, 2, 5, **3**, 9



Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

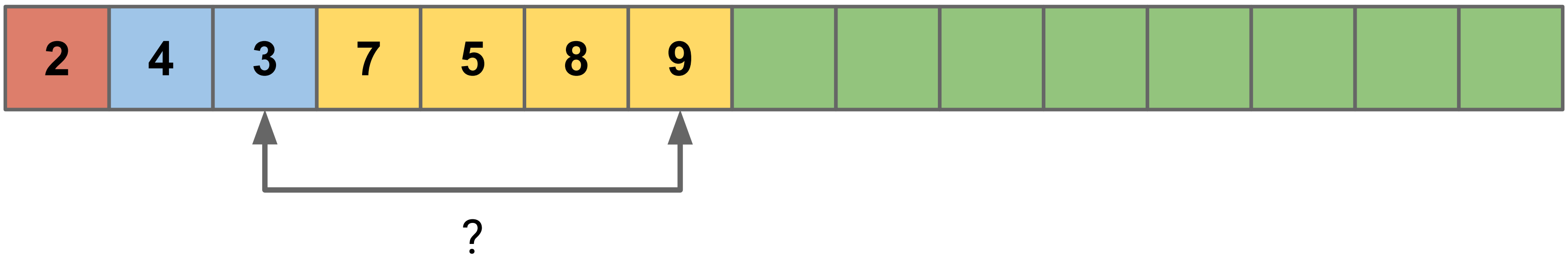
7, 4, 8, 2, 5, 3, **9**

2	4	3	7	5	8	9								

Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

7, 4, 8, 2, 5, 3, **9**



Heap Sort

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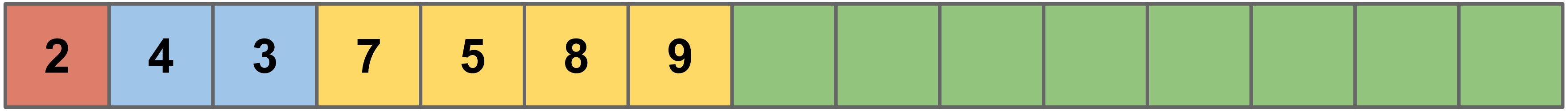
7, 4, 8, 2, 5, 3, 9



Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

7, 4, 8, 2, 5, 3, 9

	4	**3**	**7**	**5**	**8**	**9**								

2

Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

7, 4, 8, 2, 5, 3, 9

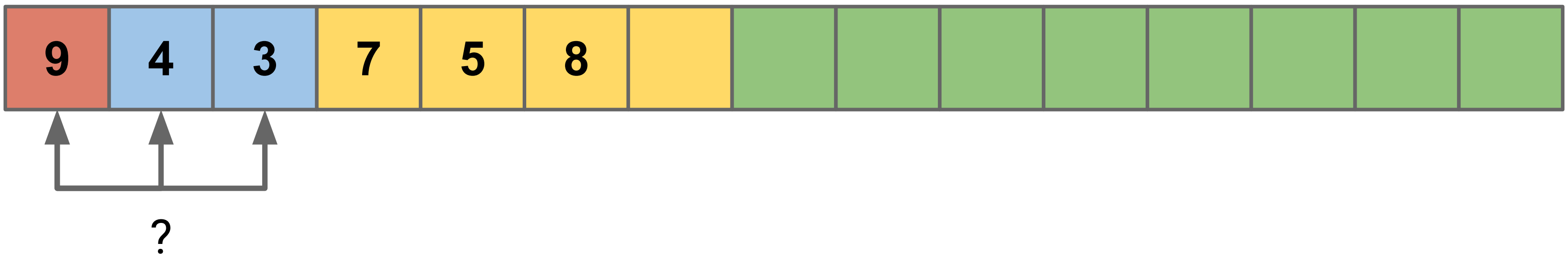
9	4	3	7	5	8									

2

Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

7, 4, 8, 2, 5, 3, 9

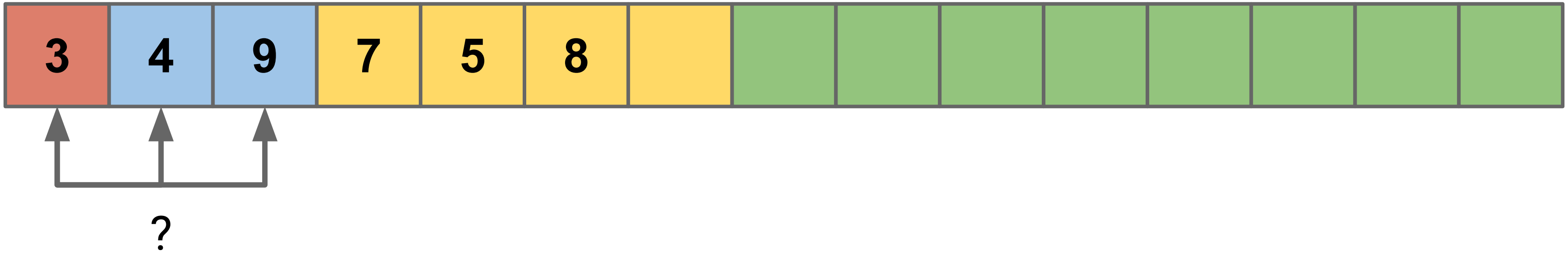


2

Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

7, 4, 8, 2, 5, 3, 9



2

Heap Sort

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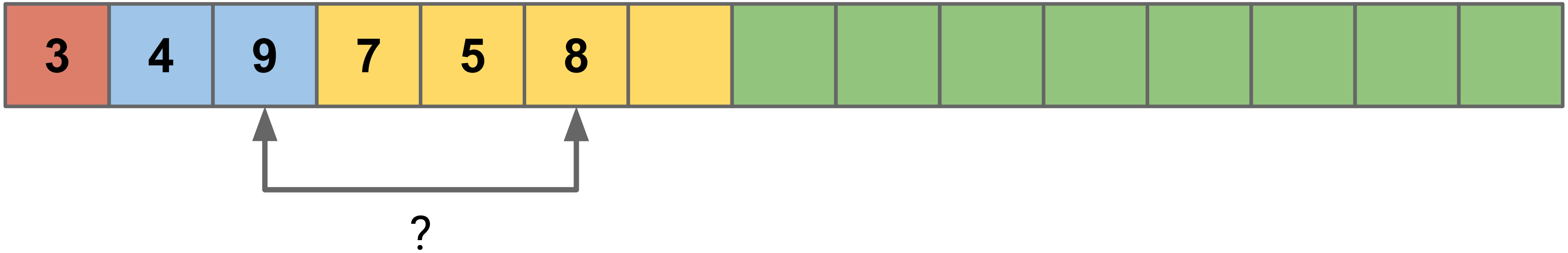
2. Reconstruct sequence with dequeue

7, 4, 8, 2, 5, 3, 9

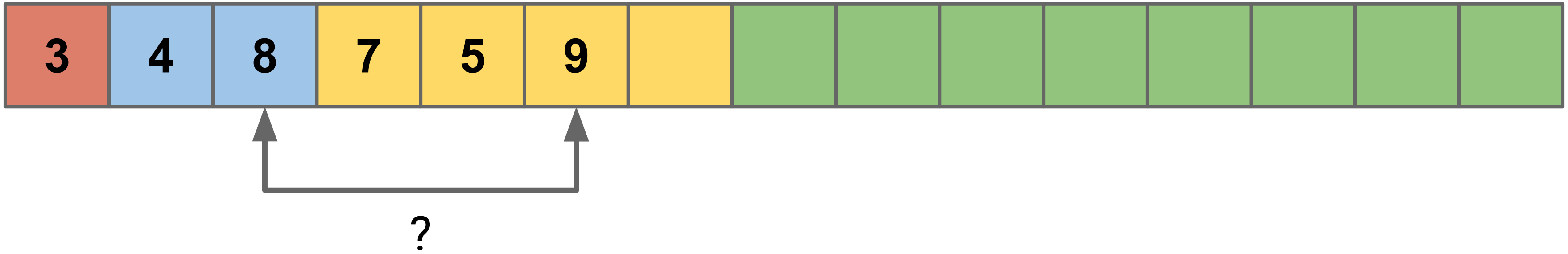


2

Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

7, 4, 8, 2, 5, 3, 9



2

Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

7, 4, 8, 2, 5, 3, 9

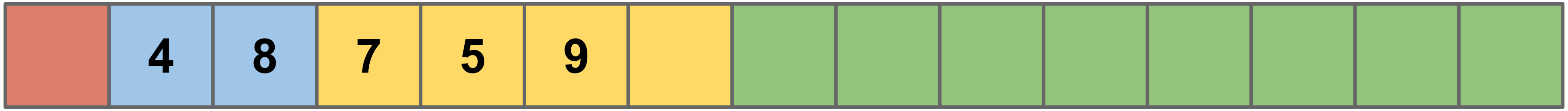
3	**4**	**8**	**7**	**5**	**9**									

2

Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

7, 4, 8, 2, 5, 3, 9

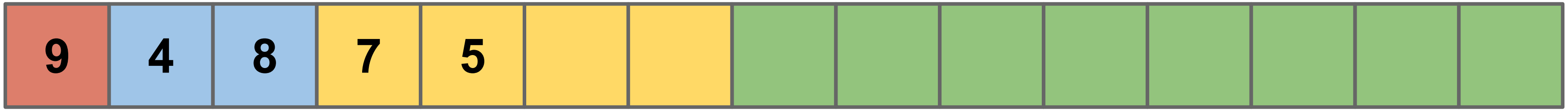


2, 3

Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

7, 4, 8, 2, 5, 3, 9

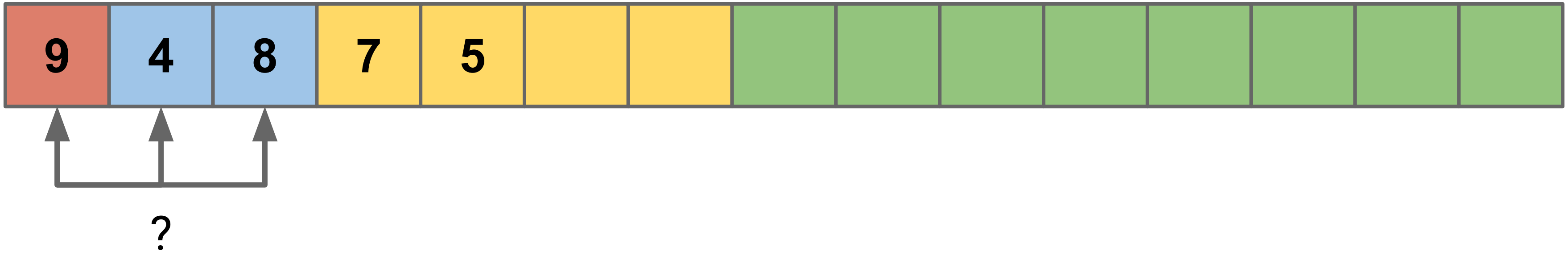


2, 3

Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

7, 4, 8, 2, 5, 3, 9

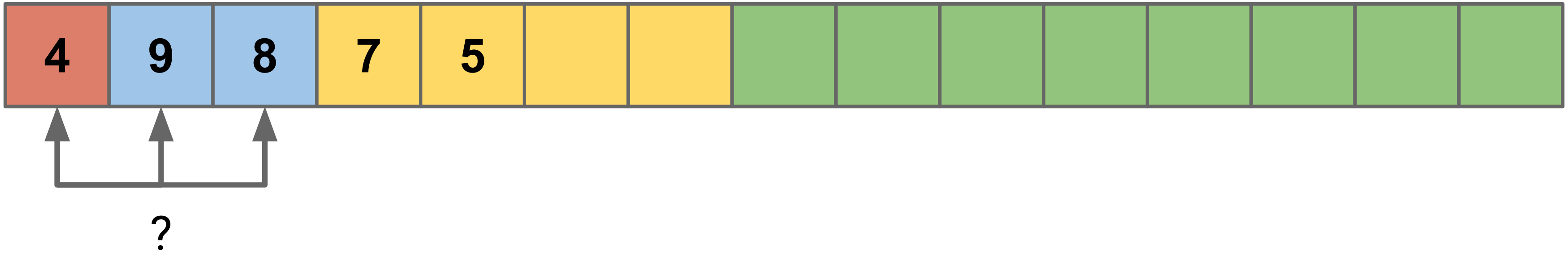


2, 3

Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

7, 4, 8, 2, 5, 3, 9

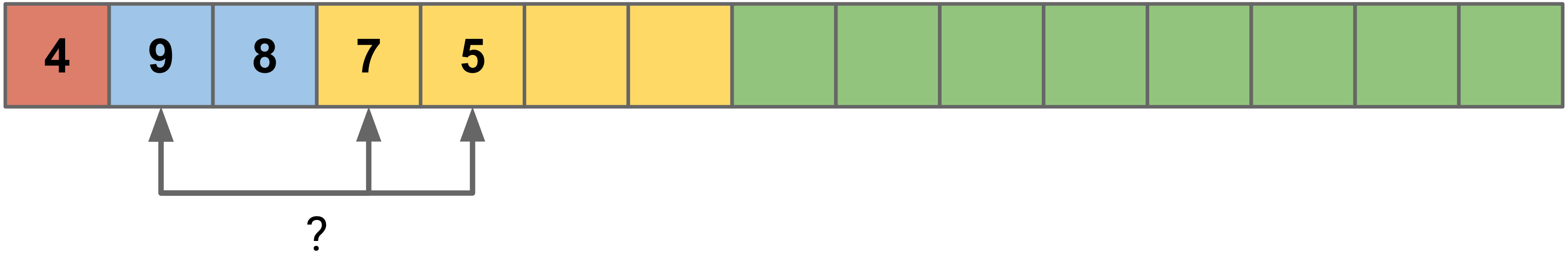


2, 3

Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

7, 4, 8, 2, 5, 3, 9

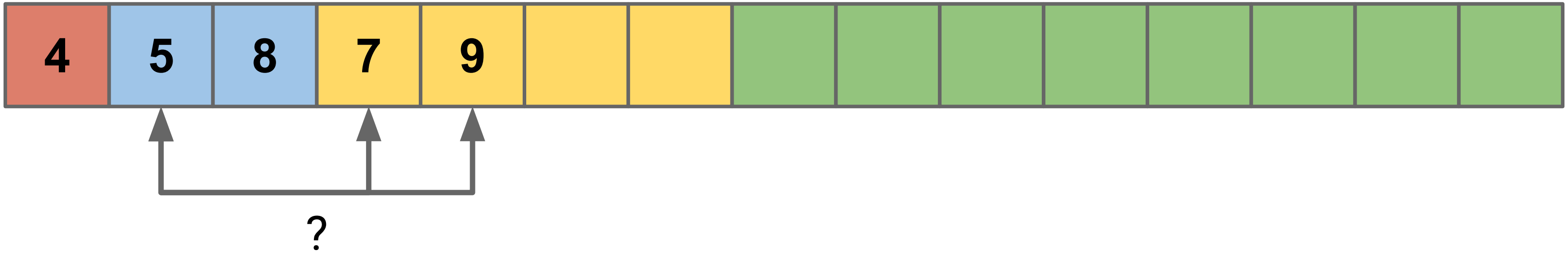


2, 3

Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

7, 4, 8, 2, 5, 3, 9



2, 3

Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

7, 4, 8, 2, 5, 3, 9

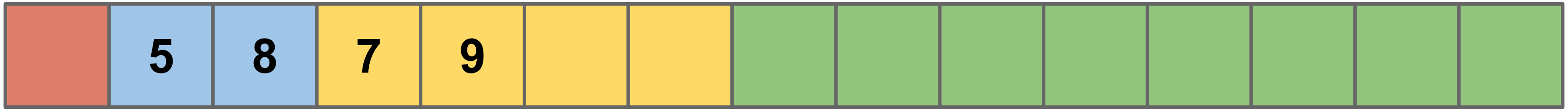
4	**5**	**8**	**7**	**9**										

2, 3

Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

7, 4, 8, 2, 5, 3, 9

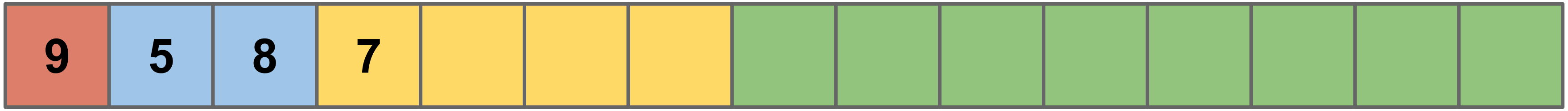


2, 3, 4

Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

7, 4, 8, 2, 5, 3, 9



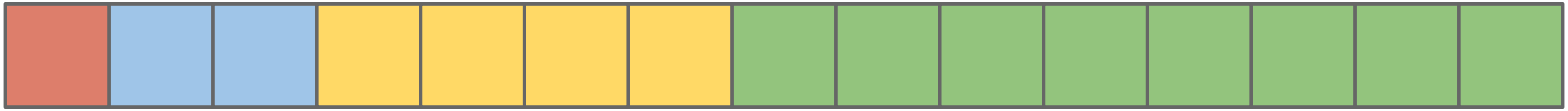
2, 3, 4

A few moments later…

Heap Sort

1. Insert items into heap
2. Reconstruct sequence with dequeue

7, 4, 8, 2, 5, 3, 9



2, 3, 4, 5, 7, 8, 9

Heap Sort

Heap Sort

Enqueue element *i*: $O(\log(i))$

Heap Sort

Enqueue element *i*: $O(\log(i))$

Dequeue element *i*: $O(\log(n - i))$

Heap Sort

Enqueue element *i*: $O(\log(i))$

Dequeue element *i*: $O(\log(n - i))$

$$\left(\sum_{i=1}^{n} O(\log(i))\right) + \left(\sum_{i=1}^{n} O(\log(n - i))\right)$$

Heap Sort

Enqueue element *i*: $O(\log(i))$

Dequeue element *i*: $O(\log(n-i))$

$$\left(\sum_{i=1}^{n} O(\log(i))\right) + \left(\sum_{i=1}^{n} O(\log(n-i))\right) \quad < O(n\log(n))$$

Updating Heap Elements

What if we want to update a value in our Heap?

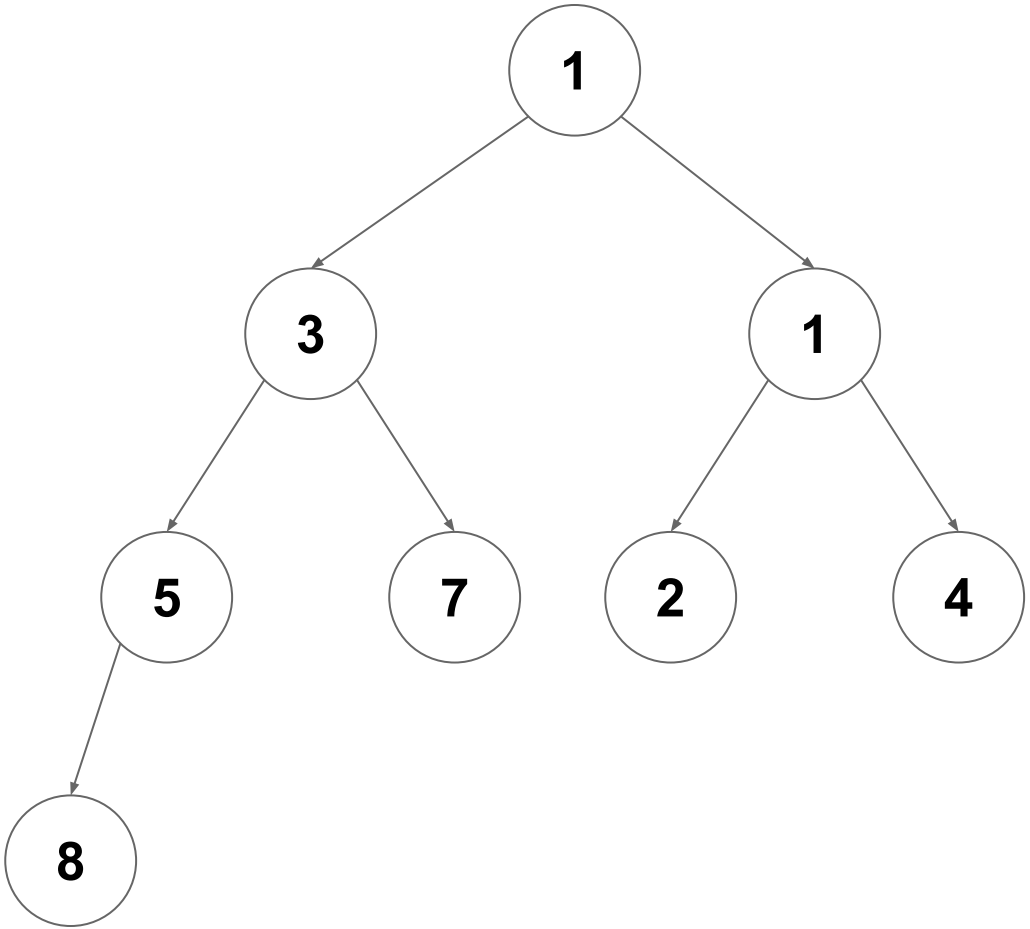
Updating Heap Elements

What if we want to update a value in our Heap?

After update we can just call **`fixUp`** or **`fixDown`** based on the new value

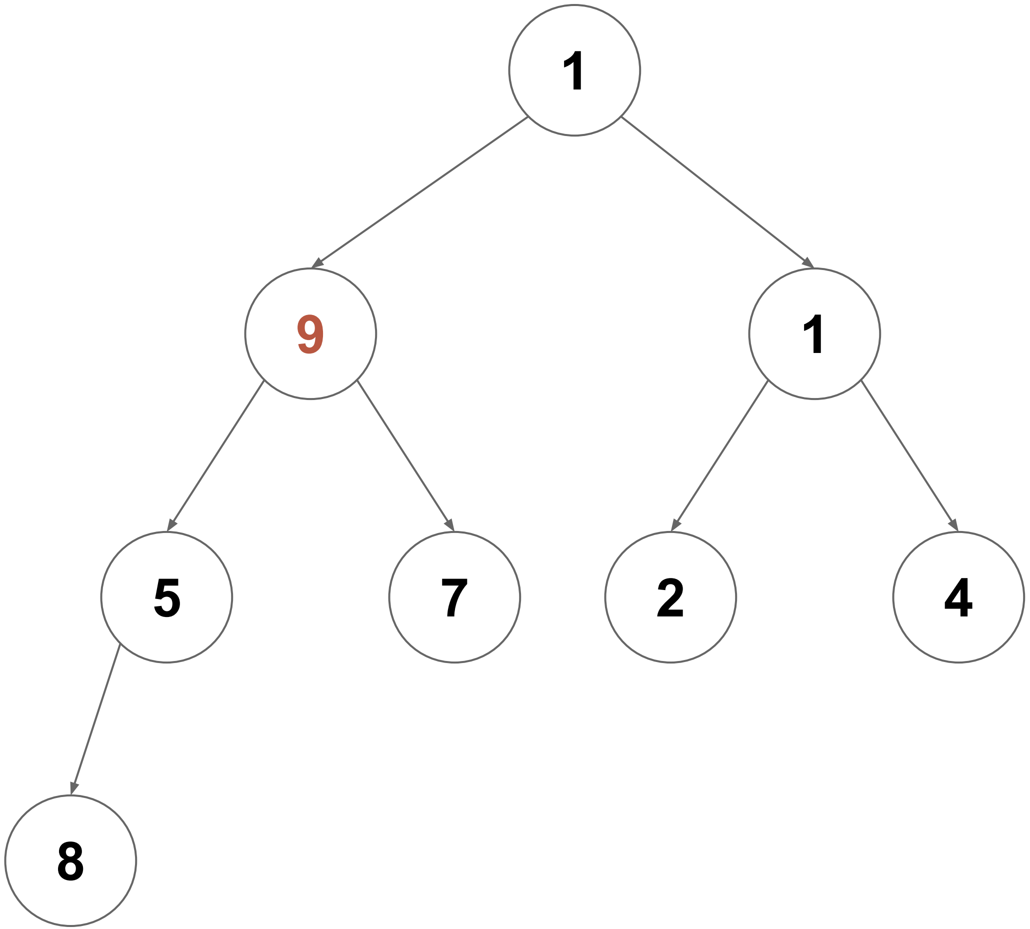
`update`

What if we change the value of the 3 node to 9?



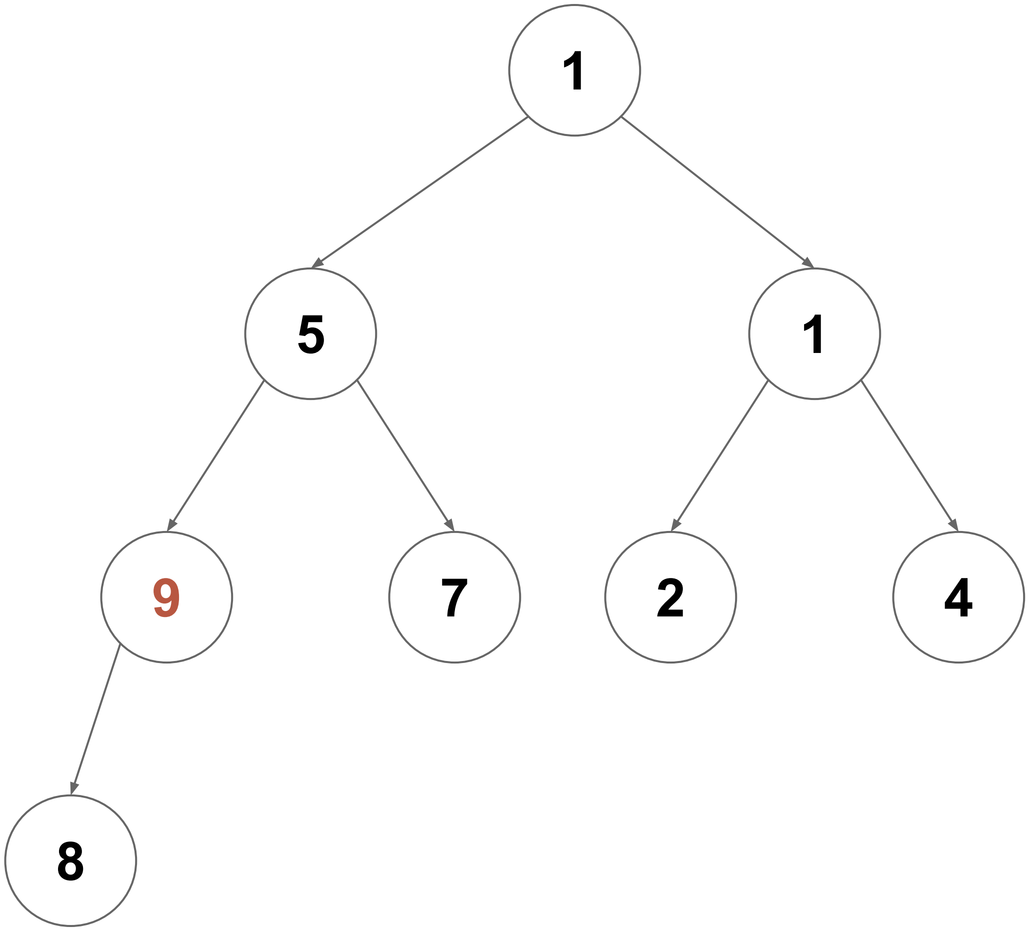
`update`

We now have to `fixUp` or `fixDown` based on the new value



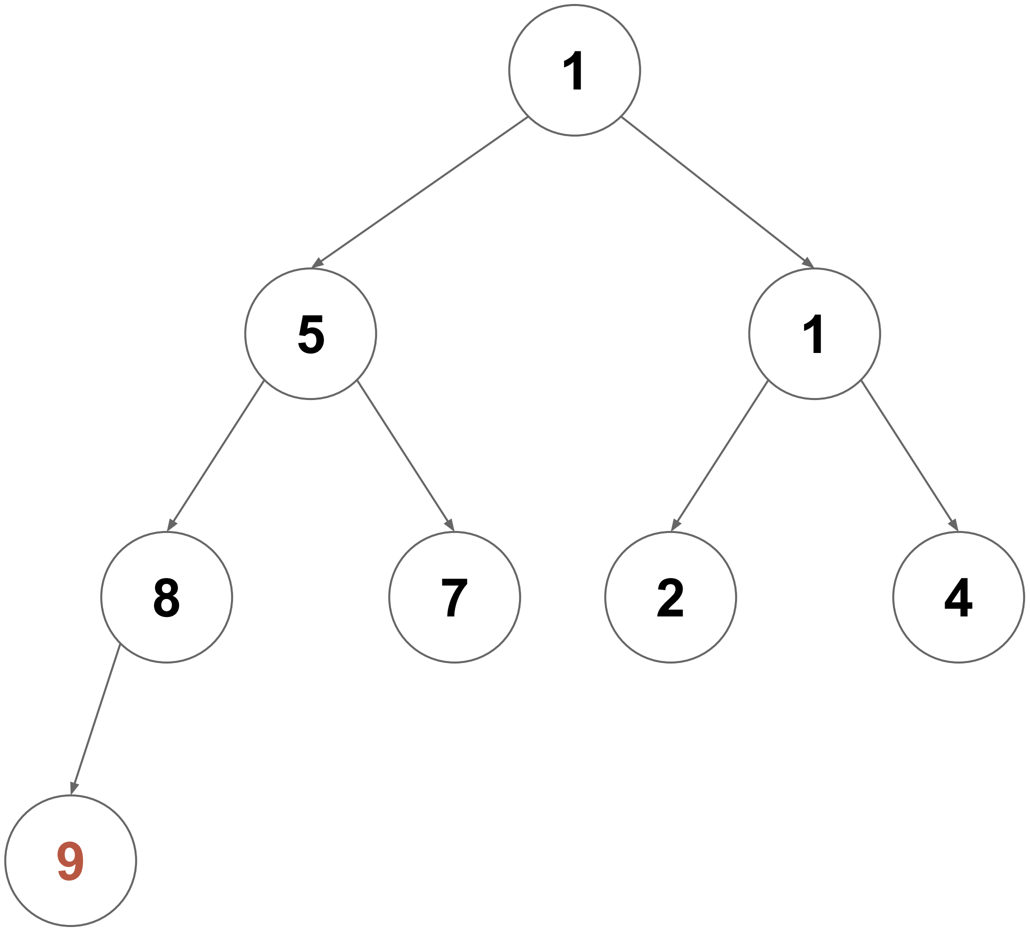
`update`

We now have to **`fixUp`** or **`fixDown`** based on the new value



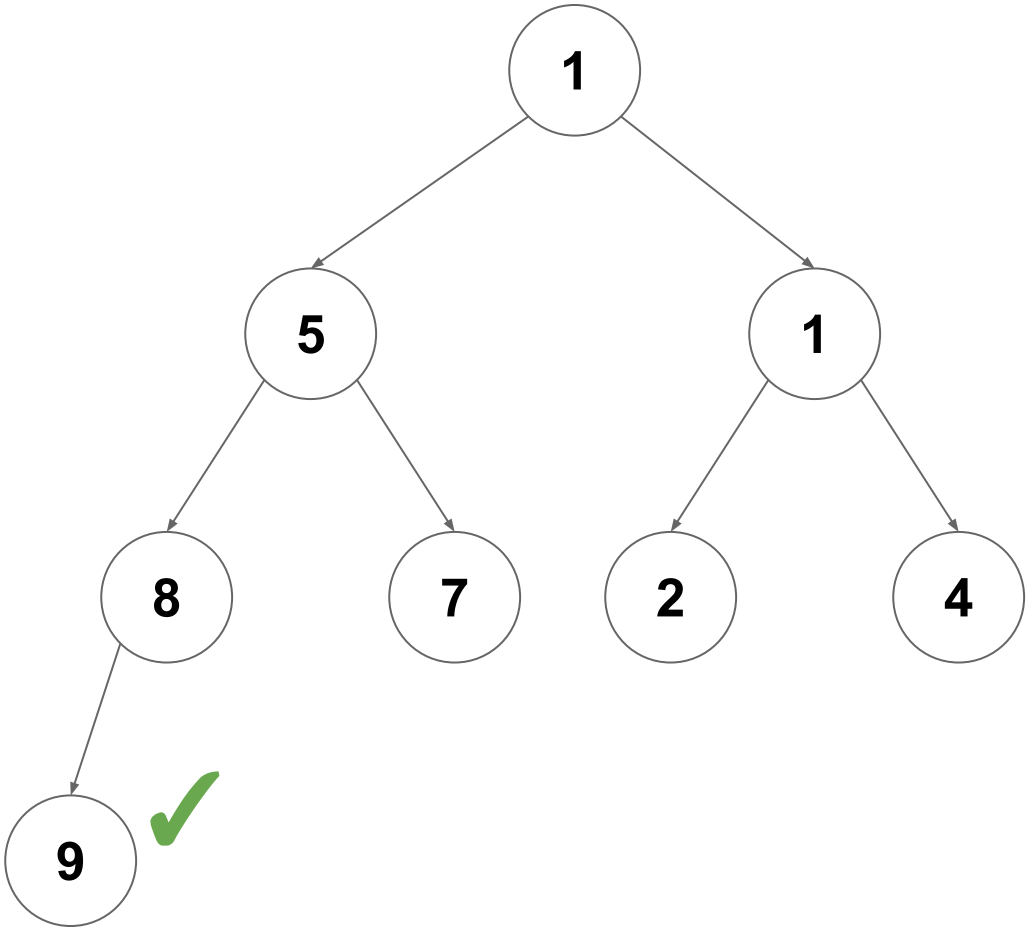
`update`

We now have to **`fixUp`** or **`fixDown`** based on the new value



`update`

We now have to **`fixUp`** or **`fixDown`** based on the new value



Updating Heap Elements

What if we want to update a value in our Heap?

After update we can just call **`fixUp`** or **`fixDown`** based on the new value

Updating Heap Elements

What if we want to update a value in our Heap?

After update we can just call **`fixUp`** or **`fixDown`** based on the new value

Can we apply this idea to an entire array?

Heapify

Input: Array

Output: Array re-ordered to be a heap

Heapify

Input: Array

Output: Array re-ordered to be a heap

Idea: **`fixUp`** or **`fixDown`** all ***n*** elements in the array

Heapify

Input: Array

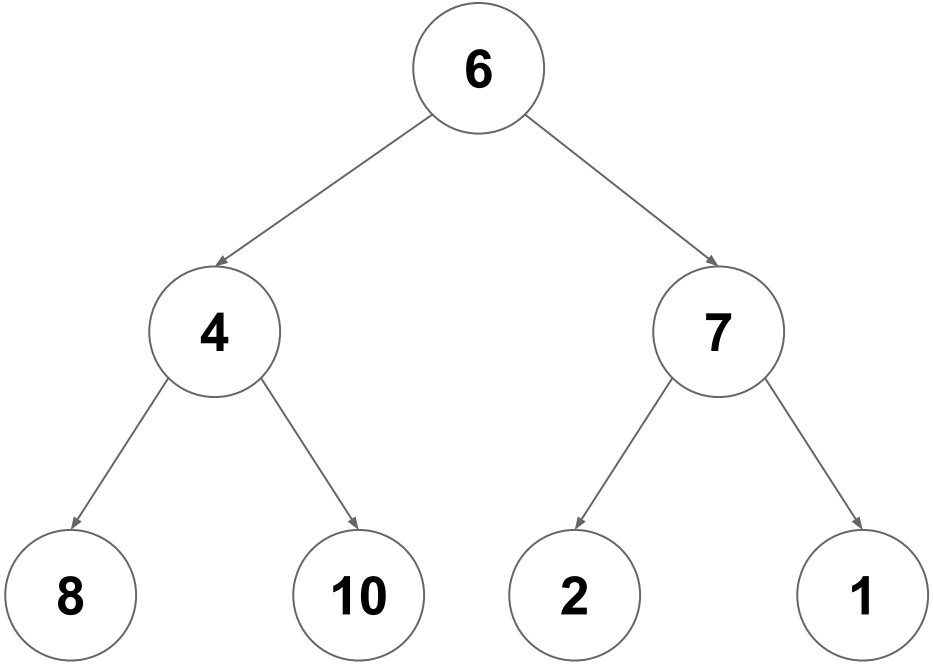
Output: Array re-ordered to be a heap

Idea: **`fixUp`** or **`fixDown`** all ***n*** elements in the array

Given the cost of **`fixUp`** *and* **`fixDown`** *what do we expect the complexity* **`Heapify`** *will be?*

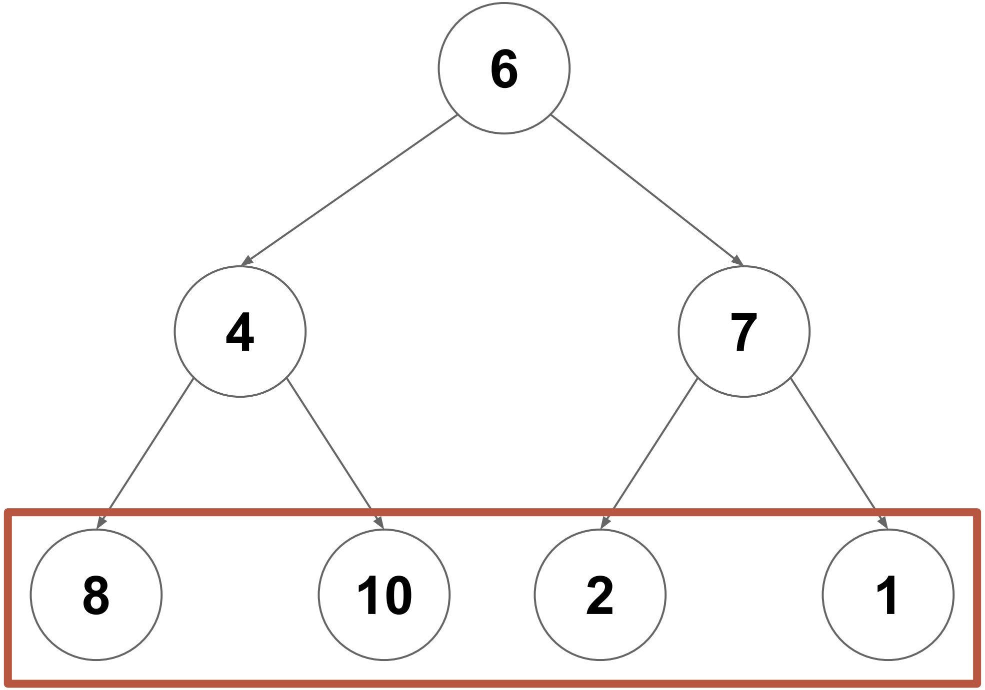
Heapify

Given an arbitrary array
(shown as a tree here)
turn it into a heap



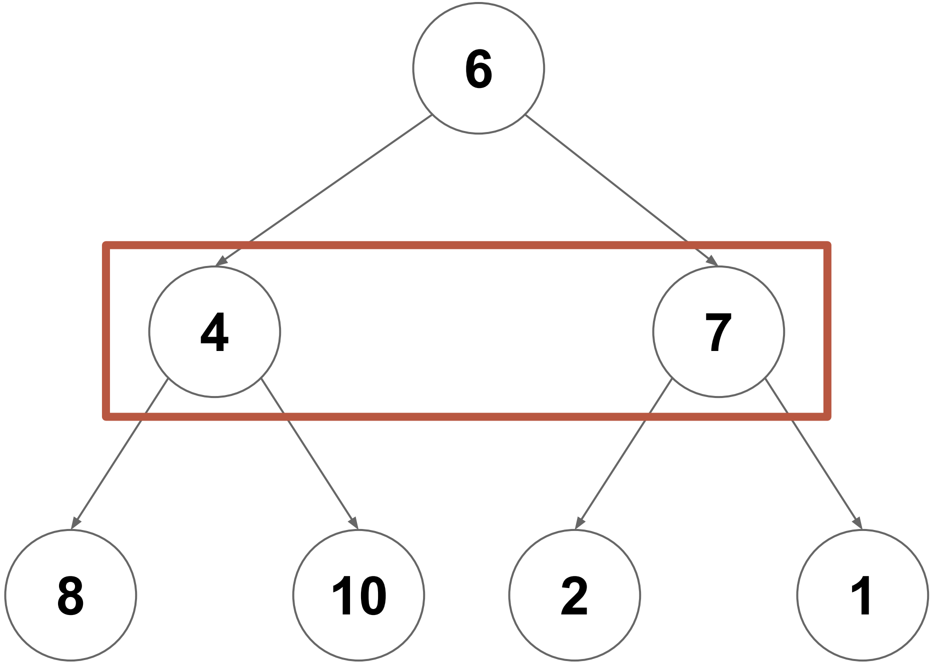
Heapify

Start at the lowest level, and call **`fixDown`** on each node (0 swaps per node)



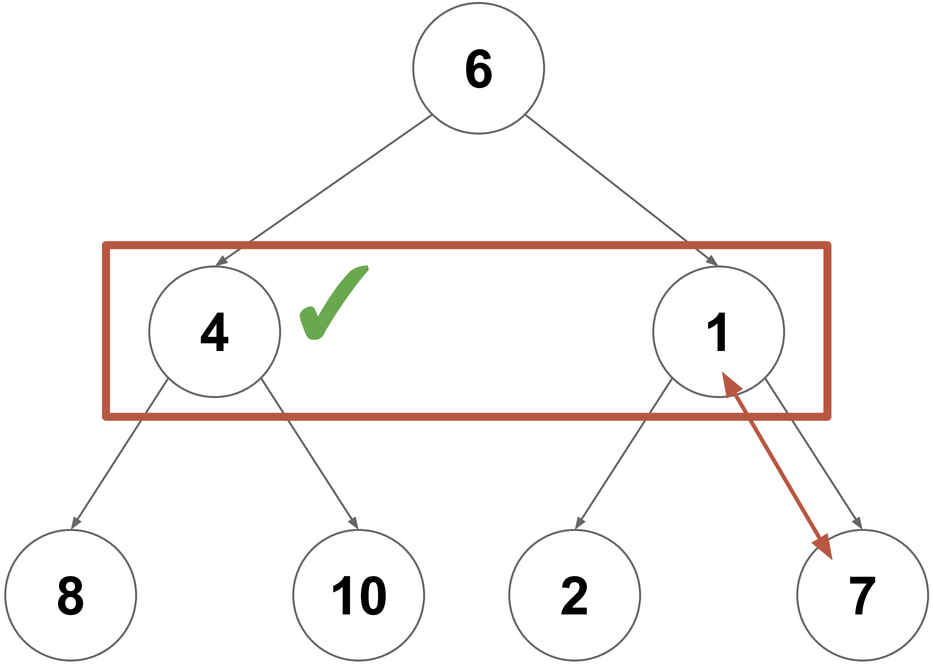
Heapify

Do the same at the next lowest level (at most one swap per node)



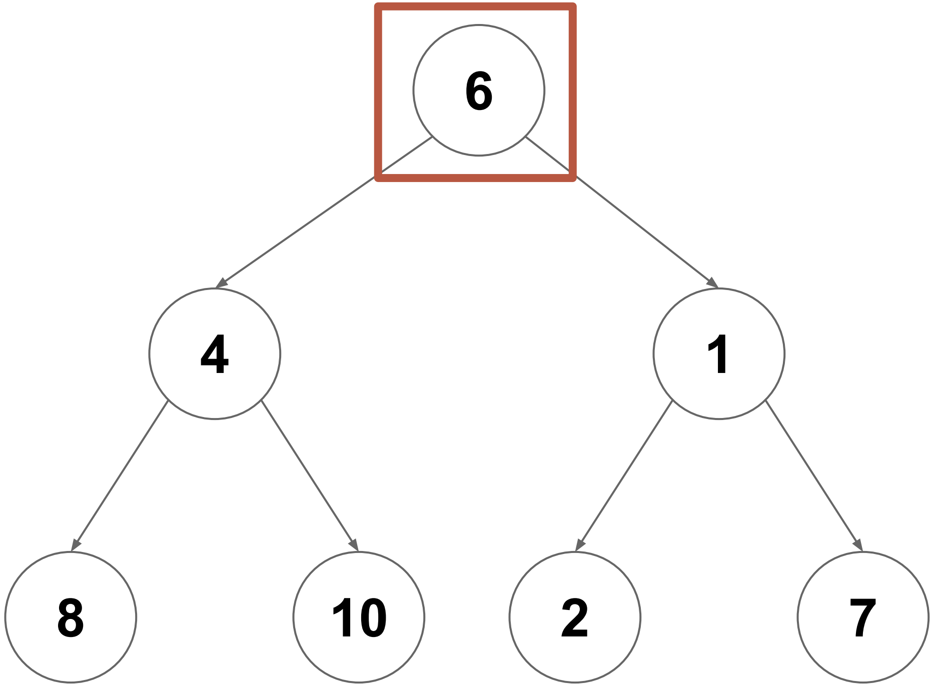
Heapify

Do the same at the next lowest level (at most one swap per node)



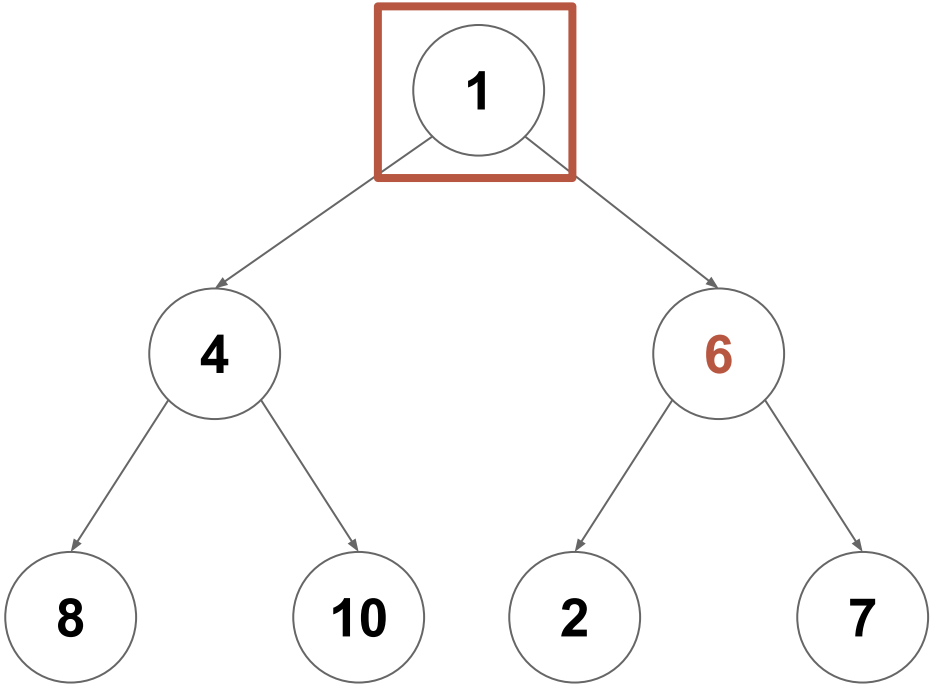
Heapify

Continue upwards (now at most 2 swaps per node)



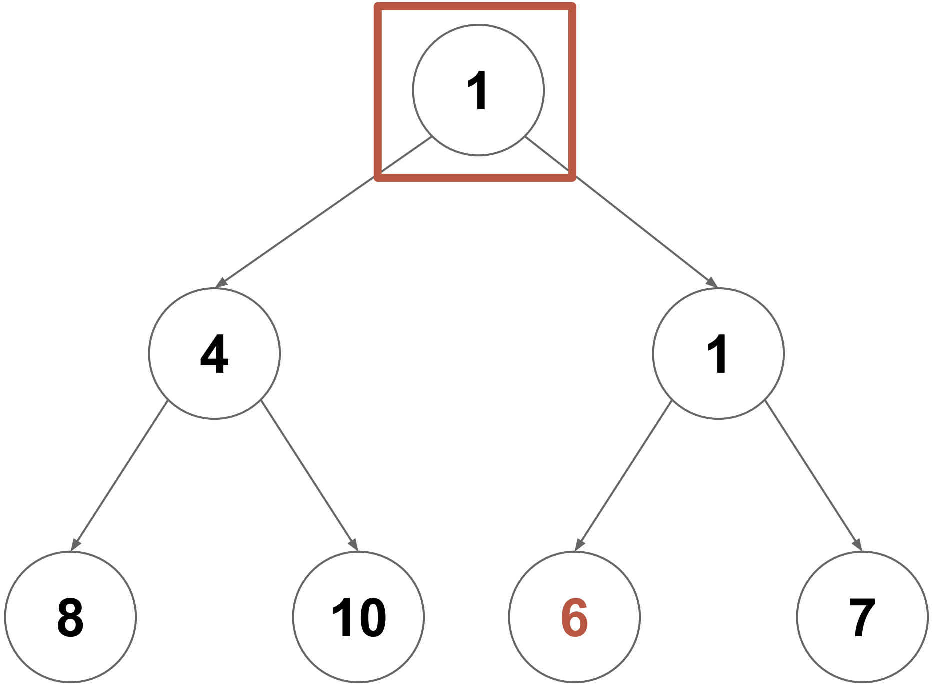
Heapify

Continue upwards (now at most 2 swaps per node)



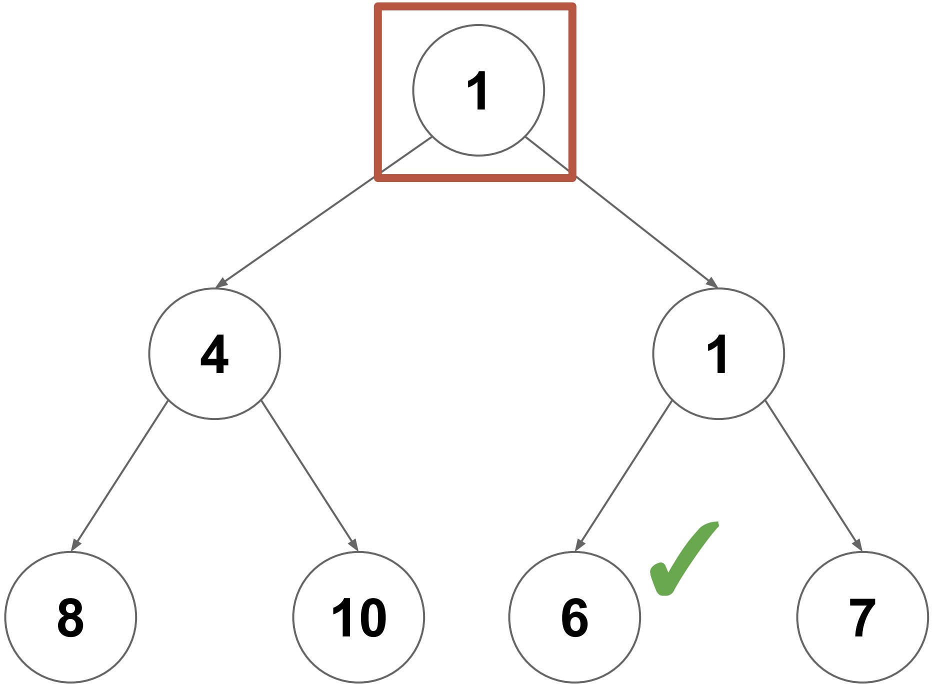
Heapify

Continue upwards (now at most 2 swaps per node)



Heapify

Continue upwards (now at most 2 swaps per node)



Heapify

Heapify

At level log(*n*): Call **`fixDown`** on all $n/2$ nodes at this level (max 0 swaps each)

Heapify

At level log(*n*): Call **`fixDown`** on all *n*/2 nodes at this level (max 0 swaps each)

At level log(*n*)-1: Call **`fixDown`** on all *n*/4 nodes at this level (max 1 swaps each)

Heapify

At level log(*n*): Call **`fixDown`** on all *n*/2 nodes at this level (max 0 swaps each)

At level log(*n*)-1: Call **`fixDown`** on all *n*/4 nodes at this level (max 1 swaps each)

At level log(*n*)-2: Call **`fixDown`** on all *n*/8 nodes at this level (max 2 swaps each)

Heapify

At level log(*n*): Call **`fixDown`** on all *n*/2 nodes at this level (max 0 swaps each)

At level log(*n*)-1: Call **`fixDown`** on all *n*/4 nodes at this level (max 1 swaps each)

At level log(*n*)-2: Call **`fixDown`** on all *n*/8 nodes at this level (max 2 swaps each)

…

At level 1: Call **`fixDown`** on all 1 nodes at this level (max log(*n*) swaps each)

Heapify

Sum the number of swaps required by each level

$$O\left(\sum_{i=1}^{\log(n)} \frac{n}{2^i} \cdot (i+1)\right)$$

Heapify

Pull out the *n* as a constant and distribute multiplication

$$O\left(\sum_{i=1}^{\log(n)} \frac{n}{2^i} \cdot (i+1)\right)$$

$$O\left(n \sum_{i=1}^{\log(n)} \frac{i}{2^i} + \frac{1}{2^i}\right)$$

Heapify

Focus on the dominant term only

$$O\left(\sum_{i=1}^{\log(n)} \frac{n}{2^i} \cdot (i+1)\right)$$

$$O\left(n \sum_{i=1}^{\log(n)} \frac{i}{2^i} + \frac{1}{2^i}\right)$$

$$O\left(n \sum_{i=1}^{\log(n)} \frac{i}{2^i}\right)$$

Heapify

Change log(n) to infinity (can only increase complexity class if anything)

$$O\left(\sum_{i=1}^{\log(n)} \frac{n}{2^i} \cdot (i+1)\right)$$

$$O\left(n\sum_{i=1}^{\log(n)} \frac{i}{2^i} + \frac{1}{2^i}\right)$$

$$O\left(n\sum_{i=1}^{\log(n)} \frac{i}{2^i}\right)$$

$$O\left(n\sum_{i=1}^{\infty} \frac{i}{2^i}\right)$$

Heapify

We can now treat the sum as a constant

$$O\left(\sum_{i=1}^{\log(n)} \frac{n}{2^i} \cdot (i+1)\right)$$

$$O\left(n \sum_{i=1}^{\log(n)} \frac{i}{2^i} + \frac{1}{2^i}\right)$$

$$O\left(n \sum_{i=1}^{\log(n)} \frac{i}{2^i}\right)$$

$$O\left(n \boxed{\sum_{i=1}^{\infty} \frac{i}{2^i}}\right)$$

This is known to converge to a constant

Heapify

Therefore we can heapify an array of size *n* in *O*(*n*)

$$O\left(\sum_{i=1}^{\log(n)} \frac{n}{2^i} \cdot (i+1)\right)$$

$$O\left(n \sum_{i=1}^{\log(n)} \frac{i}{2^i} + \frac{1}{2^i}\right)$$

$$O\left(n \sum_{i=1}^{\log(n)} \frac{i}{2^i}\right)$$

$$O\left(n \sum_{i=1}^{\infty} \frac{i}{2^i}\right) = O(n)$$

Heapify

Therefore we can heapify an array of size *n* in *O*(*n*)

(but heap sort still requires *n* log(*n*) due to dequeue costs)

$$O\left(\sum_{i=1}^{\log(n)} \frac{n}{2^i} \cdot (i+1)\right)$$

$$O\left(n \sum_{i=1}^{\log(n)} \frac{i}{2^i} + \frac{1}{2^i}\right)$$

$$O\left(n \sum_{i=1}^{\log(n)} \frac{i}{2^i}\right)$$

$$O\left(n \sum_{i=1}^{\infty} \frac{i}{2^i}\right) = O(n)$$