CSE 250
Data Structures

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Lec 31: Hash Tables

Announcements

- WA4 due tonight
- PA3 posted

Picking a Hash Function

What function could we use that would evenly distribute values to buckets?

Picking a Hash Function

What function could we use that would evenly distribute values to buckets?

Wacky Idea: Have $h(x)$ return a random value in $[0,N)$

(This makes apply impossible…but bear with me)

Random Hash Function

$$n = \text{number of elements in any bucket}$$

$$N = \text{number of buckets}$$

$$b_{i,j} = \begin{cases} 1 & \textbf{if} \text{ element } i \text{ is assigned to bucket } j \\ 0 & otherwise \end{cases}$$

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Only true if $b_{i,j}$ and $b_{i',j}$ are uncorrelated for any i ≠ i'

(h(i) can't be related to h(i'))

$$\mathbb{E}\left[\sum_{i=0}^{n} b_{i,j}\right] = \frac{n}{N}$$

The **expected** number of elements in any bucket j

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The **expected**
number of elements
in any bucket j

...given this information, what do the runtimes of our operations look like?

Random Hash Function

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$$N = \text{number of buckets}$$

$$b_{i,j} = \begin{cases} 1 & \textbf{if } \text{element } i \text{ is assigned to bucket } j \\ 0 & otherwise \end{cases}$$

Expected runtime of `insert`, `apply`, `remove`: ***O*(*n*/*N*)**

Worst-Case runtime of `insert`, `apply`, `remove`: ***O*(*n*)**

Hash Functions In the Real-World

Examples

- SHA256 ← Used by GIT
- MD5, BCRYPT ← Used by unix login, apt
- MurmurHash3 ← Used by Scala

hash(*x*) is pseudo-random

- **hash(*x*)** ~ uniform random value in [0, INT_MAX)
- **hash(*x*)** always returns the same value for the same ***x***
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We then use modulus to fit this random value into the size of our hash table

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Refresher on Modulus

The modulus function takes any integers ***n*** and ***d***, and returns a number ***r*** in the range [0, ***d***), such that ***n* = *q* * *d* + *r***.

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7	8	9	10	11	12	13

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14	15	16	17	18	19	20

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If my hash table has 7 buckets, and I insert an element with hash code 73, what bucket would it go in?

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If my hash table has 7 buckets, and I insert an element with hash code 73, what bucket would it go in? **73 % 7 = 3**

Hash Function Recap

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  - They act as if they are uniformly random
    - Will evenly distribute elements to buckets
    - hash(***x***) is uncorrelated with hash(***y***)
  - They are deterministic (hash(***x***) will always return the same value)
- We can use these hash functions to determine which bucket an arbitrary element belongs in in ***O*(1)** time
- There are expected to be ***n/N*** elements in that bucket
  - So runtime for all operations is **expected *O*(1) + *O*(*n*/*N*) = expected *O*(*n*)**

Hash Functions + Buckets

Everything is: $O\left(\frac{n}{N}\right)$

Let's call $\alpha = \frac{n}{N}$ the load factor.

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What do we do when this constraint is violated?

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Idea: Make α a constant

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What do we do when this constraint is violated? **Resize!**

Rehashing

When we insert an element that would exceed the load factor we:

1. Resize the underlying array from N_{old} to N_{new}
2. Rehash all of the elements from their old bucket to their new bucket
    a. Element x moves from hash(x) % N_{old} to hash(x) % N_{new}

Rehashing

Let's say we have a hash table of size 6, and hash(***x***) = 65

What bucket does it belong in?

0	**1**	**2**	**3**	**4**	**5**

Rehashing

Let's say we have a hash table of size 6, and hash(***x***) = 65

What bucket does it belong in? 65 % 6 = 5

0	1	2	3	4	5 X

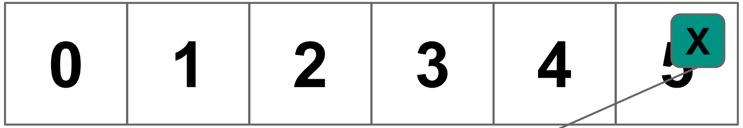
Now we want to resize the array to size 8. Where do we move ***x***?

0	1	2	3	4	5	6	7

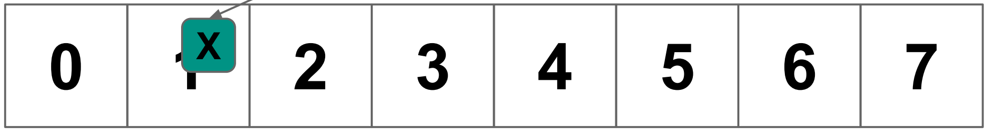
Rehashing

Let's say we have a hash table of size 6, and hash(***x***) = 65

What bucket does it belong in? 65 % 6 = 5



Now we want to resize the array to size 8. Where do we move ***x***? 65 % 8 = 1

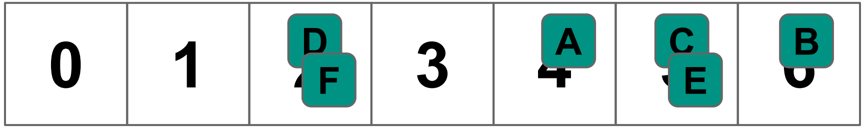


Rehashing

How long will it take to rehash every element after we resize?

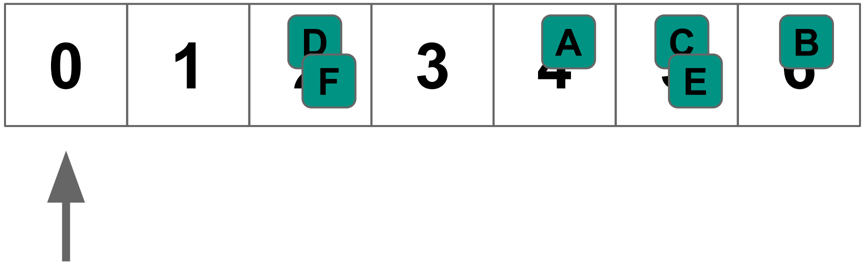
Related Question: *How do we iterate through a hash table?*

Iterating over a Hash Table



Iterating over a Hash Table

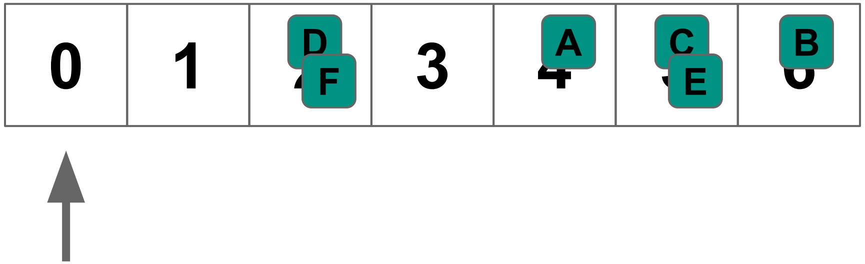
Start at the first bucket



Iterating over a Hash Table

Start at the first bucket

Iterate through that bucket



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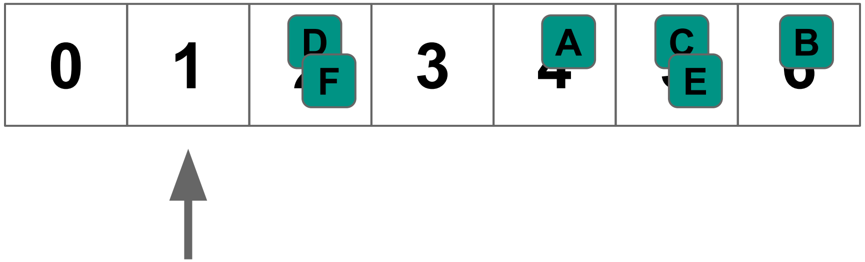

Iterating over a Hash Table

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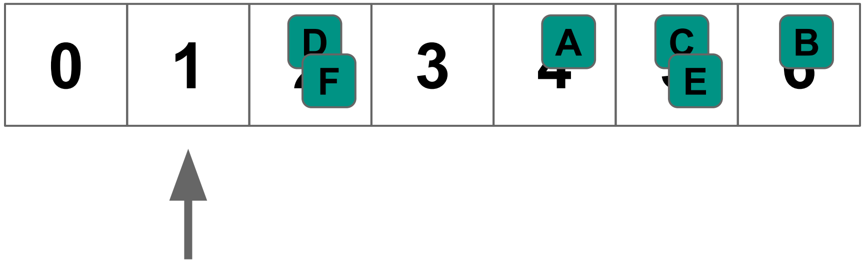
Iterate through that bucket

Move to the next bucket

…and repeat



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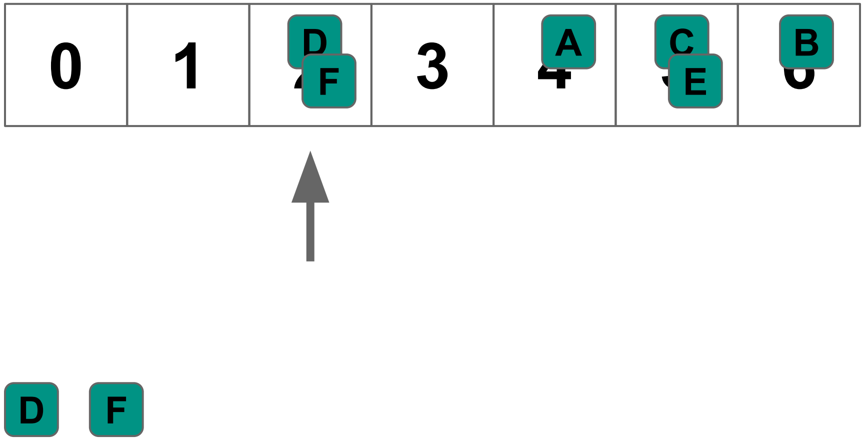

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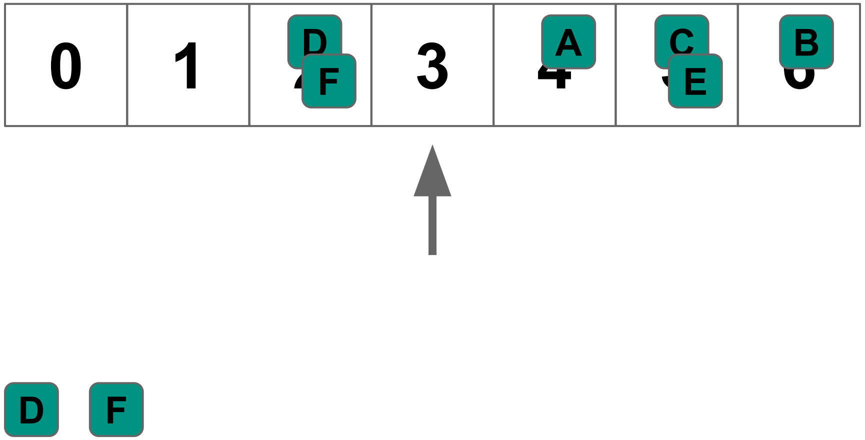

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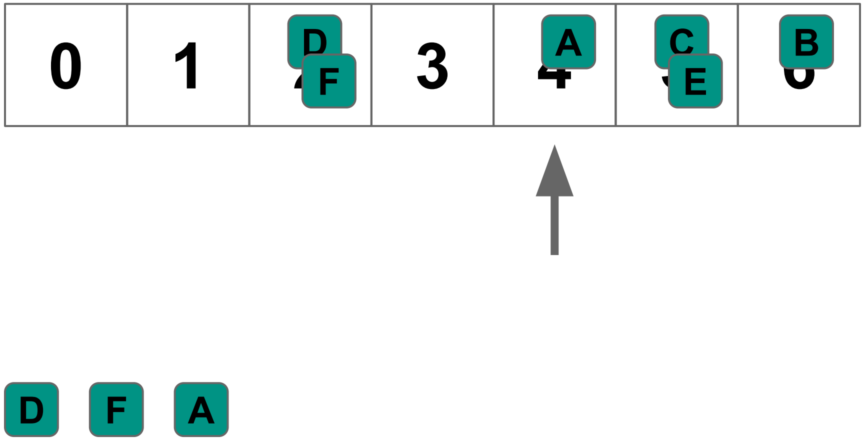

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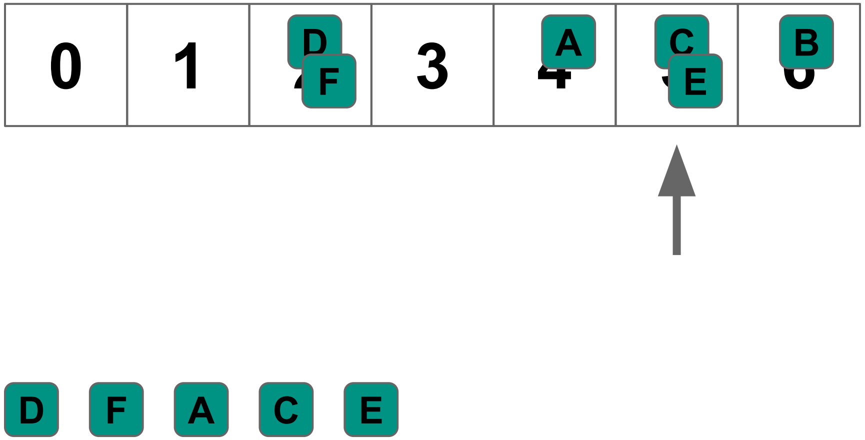

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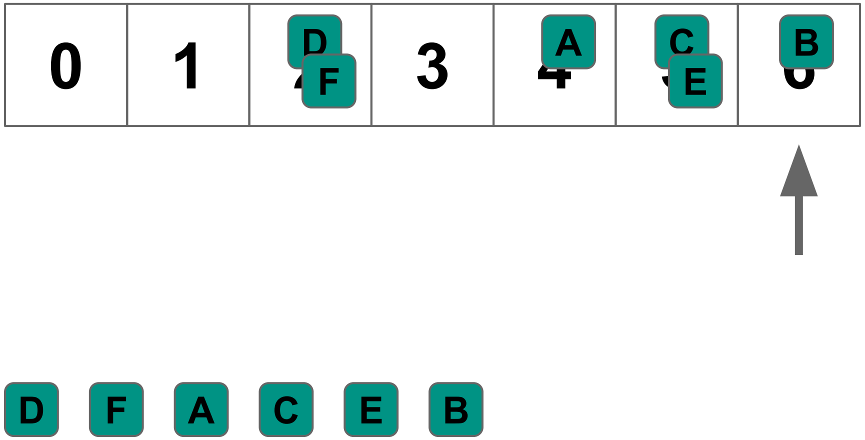

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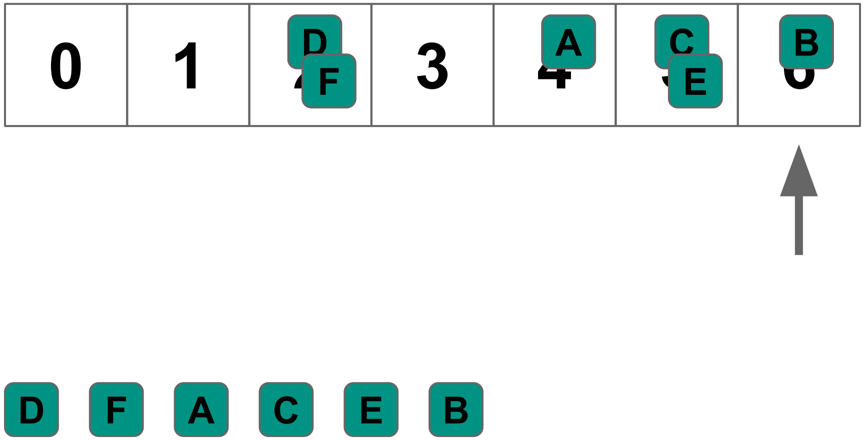
How long does it take?



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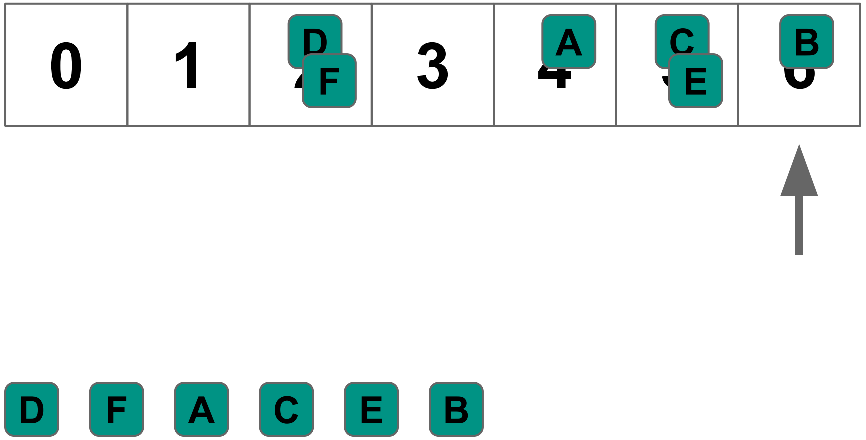

How long does it take? ***O*(*N* + *n*)**

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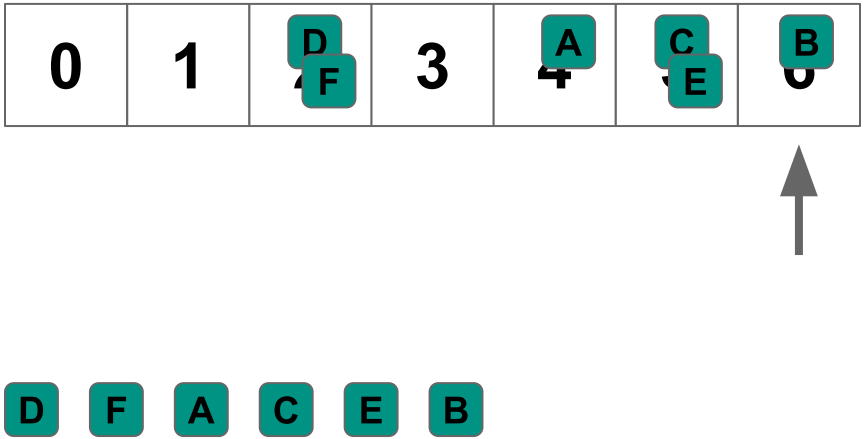

How long does it take? $O(N + n)$

Visit every bucket

Iterating over a Hash Table

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Move to the next bucket

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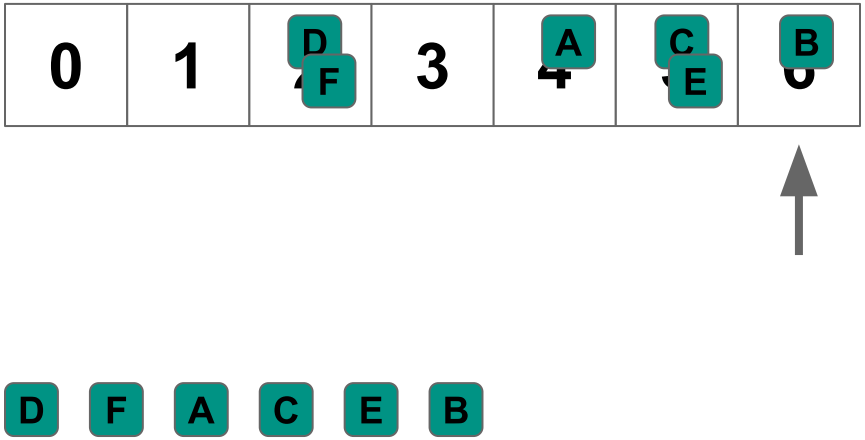

How long does it take? $\boldsymbol{O(N + n)}$

Visit every bucket

Visit every element in each bucket

Rehashing

So how long does it take to rehash an entire hash table with ***n*** elements and ***N*** buckets?

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Iterating through each element costs ***O*(*N* + *n*)**

Rehashing

So how long does it take to rehash an entire hash table with n elements and N buckets?

Rehashing an individual element costs $O(1)$

Iterating through each element costs $O(N + n)$

Rehashing costs: $O(N + n)$

Rehashing

When we insert an element that would exceed the load factor we:

1. Resize the underlying array from $\boldsymbol{N_{old}}$ to $\boldsymbol{N_{new}}$
2. Rehash all of the elements from their old bucket to their new bucket
   a. Element ***x*** moves from hash(***x***) % $\boldsymbol{N_{old}}$ to hash(***x***) % $\boldsymbol{N_{new}}$

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How long does this take?

1. Allocate the new array: $\boldsymbol{O(1)}$
2. Rehash every element from the old array to the new: $\boldsymbol{O(N_{old} + n)}$
3. Free the old array: $\boldsymbol{O(1)}$

Total: $\boldsymbol{O(N_{old} + n)}$

Rehashing

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How long does this take?

How do we pick N_{new}?

1. Allocate the new array: $\boldsymbol{O(1)}$
2. Rehash every element from the old array to the new: $\boldsymbol{O(N_{old} + n)}$
3. Free the old array: $\boldsymbol{O(1)}$

Total: $O(N_{old} + n)$

Rehashing

Whenever $\alpha > \alpha_{max}$, double the size of the array (remember ArrayLists)

If we start with ***N*** buckets and insert ***n*** elements:

1. First rehash happens at $n_1 = \alpha_{max} \times N$: goes from ***N*** to **2*N***

Rehashing

Whenever $\alpha > \alpha_{max}$, double the size of the array (remember ArrayLists)

If we start with N buckets and insert n elements:

1. First rehash happens at $n_1 = \alpha_{max} \times N$: goes from N to $2N$
2. Second rehash happens at $n_2 = \alpha_{max} \times 2N$: goes from $2N$ to $4N$

Rehashing

Whenever $\alpha > \alpha_{max}$, double the size of the array (remember ArrayLists)

If we start with ***N*** buckets and insert ***n*** elements:

1. First rehash happens at $\boldsymbol{n_1} = \alpha_{max} \times \boldsymbol{N}$**:** goes from ***N*** to **2*N***
2. Second rehash happens at $\boldsymbol{n_2} = \alpha_{max} \times \boldsymbol{2N}$**:** goes from **2*N*** to **4*N***
3. Third rehash happens at $\boldsymbol{n_3} = \alpha_{max} \times \boldsymbol{4N}$**:** goes from **4*N*** to **8*N***

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   ...

j. jth rehash happens at $\boldsymbol{n_j} = \alpha_{max} \times \boldsymbol{2^{j-1}N}$: goes from $\boldsymbol{2^{j-1}N}$ to $\boldsymbol{2^{j}N}$

Total Work

With n insertions, choose j s.t. $n = 2^j \alpha_{max}$

$$2^j = n / \alpha_{max}$$

$$j = \log(n / \alpha_{max})$$

$$j = \log(n) - \log(\alpha_{max})$$

$$j \leq \log(n) \quad \leftarrow \text{Number of rehashes}$$

Total Work

Rehashes required: **≤ log(n)**

The ith rehash: ***O*(2^{i}N)**

$$\sum_{i=0}^{\log(n)} O(2^i N) = O\left(N \sum_{i=0}^{\log(n)} 2^i\right) = O(2^{\log(n)+1} - 1) = O(n)$$

So ***O*(*n*)** work is required to do ***n*** insertions → Insert cost is **amortized *O*(1)**