CSE 250: Asymptotic Analysis

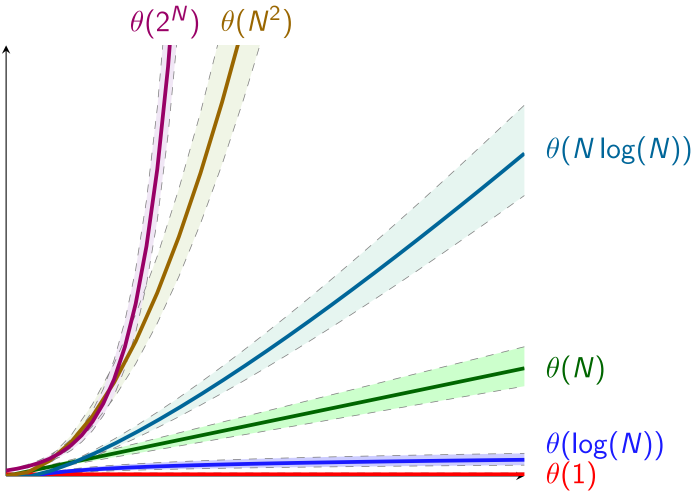
Lecture 6

Sept 9, 2023

Reminders

- PA1 Tests due Sun, Sept 15 at 11:59 PM
  - Recitations will cover writing good test cases.
- PA1 Implementation due Sun, Sept 22 at 11:59 PM
  - Implement a Sorted Linked List

Complexity Classes



Scaling Up

At $\frac{1}{4}$ ns per 'step' (4 GHz):

$f(n)$	10	20	50	100	1000
$\log(\log(n))$	0.43 ns	0.52 ns	0.62 ns	0.68 ns	0.82 ns
$\log(n)$	0.83 ns	1.01 ns	1.41 ns	1.66 ns	2.49 ns
n	2.5 ns	5 ns	12.5 ns	25 ns	0.25 µs
$n\log(n)$	8.3 ns	22 ns	71 ns	0.17 µs	2.49 µs
n^2	25 ns	0.1 µs	0.63 µs	2.5 µs	0.25 ms
n^5	25 µs	0.8 ms	78 ms	2.5 s	2.9 days
2^n	0.25 µs	0.26 ms	3.26 days	1013 years	10284 years
$n!$	0.91 ms	19 years	1047 years	10141 years	[yeah, no]

Examples

What is the asymptotic runtime of:

- Count the number of items in an N-item linked list.

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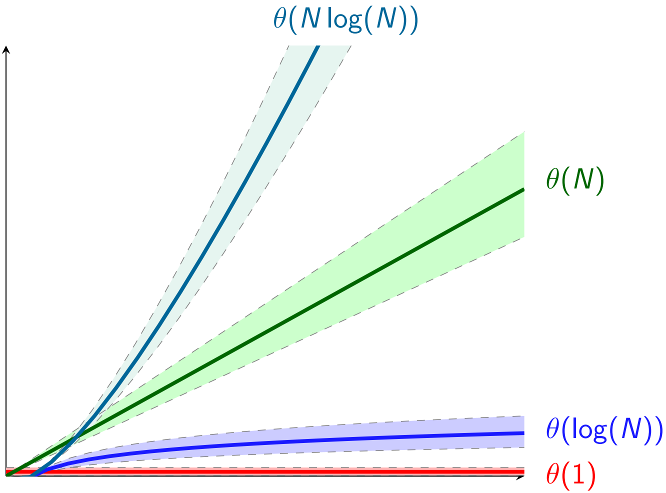
- Count the number of items in an N-item linked list. $(\theta(N))$
- Count the number of times `x` appears in a linked list. $(\theta(N))$
- Compute $x!$ $(= x \cdot (x-1) \cdot (x-2) \cdot \ldots \cdot 1)$.

Examples

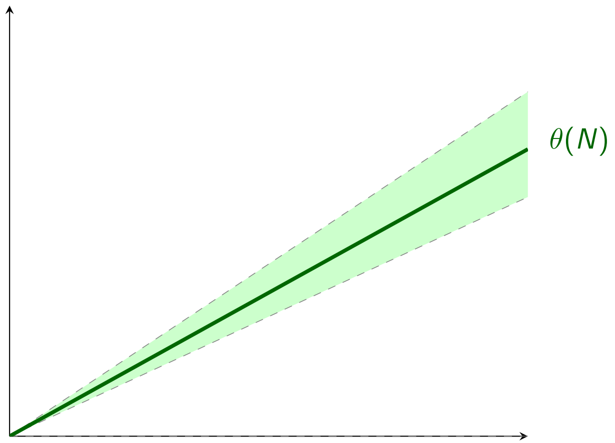
What is the asymptotic runtime of:

- Count the number of items in an N-item linked list. ($\theta(N)$)
- Count the number of times `x` appears in a linked list. ($\theta(N)$)
- Compute $x!$ $(= x \cdot (x-1) \cdot (x-2) \cdot \ldots \cdot 1)$. ($\theta(x)$)

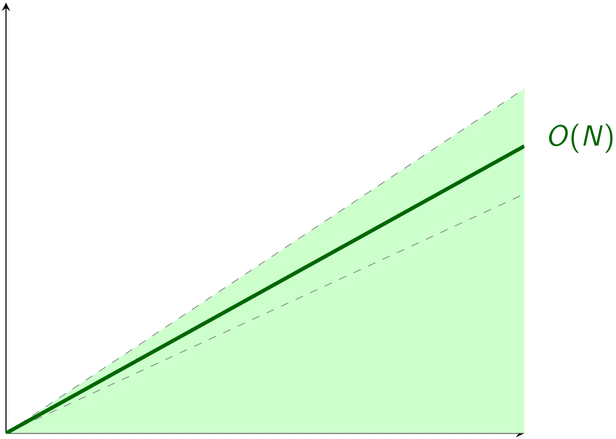
Complexity Bounds



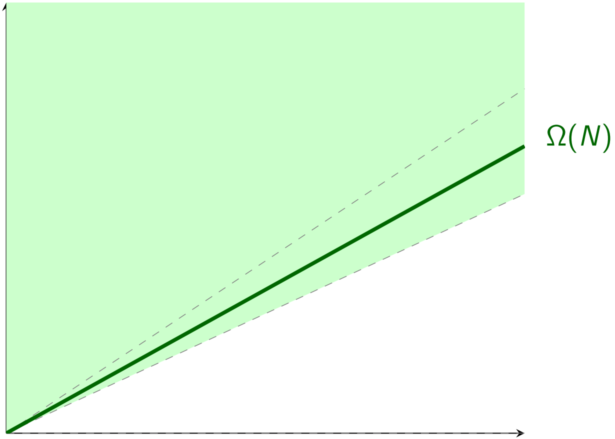
Complexity Bounds



Complexity Bounds



Complexity Bounds



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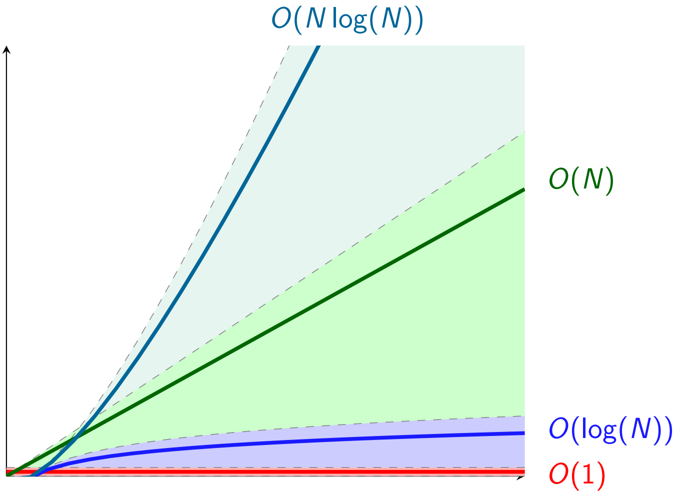
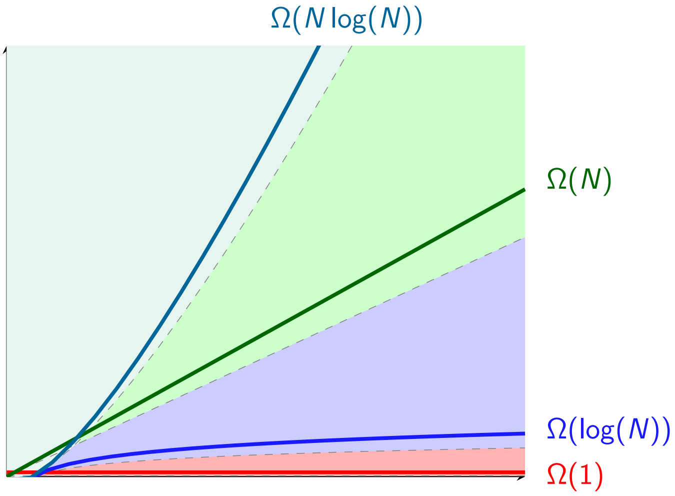


Complexity Bounds



Complexity Bounds

$g(N) \in O(f(N))$ if and only if:

- You can pick an N_0
- You can pick a c
- For all $N > N_0$: $g(N) \leq c \cdot f(N)$

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- You can pick an N_0
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- For all $N > N_0$: $g(N) \leq c \cdot f(N)$

$g(N) \in \Omega(f(N))$ if and only if:

- You can pick an N_0
- You can pick a c
- For all $N > N_0$: $g(N) \geq c \cdot f(N)$

Tight Bounds

So... along those lines: $N \in O(N^2)$

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We call this a **loose** bound.

Tight Bounds

So... along those lines: $N \in O(N^2)$
We call this a **loose** bound.
$g(N) \in O(f(N))$ is a **tight** bound if there is no $f'(N)$ in a **smaller** complexity class where $g(N) \in O(f'(N))$.

Examples

$$g(N) = N + 2N^2 \qquad f(N) = N^2$$

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$$1 + 2N \stackrel{?}{\leq} c \cdot N$$

Examples

$$g(N) = N + 2N^2 \qquad f(N) = N^2$$

$$N + 2N^2 \overset{?}{\leq} c \cdot N^2$$

$$1 + 2N \overset{?}{\leq} c \cdot N$$

$$1 + 2N \overset{?}{\leq} (a + b) \cdot N$$

Define $c = a + b$

Examples

$$g(N) = N + 2N^2 \qquad f(N) = N^2$$

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$$1 \stackrel{?}{\leq} a \cdot N$$

$$2N \stackrel{?}{\leq} b \cdot N$$

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$$2N \stackrel{?}{\leq} b \cdot N$$

$$2 \stackrel{?}{\leq} b$$

Define $c = a + b$

Examples

$$1 \overset{?}{\leq} a \cdot N \quad (1)$$

$$2 \overset{?}{\leq} b \quad (2)$$

Is there an $a + b = c > 0$ and $N_0 > 0$ you can plug in to make this equation true for all $N \geq N_0$?

Examples

$$g(N) = 3N + 1 \qquad f(N) = N^2$$

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$$3N + 1 \stackrel{?}{\leq} c \cdot N^2$$

$$3 + \frac{1}{N} \stackrel{?}{\leq} c \cdot N$$

Examples

$$g(N) = 3N + 1 \qquad f(N) = N^2$$

$$3N + 1 \overset{?}{\leq} c \cdot N^2$$

$$3 + \frac{1}{N} \overset{?}{\leq} c \cdot N$$

If $X < Y$ and $Y < Z$, then $X < Z$:

Examples

$$g(N) = 3N + 1 \qquad f(N) = N^2$$

$$3N + 1 \overset{?}{\leq} c \cdot N^2$$

$$3 + \frac{1}{N} \overset{?}{\leq} c \cdot N$$

If $X < Y$ and $Y < Z$, then $X < Z$:

$$3 + \frac{1}{N} \leq Y \overset{?}{\leq} c \cdot N$$

Examples

$$g(N) = 3N + 1 \qquad f(N) = N^2$$

$$3N + 1 \stackrel{?}{\leq} c \cdot N^2$$

$$3 + \frac{1}{N} \stackrel{?}{\leq} c \cdot N$$

If $X < Y$ and $Y < Z$, then $X < Z$:

$$3 + \frac{1}{N} \leq Y \stackrel{?}{\leq} c \cdot N$$

$$3 + \frac{1}{N} \leq 3 + 1 \stackrel{?}{\leq} c \cdot N$$

Examples

$$3 + \frac{1}{N} \leq 4 \stackrel{?}{\leq} c \cdot N$$

Examples

$$3 + \frac{1}{N} \leq 4 \stackrel{?}{\leq} c \cdot N$$

Is there a $c > 0$ and $N_0 \geq 1$ you can plug in to make this equation true for all $N \geq N_0$?

Examples

$$g(N) = 1 \qquad f(N) = N^2$$

Examples

$$g(N) = 1 \qquad f(N) = N^2$$

$$1 \overset{?}{\leq} c \cdot N^2$$

Examples

$$g(N) = 1 \qquad f(N) = N^2$$

$$1 \overset{?}{\leq} \; c \cdot N^2$$

Is there a $c > 0$ and $N_0 > 0$ you can plug in to make this equation true for all $N \geq N_0$?

Examples

$$g(N) = 1 \qquad f(N) = N^2$$

$$1 \overset{?}{\leq} c \cdot N^2$$

Is there a $c > 0$ and $N_0 > 0$ you can plug in to make this equation true for all $N \geq N_0$?

$$1 \in O(N^2)$$

Examples

$$g(N) = 1 \qquad f(N) = N^2$$

$$1 \overset{?}{\leq} \; c \cdot N^2$$

Is there a $c > 0$ and $N_0 > 0$ you can plug in to make this equation true for all $N \geq N_0$?

$$1 \in O(N^2)$$

$O(f(N))$ is every mathematical function in the complexity class of $f(N)$ or a lesser class.

Rules of Thumb

$\theta(1)$: Constant
$< \theta(\log(N))$: Logarithmic
$< \theta(N)$: Linear
$< \theta(N \log(N))$: Log-Linear
$< \theta(N^2)$: Quadratic
$< \theta(2^N)$: Exponential

Rules of Thumb

$$O(1) \subset O(\log(N))$$

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$$O(N) \subset O(N \log(N))$$

$$O(N) \subset O(N^2)$$

...

Rules of Thumb

- $O(f(N))$ (Big-O): The complexity class of $f(N)$ and every lesser class.
- $\theta(f(N))$ (Big-θ): The complexity class of $f(N)$.
- $\Omega(f(N))$ (Big-Ω): The complexity class of $f(N)$ and every greater class.

Rules of Thumb

$$\theta$$

Rules of Thumb

$$O$$

$$\theta$$

Rules of Thumb

$$O$$

$$\theta$$

$$\Omega$$

Rules of Thumb

$$O$$

$$\theta$$

$$\Omega$$

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Rules of Thumb

$$F(N) = f_1(N) + f_2(N) + \ldots + f_k(N)$$

What complexity class is $F(N)$ in?

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$f_1(N) + f_2(N)$ is in the greater of $\theta(f_1(N))$ and $\theta(f_2(N))$.

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$f_1(N) + f_2(N)$ is in the greater of $\theta(f_1(N))$ and $\theta(f_2(N))$.

$F(N)$ is in the greatest of any $\theta(f_i(N))$

Rules of Thumb

$$F(N) = f_1(N) + f_2(N) + \ldots + f_k(N)$$

What complexity class is $F(N)$ in?

$f_1(N) + f_2(N)$ is in the greater of $\theta(f_1(N))$ and $\theta(f_2(N))$.

$F(N)$ is in the greatest of any $\theta(f_i(N))$

We say the biggest f_i is the dominant term.

Formalizing θ

When is $g(N) \in \theta(f(N))$?

Idea 1:
There exists some c_1 and c_2 that makes: $g(N) = c_1 + c_2 \cdot f(N)$

Formalizing θ

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There exists some c_1 and c_2 that makes: $g(N) = c_1 + c_2 \cdot f(N)$
Problem: We want $N^2 + N \in \theta(N^2)$

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Idea 2

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Idea 2 (Use O, Ω)

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Idea 2 (Use O, Ω)

$$g(N) \in O(f(N)) \textbf{ and } g(N) \in \Omega(f(N))$$

$$\updownarrow$$

$$g(N) \in \theta(f(N))$$

Examples

$$n^2 + 4n \stackrel{?}{\in} \theta(n^2)$$

Examples

$$n^2 + 4n \overset{?}{\in} \theta(n^2)$$

$$2^n + 4n \overset{?}{\in} \theta(n^2)$$

Examples

$$n^2 + 4n \stackrel{?}{\in} \theta(n^2)$$

$$2^n + 4n \stackrel{?}{\in} \theta(n^2)$$

$$1000 \cdot n \log(n) + 5n \stackrel{?}{\in} \theta(n \log(n))$$

Examples

$$n^2 + 4n \stackrel{?}{\in} \theta(n^2)$$

$$2^n + 4n \stackrel{?}{\in} \theta(n^2)$$

$$1000 \cdot n \log(n) + 5n \stackrel{?}{\in} \theta(n \log(n))$$

Shortcut: Find the dominant term being summed, and compare it.

Tight Bounds

If $g(N) \in \theta(f(N))$:

- $g(N) \in O(f(N))$ is a **tight bound**.
- $g(N) \in \Omega(f(N))$ is a **tight bound**.

Examples

```
public void updateUsers(User[] users)
{
  x = 1;
  for(user : users)
  {
    user.id = x;
  }
}
```

$$1 + \sum_{\texttt{user} \in \texttt{users}} 2 \text{ steps} =$$

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$$1 + \sum_{\texttt{user} \in \texttt{users}} 2 \text{ steps} = 1 + 2 \times |\texttt{users}| \in \theta(N)$$

Examples

```
public void userFullName(User[] users, int id)
{
  User user = users[id];
  String fullName = user.firstName + user.lastName;
  return fullName;
}
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3

Examples

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  User user = users[id];
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}
```

$$3 \in \theta(1)$$

Count the Steps

```
public void totalReads(User[] users, Post[] posts)
{
  int totalReads = 0;
  for(post : posts)
  {
    int userReads = 0;
    for(user : users)
    {
      if(user.readPost(post)){ userReads += 1; }
    }
    totalReads += userReads;
  }
}
```

$$1 + \sum_{\texttt{post} \in \texttt{posts}} \left(3 + \sum_{\texttt{user} \in \texttt{users}} 2 \right)$$

Count the Steps

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Count the Steps

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$$= 1 + \sum_{\texttt{post} \in \texttt{posts}} \left(3 + 2 \cdot |\texttt{users}| \right)$$

$$= 1 + \left(\sum_{\texttt{post} \in \texttt{posts}} 3 \right) + \left(\sum_{\texttt{post} \in \texttt{posts}} 2 \cdot |\texttt{users}| \right)$$

Count the Steps

$$1 + \sum_{\texttt{post} \in \texttt{posts}} \left(3 + \sum_{\texttt{user} \in \texttt{users}} 2\right)$$

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$$= 1 + (3 \cdot |\texttt{posts}|) + \left(\sum_{\texttt{post} \in \texttt{posts}} 2 \cdot |\texttt{users}|\right)$$

Count the Steps

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Count the Steps

$$1 + \sum_{\texttt{post} \in \texttt{posts}} \left(3 + \sum_{\texttt{user} \in \texttt{users}} 2 \right)$$

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$$= 1 + (3 \cdot |\texttt{posts}|) + \left(\sum_{\texttt{post} \in \texttt{posts}} 2 \cdot |\texttt{users}| \right)$$

$$= 1 + (3 \cdot |\texttt{posts}|) + (2 \cdot |\texttt{users}| \cdot |\texttt{posts}|)$$

$$\in \theta(|\texttt{users}| \cdot |\texttt{posts}|)$$

Another Example

```
public int myAlgorithm(int[] input)
{
  if(input.size % 2 == 0){
    return 12345;
  } else {
    var total = 0;
    for(i : input)
    {
      total += i;
    }
    return total;
  }
}
```

Another Example

```
public int myAlgorithm(int[] input)
{
  if(input.size % 2 == 0){
    θ(1)
  } else {
    θ(1)
    for(i : input)
    {
      θ(1)
    }
    θ(1)
  }
}
```

Another Example

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public int myAlgorithm(int[] input)
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  if(input.size % 2 == 0){
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    for(i : input)
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      θ(1)
    }
    θ(1)
  }
}
```

Let's call $|input| = N$

Another Example

```
public int myAlgorithm(int[] input)
{
  if(input.size % 2 == 0){
    θ(1)
  } else {
    θ(1)
    θ(N · 1)
    θ(1)
  }
}
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Another Example

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public int myAlgorithm(int[] input)
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  if(input.size % 2 == 0){
    θ(1)
  } else {
    θ(N)
  }
}
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public int myAlgorithm(int[] input)
{
  if(input.size % 2 == 0){
    θ(1)
  } else {
    θ(N)
  }
}
```

$\theta(1)$ if N is even **OR** $\theta(N)$ if N is odd.

Multi-Class Functions

$$T(N) = \begin{cases} \theta(1) & \textbf{if } N \text{ is even} \\ \theta(N) & \textbf{if } N \text{ is odd} \end{cases}$$

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Multi-Class Functions

$$T(N) = \begin{cases} \theta(1) & \textbf{if } N \text{ is even} \\ \theta(N) & \textbf{if } N \text{ is odd} \end{cases}$$

What is the complexity class of $T(N)$?

- $T(N) \in O(N)$ is a **tight** bound.
- $T(N) \in \Omega(1)$ is a **tight** bound.

Multi-Class Functions

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If the tight Big-O and Big-Ω bounds are different, the function is not in ANY complexity class.

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- $T(N) \in O(N)$ is a **tight** bound.
- $T(N) \in \Omega(1)$ is a **tight** bound.

If the tight Big-O and Big-Ω bounds are different, the function is not in ANY complexity class. (Big-Theta doesn't exist).

Does Big-Theta Exist?

$N + 2N^2$ belongs to one complexity class. $(\theta(N^2))$

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$N + 2N^2$ belongs to one complexity class. ($\theta(N^2)$)

$5N + 10N^2 + 2^N$ belongs to one complexity class ($\theta(2^N)$)

Does Big-Theta Exist?

$N + 2N^2$ belongs to one complexity class. ($\theta(N^2)$)

$5N + 10N^2 + 2^N$ belongs to one complexity class ($\theta(2^N)$)

$\begin{cases} 2^N & \textbf{if } \texttt{rand()} > 0.5 \\ N & \textbf{otherwise} \end{cases}$ does **not** belong to one complexity class.

Does Big-Theta Exist?

$N + 2N^2$ belongs to one complexity class. ($\theta(N^2)$)

$5N + 10N^2 + 2^N$ belongs to one complexity class ($\theta(2^N)$)

$\begin{cases} 2^N & \textbf{if } \texttt{rand()} > 0.5 \\ N & \textbf{otherwise} \end{cases}$ does **not** belong to one complexity class.

- **Usually $\theta(f_1(N) + f_2(N) + \ldots)$ is based on the dominant term**
- **If you see cases (i.e., '{'), it's probably multi-class.**