CSE 250: Asymptotic Analysis

Lecture 4

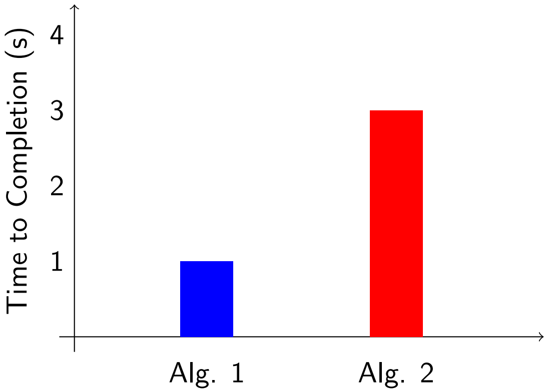
September 4, 2024

Reminders

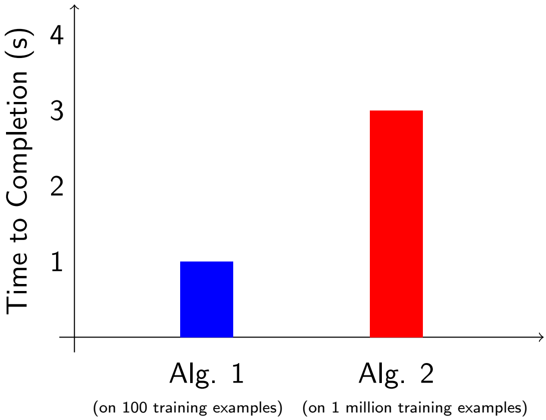
- AI Quiz due Sun, Sept 8 at 11:59 PM.
  - Your final submission must have a score of 1.0 to pass the class.
  - If you can't submit in autolab, let course staff know ASAP.
- PA0 due Sun, Sept 8 at 11:59 PM.
  - All you need to do is make sure you have a working environment.
  - If you can't submit in autolab, let course staff know ASAP.
- WA1 due Sun, Sept 8 at 11:59 PM.
  - Summations, Limits, Exponentials; Friday's Lecture

How "fast" is an algorithm?

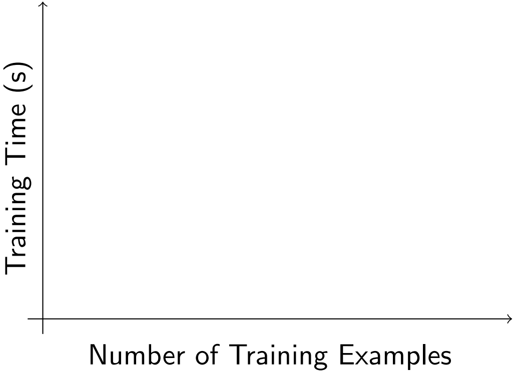
Runtime



Runtime


Time to Completion (s)
4
3
2
1
Alg. 1
(on 100 training examples)
Alg. 2
(on 1 million training examples)

Runtime



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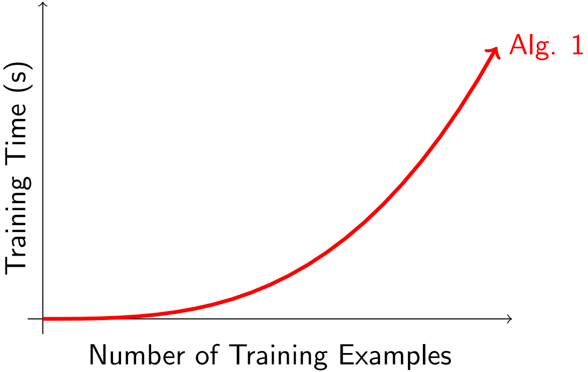

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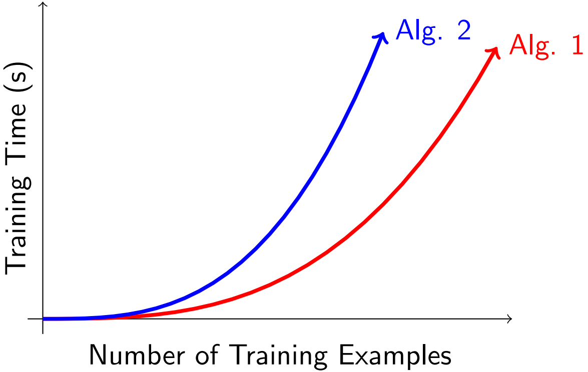

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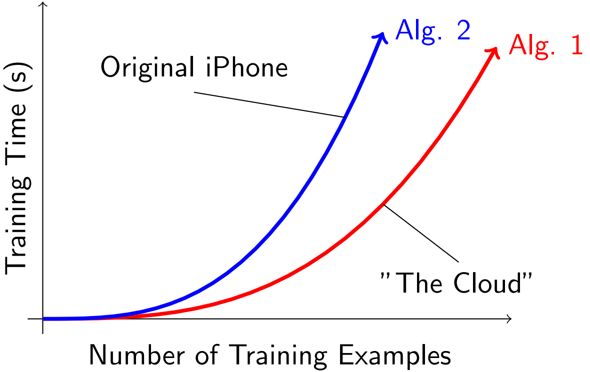
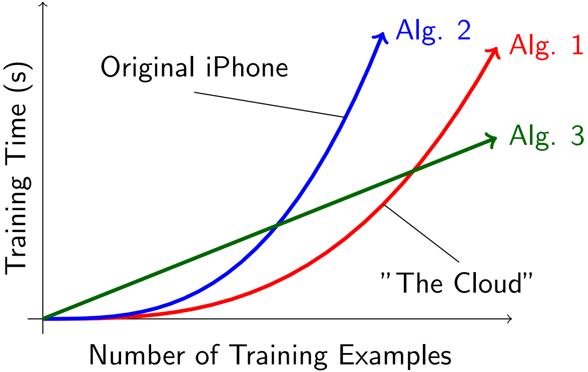

Alg. 2
Alg. 1
Original iPhone
Training Time (s)
"The Cloud"
Number of Training Examples

Runtime


Alg. 2
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Training Time (s)
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Implementation Variation

- How much data does it process?
- What hardware is it running on?
- How cleverly has the implementation been optimized?

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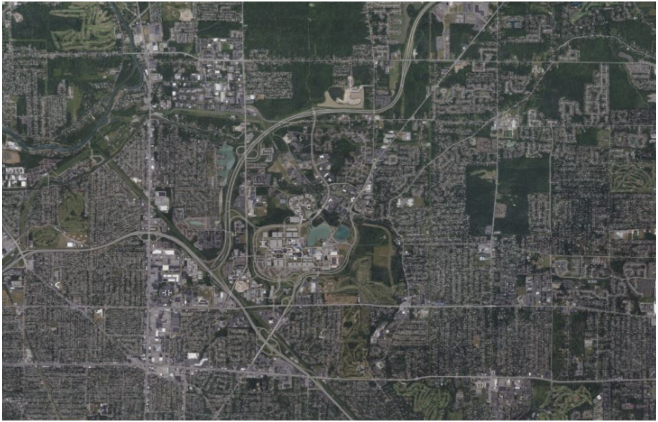
- What hardware is it running on?
- How cleverly has the implementation been optimized?

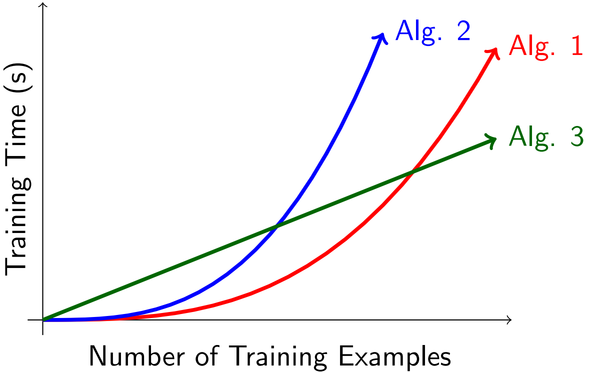
These are all (brittle) low-level details.

The Big Picture



©Earthstar Geographics SIO; via Bing Maps

Runtime



Scaling

Idea

Identify algorithms by their ???

Scaling

Idea

Identify algorithms by their "shape"

Scaling

Idea

Identify algorithms by their **"Complexity Class"**

Scaling

Idea

Identify algorithms by their **"Complexity Class"**

Quadratic is generally worse than *linear*.

- Algorithm 1 is quadratic
- Algorithm 3 is linear

Scaling

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Identify algorithms by their **"Complexity Class"**

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Some Notation

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- N: The input "size"
  - How many students I have to email.
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- $T(N)$: The runtime of 'some' implementation of the algorithm.
  - Some... correct implementation.

We care about the "shape" of $T(N)$ when you plot it.

Thinking in Steps

Instead of runtime, let's count the **'steps'**

Count the Steps

```
public void updateUsers(User[] users)
{
  x = 1;
  for(user : users)
  {
    user.id = x;
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1

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$$1 + \sum_{\texttt{user} \in \texttt{users}} 2 \text{ steps} = 1 + 2 \times |\texttt{users}|$$

... where $|\texttt{users}|$ means the size of the users array.

Count the Steps

```
public void userFullName(User[] users, int id)
{
  User user = users[id];
  String fullName = user.firstName + user.lastName;
  return fullName;
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Count the Steps

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3 steps

Count the Steps

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```

3 steps[1]

[1] This is actually a lie, but more on that in later lectures

Count the Steps

```
public void totalReads(User[] users, Post[] posts)
{
  int totalReads = 0;
  for(post : posts)
  {
    int userReads = 0;
    for(user : users)
    {
      if(user.readPost(post)){ userReads += 1; }
    }
    totalReads += userReads;
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$$1 + \sum_{\texttt{post} \in \texttt{posts}} \Bigg(3$$

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$$1 + \sum_{\texttt{post} \in \texttt{posts}} \left(3 + \sum_{\texttt{user} \in \texttt{users}} 2 \right)$$

Comparing Step Counts

Which is better?

1. An algorithm that takes $5 + (|\texttt{users}| \times 3)$ steps
2. An algorithm that takes $\frac{1}{2}(|\texttt{users}|^2)$ steps

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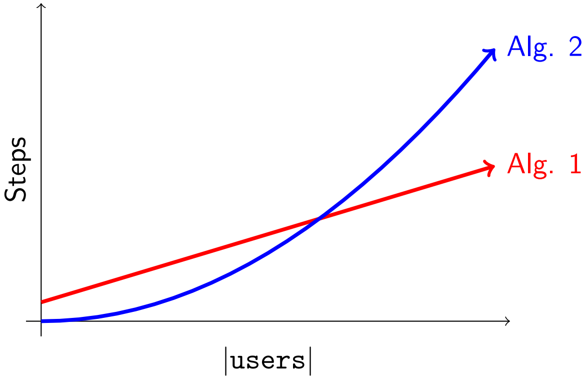

Comparing Step Counts

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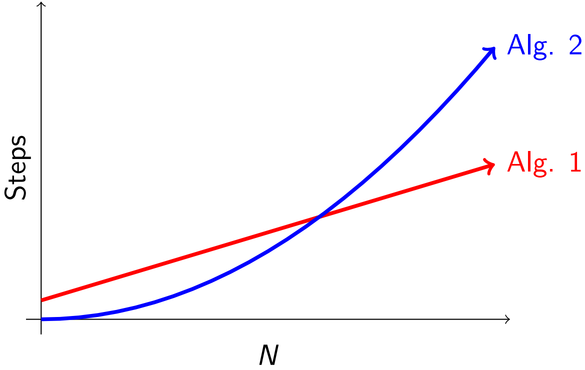
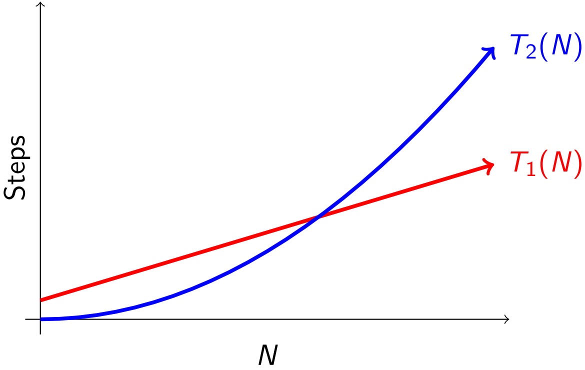


Comparing Step Counts

Which is better?

1. $T_1(N) = 5 + (N \times 3)$ steps
2. $T_2(N) = \frac{1}{2}(N^2)$ steps



Comparing Step Counts

$T_1(N) \ll T_2(N)$ (for "big enough" N).

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So... <u>to us</u> an algorithm that takes $T_1(N)$ steps is better/faster/stronger than $T_2(N)$.

Additive Factors

Which is better?

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2. $T_2(N) = 10 + (N \times 3)$

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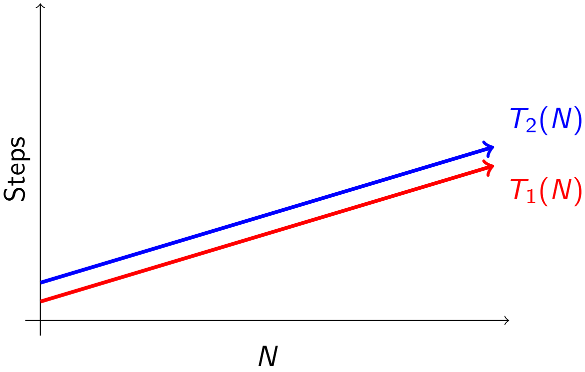

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In This Class

$T_1(N)$ and $T_2(N)$ are the same.

Multiplicative Factors

Which is better?

1. $T_1(N) = 3 + (N \times 3)$
2. $T_2(N) = 4 + (N \times 4)$

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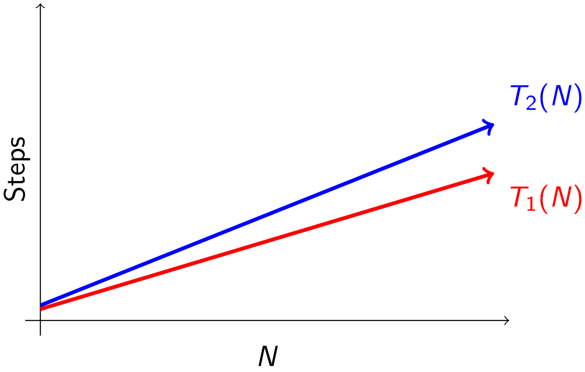

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$T_1(N)$ is within a constant multiplicative factor of $T_2(N)$ (i.e., $T_1(N) = c \times T_2(N)$)

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Complexity

If there's a c_1 and c_2 so that $T_1(N) = c_2 + (c_1 \times T_2(N))$ then we say that T_1 is in the same **complexity class** as $T_2(N)$.

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On an input of size N the algorithm finishes in <u>exactly</u> $T(N)$ steps.

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But...

How many Steps?

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x = 10;
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vs

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The exact number of steps doesn't matter.

Steps

A step is <u>any</u> computation that always has the same runtime.

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A step is any computation that always[3] has the same runtime.

[3] Offer void where prohibited, some approximations may apply.

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We call any function T with these properties a **growth function**.

Growth Functions

When I say a **function**, I mean a mathematical expression like $1 + 2N$ (not a bit of code).

Shorthands

$$\theta(f(N))$$

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(all the mathematical functions in $f(N)$'s complexity class)

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$\theta(2 + (3 \times N)) = \{$

- $5 + (10 \times N)$
- N
- $2 \times N$
- $\ldots$

$\}$

Shorthands

$$\theta(f(N))$$

(all the mathematical functions in $f(N)$'s complexity class)

$\theta(2 + (3 \times N)) = \{$

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- ...

$\}$

$g(N) \in \theta(f(N))$ means g and f are in the same complexity class

Shorthands

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- $g(N)$ is in $\theta(f(N))$:
  Common shorthand for $g(N) \in \theta(f(N))$
- Algorithm Foo is in $\theta(f(N))$:
  Common shorthand for $T(N) \in \theta(f(N))$ where $T(N)$ is the <u>runtime</u> of Foo.

Class Names

- $\theta(1)$: Constant
- $\theta(\log(N))$: Logarithmic
- $\theta(N)$: Linear
- $\theta(N \log(N))$: Log-Linear
- $\theta(N^2)$: Quadratic
- $\theta(N^k)$ (for any $k \geq 1$): Polynomial
- $\theta(2^N)$: Exponential

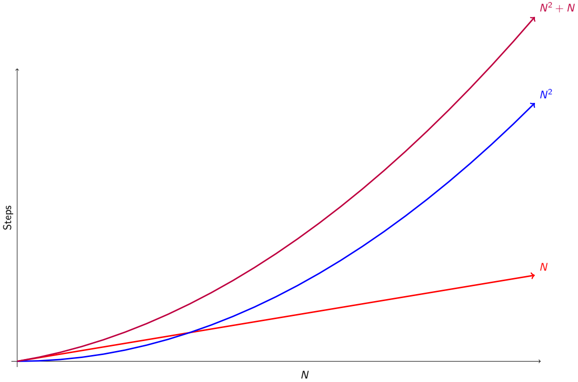
Moving forward:

- $f(N)$, $g(N)$, $f_1(N)$, $f_2(N)$, ...: Any mathematical function that's a growth function.
- $T(N)$: The growth function for a specific algorithm

Combining Classes

What class is $g(N) = N + N^2$ in?

Combining Classes



Combining Classes

For big N, $N + N^2$ looks a lot more like N^2 than N.
But it's not a constant factor different.

$$N + N^2 \neq c_1 + N^2 \times c_2$$

Combining Classes

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$$
\begin{aligned}
N^2 + N &\overset{?}{\leq} 2N^2 \\
N &\overset{?}{\leq} N^2
\end{aligned}
$$

Combining Classes

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$$\begin{aligned} N^2 + N &\stackrel{?}{\leq} 2N^2 \\ N &\stackrel{?}{\leq} N^2 \\ 1 &\leq N \end{aligned}$$

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$$N^2 \leq N^2 + N \leq 2N^2$$

Complexity Bounds

$$N^2 \leq \quad N^2 + N \quad \leq 2N^2$$

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$N^2 + N$ should probably be in $\theta(N^2)$ too.

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- g is bounded from above by something f-shaped

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  $g(N) \in O(f(N))$
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  $g(N) \in O(f(N))$
- g is bounded from below by something f-shaped
  $g(N) \in \Omega(f(N))$