CSE 250: Induction

Lecture 12

Sept 23, 2024

Reminders

- WA2 due Sun, Sept 29 at 11:59 PM
- Midterm 1 in class on Fri, Oct 04.
  - Covers: Asymptotics, Sequences/Lists, Arrays, Linked Lists, Recursion
  - Bounds: Tight Upper/Lower, Unqualified vs Amortized

Sorted Lists

Finding an item

- **Regular List**: Sequential Scan
- **Sorted List**: Binary Search

Sorted Lists

Finding an item

- **Regular List**: Sequential Scan $O(N)$
- **Sorted List**: Binary Search

Sorted Lists

Finding an item

- **Regular List**: Sequential Scan — $O(N)$
- **Sorted List**: Binary Search — $\theta(log(N))$ calls to `get()`

Sorted Lists

Finding an item

- **Regular List**: Sequential Scan $O(N)$
- **Sorted List**: Binary Search $\theta(log(N))$ calls to `get()`

So how do we get a sorted list?

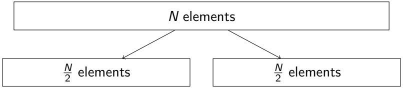
Bubble Sort

$O(N^2)$

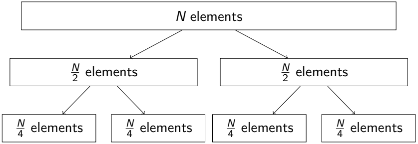
Merge Sort

N elements

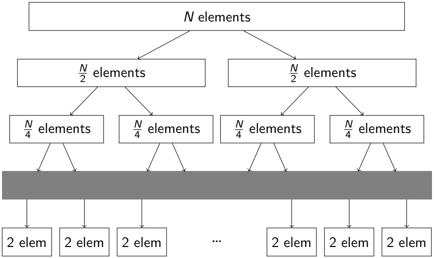
Merge Sort



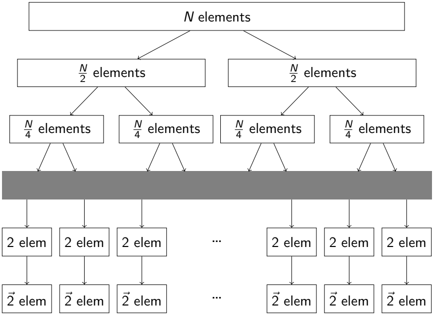
Merge Sort


N elements
$\frac{N}{2}$ elements
$\frac{N}{2}$ elements
$\frac{N}{4}$ elements
$\frac{N}{4}$ elements
$\frac{N}{4}$ elements
$\frac{N}{4}$ elements

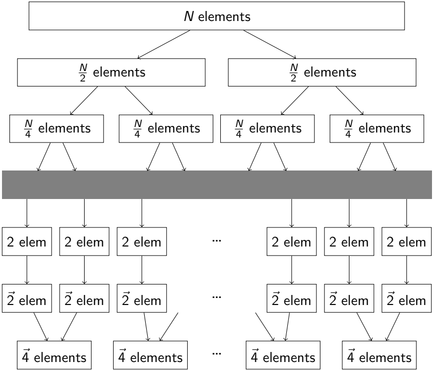
Merge Sort


N elements
$\frac{N}{2}$ elements
$\frac{N}{2}$ elements
$\frac{N}{4}$ elements
$\frac{N}{4}$ elements
$\frac{N}{4}$ elements
$\frac{N}{4}$ elements
2 elem
2 elem
2 elem
...
2 elem
2 elem
2 elem

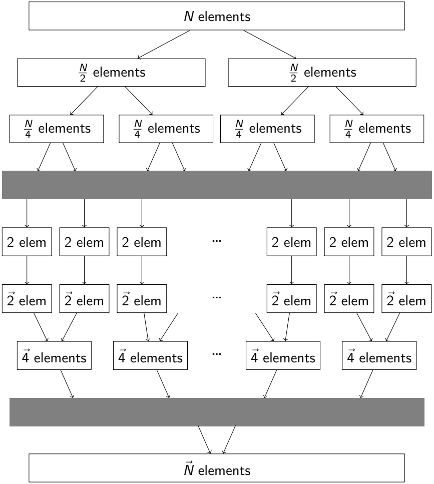
Merge Sort


N elements
N/2 elements
N/2 elements
N/4 elements
N/4 elements
N/4 elements
N/4 elements
2 elem
2 elem
2 elem
...
2 elem
2 elem
2 elem
2⃗ elem
2⃗ elem
2⃗ elem
...
2⃗ elem
2⃗ elem
2⃗ elem

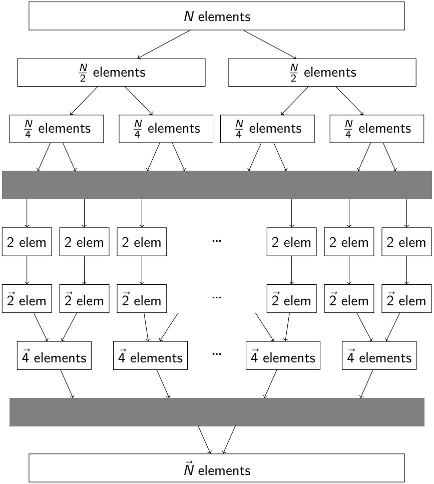
Merge Sort


N elements
N/2 elements
N/2 elements
N/4 elements
N/4 elements
N/4 elements
N/4 elements
2 elem
2 elem
2 elem
...
2 elem
2 elem
2 elem
2⃗ elem
2⃗ elem
2⃗ elem
...
2⃗ elem
2⃗ elem
2⃗ elem
4⃗ elements
4⃗ elements
...
4⃗ elements
4⃗ elements

Merge Sort

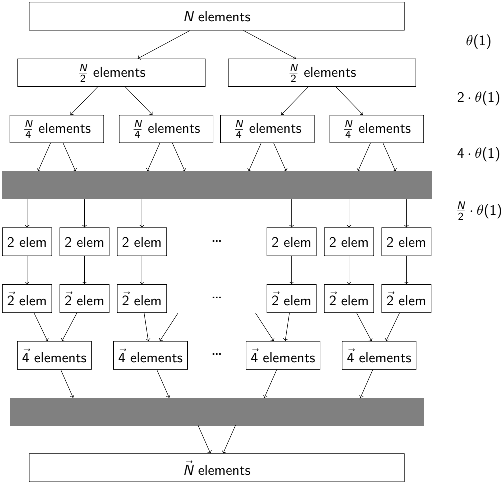

N elements
N/2 elements
N/2 elements
N/4 elements
N/4 elements
N/4 elements
N/4 elements
2 elem
2 elem
2 elem
...
2 elem
2 elem
2 elem
2⃗ elem
2⃗ elem
2⃗ elem
...
2⃗ elem
2⃗ elem
2⃗ elem
4⃗ elements
4⃗ elements
...
4⃗ elements
4⃗ elements
N⃗ elements

Merge Sort

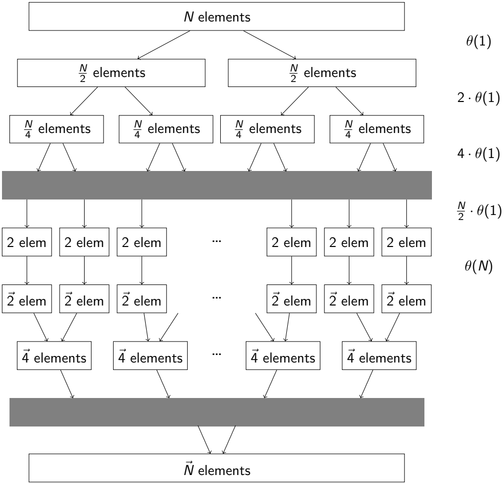


$\theta(1)$

Merge Sort


N elements
N/2 elements
N/2 elements
N/4 elements
N/4 elements
N/4 elements
N/4 elements
2 elem
2 elem
2 elem
...
2 elem
2 elem
2 elem
2⃗ elem
2⃗ elem
2⃗ elem
...
2⃗ elem
2⃗ elem
2⃗ elem
4⃗ elements
4⃗ elements
...
4⃗ elements
4⃗ elements
N⃗ elements
θ(1)
2 · θ(1)
4 · θ(1)
N/2 · θ(1)

Merge Sort


N elements
N/2 elements
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N/4 elements
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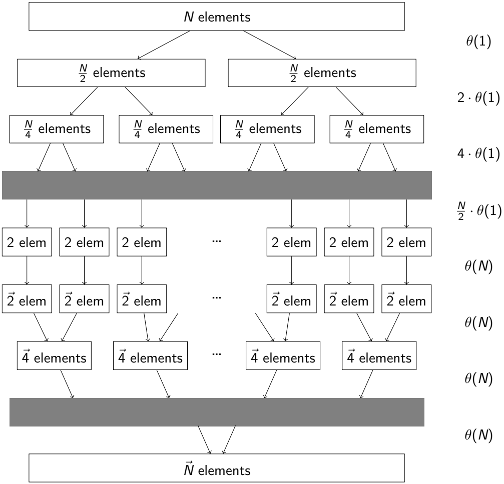
N/4 elements
2 elem
2 elem
2 elem
...
2 elem
2 elem
2 elem
2⃗ elem
2⃗ elem
2⃗ elem
...
2⃗ elem
2⃗ elem
2⃗ elem
4⃗ elements
4⃗ elements
...
4⃗ elements
4⃗ elements
N⃗ elements
θ(1)
2 · θ(1)
4 · θ(1)
N/2 · θ(1)
θ(N)

Merge Sort


N elements
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2 elem
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...
2 elem
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2 elem
2⃗ elem
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2⃗ elem
...
2⃗ elem
2⃗ elem
2⃗ elem
4⃗ elements
4⃗ elements
...
4⃗ elements
4⃗ elements
N⃗ elements
θ(1)
2 · θ(1)
4 · θ(1)
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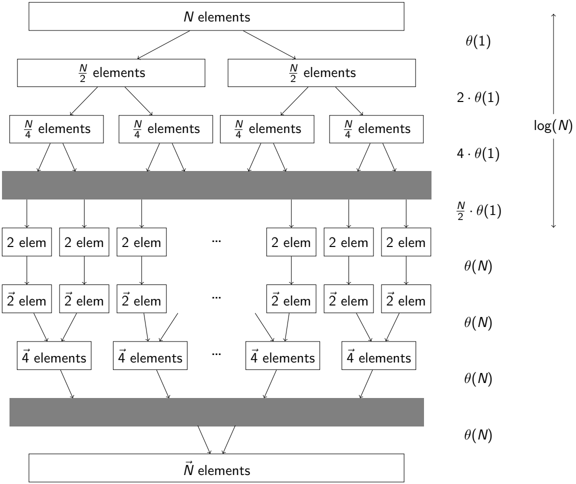
θ(N)
θ(N)
θ(N)
θ(N)

Merge Sort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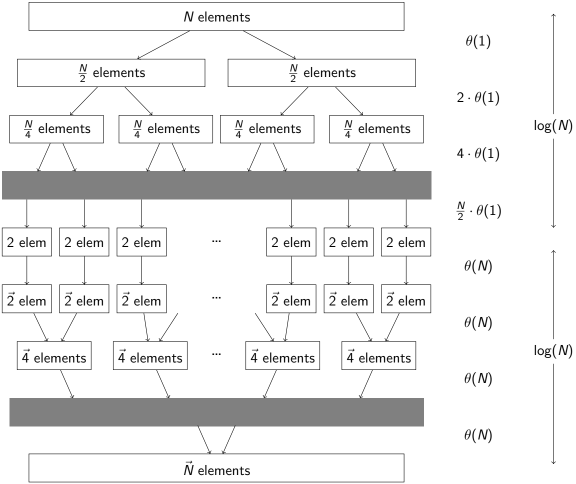
$\frac{N}{4}$ elements
$\frac{N}{4}$ elements
$\frac{N}{4}$ elements
2 elem
2 elem
2 elem
...
2 elem
2 elem
2 elem
$\vec{2}$ elem
$\vec{2}$ elem
$\vec{2}$ elem
...
$\vec{2}$ elem
$\vec{2}$ elem
$\vec{2}$ elem
$\vec{4}$ elements
$\vec{4}$ elements
...
$\vec{4}$ elements
$\vec{4}$ elements
$\vec{N}$ elements
$\theta(1)$
$2 \cdot \theta(1)$
$4 \cdot \theta(1)$
$\frac{N}{2} \cdot \theta(1)$
$\theta(N)$
$\theta(N)$
$\theta(N)$
$\theta(N)$
$\log(N)$

Merge Sort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$\vec{2}$ elem
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...
$\vec{2}$ elem
$\vec{2}$ elem
$\vec{2}$ elem
$\vec{4}$ elements
$\vec{4}$ elements
...
$\vec{4}$ elements
$\vec{4}$ elements
$\vec{N}$ elements
$\theta(1)$
$2 \cdot \theta(1)$
$4 \cdot \theta(1)$
$\frac{N}{2} \cdot \theta(1)$
$\theta(N)$
$\theta(N)$
$\theta(N)$
$\theta(N)$
$\log(N)$
$\log(N)$

Merge Sort

$$\left(\sum_{i=0}^{\log(N)-1} 2^i \cdot \theta(1)\right) + \left(\sum_{i=1}^{\log(N)-1} \theta(N)\right)$$

Merge Sort

$$\left(\sum_{i=0}^{\log(N)-1} 2^i \cdot \theta(1)\right) + \left(\sum_{i=1}^{\log(N)-1} \theta(N)\right)$$

$$\left(2^{\log(N)}\theta(1)\right) + (\log(N)\theta(N))$$

Merge Sort

$$\left(\sum_{i=0}^{\log(N)-1} 2^i \cdot \theta(1)\right) + \left(\sum_{i=1}^{\log(N)-1} \theta(N)\right)$$

$$\left(2^{\log(N)}\theta(1)\right) + (\log(N)\theta(N))$$

$$\theta(N) + \theta(N\log(N))$$

Merge Sort

$$\left(\sum_{i=0}^{\log(N)-1} 2^i \cdot \theta(1)\right) + \left(\sum_{i=1}^{\log(N)-1} \theta(N)\right)$$

$$\left(2^{\log(N)}\theta(1)\right) + (\log(N)\theta(N))$$

$$\theta(N) + \theta(N\log(N))$$

Merge Sort: $\theta(N\log(N))$

Merge Sort

$$\left(\sum_{i=0}^{\log(N)-1} 2^i \cdot \theta(1)\right) + \left(\sum_{i=1}^{\log(N)-1} \theta(N)\right)$$

$$\left(2^{\log(N)}\theta(1)\right) + (\log(N)\theta(N))$$

$$\theta(N) + \theta(N\log(N))$$

Merge Sort: $\theta(N\log(N))$

Bubble Sort: $\theta(N^2)$

Induction

Can we frame this proof in terms of the code?

Merge Sort

```
public ArrayList<E> mergeSort<E>(ArrayList<E> list)
{
  if(list.size() > 2){
    int splitIndex = input.size()/2;
    ArrayList<E> left =
      mergeSort(list.subList(0, splitIndex));
    ArrayList<E> right =
      mergeSort(list.subList(splitIndex, list.size));
    return merge(left, right);
  } else {
    if((list.size() == 2) && (list.get(0) > list.get(1))){
      E tmp = list.get(0);
      list.set(0, list.get(1));
      list.set(1, tmp);
    }
    return list;
  }
}
```

Merge Sort

- If $N > 2$
  - Split
  - $2\times$ `mergeSort(`$\frac{N}{2}$`)`
  - `merge(`N`)`
- Otherwise
  - Sort 2 elements

Merge Sort

- If $N > 2$
  - Split $O(1)$
  - $2\times$ `mergeSort(`$\frac{N}{2}$`)`
  - `merge(`N`)`
- Otherwise
  - Sort 2 elements

Merge Sort

- If $N > 2$
  - Split $O(1)$
  - $2\times$ `mergeSort(`$\frac{N}{2}$`)` ???
  - `merge(`N`)`
- Otherwise
  - Sort 2 elements

Merge Sort

- If $N > 2$
  - Split $O(1)$
  - $2\times$ `mergeSort`$(\frac{N}{2})$???
  - `merge`(N) $O(N)$
- Otherwise
  - Sort 2 elements

Merge Sort

- If $N > 2$
  - Split — $O(1)$
  - $2\times$ `mergeSort(`$\frac{N}{2}$`)` — ???
  - `merge(`N`)` — $O(N)$
- Otherwise
  - Sort 2 elements — $O(1)$

Merge Sort

- If $N > 2$
  - Split $O(1)$
  - $2\times$ `mergeSort(`$\frac{N}{2}$`)` ???
  - `merge(`N`)` $O(N)$
- Otherwise
  - Sort 2 elements $O(1)$

$$T_{merge}(N) = \begin{cases} O(N) + 2 \cdot ??? & \textbf{if } N > 2 \\ O(1) & \textbf{otherwise} \end{cases}$$

Merge Sort

- If $N > 2$
  - Split — $O(1)$
  - $2\times$ `mergeSort`$(\frac{N}{2})$ — ???
  - `merge`(N) — $O(N)$
- Otherwise
  - Sort 2 elements — $O(1)$

$$T_{merge}(N) = \begin{cases} O(N) + 2 \cdot T_{merge}(\frac{N}{2}) & \textbf{if } N > 2 \\ O(1) & \textbf{otherwise} \end{cases}$$

Merge Sort

- If $N > 2$
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  - $2\times$ `mergeSort(`$\frac{N}{2}$`)` — ???
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$$T_{merge}(N) = \begin{cases} O(N) + 2 \cdot T_{merge}(\frac{N}{2}) & \textbf{if } N > 2 \\ O(1) & \textbf{otherwise} \end{cases}$$

How do we solve for $T_{merge}(N)$?

Recursive Complexity

Let's start with a simpler problem.

Factorial

$$439! = \quad 439 \cdot 438 \cdot 437 \cdot 436 \cdot 435 \cdot 434 \cdot 433 \cdot 432 \cdot \ldots$$

Factorial

$$439! = 439 \cdot 438 \cdot 437 \cdot 436 \cdot 435 \cdot 434 \cdot 433 \cdot 432 \cdot \ldots$$

$$438! = 438 \cdot 437 \cdot 436 \cdot 435 \cdot 434 \cdot 433 \cdot 432 \cdot \ldots$$

Factorial

$$\begin{aligned} 439! &= 439 \cdot 438 \cdot 437 \cdot 436 \cdot 435 \cdot 434 \cdot 433 \cdot 432 \cdot \ldots \\ &= 439 \cdot 438! \end{aligned}$$

$$438! = 438 \cdot 437 \cdot 436 \cdot 435 \cdot 434 \cdot 433 \cdot 432 \cdot \ldots$$

Factorial

- $1! = 1$
- $N! = N \cdot (N-1)!$

Factorial

- $1! = 1$ Base Case
- $N! = N \cdot (N-1)!$

Factorial

- $1! = 1$ Base Case
- $N! = N \cdot (N-1)!$ Recursive Case

Factorial

- $1! = 1$ — Base Case
- $N! = N \cdot (N-1)!$ — Recursive Case

```
public long factorial(long N)
{
  if(N <= 1){ return 1; }
  else { return N * factorial(N-1); }
}
```

Factorial

What is the complexity of Factorial?

```
public long factorial(long N)
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  if(N <= 1){ return 1; }
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Factorial

What is the complexity of Factorial?

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$$T_{factorial}(N) = \begin{cases} \theta(1) & \textbf{if } N \leq 1 \\ \theta(1) + ??? & \textbf{otherwise} \end{cases}$$

Factorial

What is the complexity of Factorial?

```
public long factorial(long N)
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$$T_{factorial}(N) = \begin{cases} \theta(1) & \textbf{if } N \leq 1 \\ \theta(1) + T_{factorial}(N-1) & \textbf{otherwise} \end{cases}$$

Factorial

$$T_{factorial}(N) = \begin{cases} \theta(1) & \textbf{if } N \leq 1 \\ \theta(1) + T_{factorial}(N-1) & \textbf{otherwise} \end{cases}$$

Solve for $T_{factorial}(N)$.

Induction

Solve for $T_{factorial}(N)$.

Induction

1. Generate a hypothesis.
2. Prove the hypothesis for the base case.
3. Prove the hypothesis inductively.

Generate a Hypothesis

Hypothesis: $T(N) \in \theta(N)$

Generate a Hypothesis

Hypothesis: $T(N) \in \theta(N)$

- There is some $c_{high} > 0$ such that $T(n) \leq c_{high} \cdot n$
- There is some $c_{low} > 0$ such that $T(n) \geq c_{low} \cdot n$

Generate a Hypothesis

Hypothesis: $T(N) \in \theta(N)$

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Hypothesis: $T(N) \in \theta(N)$

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Factorial

$$T_{factorial}(N) = \begin{cases} \theta(1) & \textbf{if } N \leq 1 \\ \theta(1) + T_{factorial}(N-1) & \textbf{otherwise} \end{cases}$$

Factorial

$$T_{factorial}(N) = \begin{cases} c_1 & \textbf{if } N \leq 1 \\ c_2 + T_{factorial}(N-1) & \textbf{otherwise} \end{cases}$$

Factorial

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Goal: Show that $T_{factorial}(N) \leq c \cdot N$ for some $c > 0$

Factorial

$$T_{factorial}(N) = \begin{cases} c_1 & \textbf{if } N \leq 1 \\ c_2 + T_{factorial}(N-1) & \textbf{otherwise} \end{cases}$$

Goal: Show that $T_{factorial}(N) \leq c \cdot N$ for some $c > 0$ $(N > N_0)$

Factorial

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Goal: Show that $T_{factorial}(N) \leq c \cdot N$ for some $c > 0$ $(N > N_0)$

$$T_{factorial}(1) \overset{?}{\leq} 1 \cdot c$$

Factorial

$$T_{factorial}(N) = \begin{cases} c_1 & \textbf{if } N \leq 1 \\ c_2 + T_{factorial}(N-1) & \textbf{otherwise} \end{cases}$$

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$$T_{factorial}(1) \overset{?}{\leq} 1 \cdot c$$

$$c_1 \overset{?}{\leq} 1 \cdot c \checkmark$$

Factorial

$$T_{factorial}(N) = \begin{cases} c_1 & \textbf{if } N \leq 1 \\ c_2 + T_{factorial}(N-1) & \textbf{otherwise} \end{cases}$$

Goal: Show that $T_{factorial}(N) \leq c \cdot N$ for some $c > 0$ $(N > N_0)$

$$T_{factorial}(2) \overset{?}{\leq} 2 \cdot c$$

Factorial

$$T_{factorial}(N) = \begin{cases} c_1 & \textbf{if } N \leq 1 \\ c_2 + T_{factorial}(N-1) & \textbf{otherwise} \end{cases}$$

Goal: Show that $T_{factorial}(N) \leq c \cdot N$ for some $c > 0$ $(N > N_0)$

$$T_{factorial}(2) \overset{?}{\leq} 2 \cdot c$$

$$c_2 + T_{factorial}(1) \overset{?}{\leq} 2 \cdot c$$

Factorial

$$T_{factorial}(N) = \begin{cases} c_1 & \textbf{if } N \leq 1 \\ c_2 + T_{factorial}(N-1) & \textbf{otherwise} \end{cases}$$

Goal: Show that $T_{factorial}(N) \leq c \cdot N$ for some $c > 0$ $(N > N_0)$

$$T_{factorial}(2) \stackrel{?}{\leq} 2 \cdot c$$

$$c_2 + T_{factorial}(1) \stackrel{?}{\leq} 2 \cdot c$$

$$c_2 + c_1 \stackrel{?}{\leq} 2 \cdot c$$

Factorial

$$T_{factorial}(N) = \begin{cases} c_1 & \textbf{if } N \leq 1 \\ c_2 + T_{factorial}(N-1) & \textbf{otherwise} \end{cases}$$

Goal: Show that $T_{factorial}(N) \leq c \cdot N$ for some $c > 0$ $(N > N_0)$

$$T_{factorial}(2) \stackrel{?}{\leq} 2 \cdot c$$

$$c_2 + T_{factorial}(1) \stackrel{?}{\leq} 2 \cdot c$$

$$c_2 + c_1 \stackrel{?}{\leq} 2 \cdot c \checkmark$$

Factorial

$$T_{factorial}(N) = \begin{cases} c_1 & \textbf{if } N \leq 1 \\ c_2 + T_{factorial}(N-1) & \textbf{otherwise} \end{cases}$$

Goal: Show that $T_{factorial}(N) \leq c \cdot N$ for some $c > 0$ $(N > N_0)$

$$T_{factorial}(2) \stackrel{?}{\leq} 2 \cdot c$$

$$c_2 + T_{factorial}(1) \stackrel{?}{\leq} 2 \cdot c$$

Factorial

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$$T_{factorial}(2) \stackrel{?}{\leq} 2 \cdot c$$

$$c_2 + T_{factorial}(1) \stackrel{?}{\leq} 2 \cdot c$$

$$c_2 + T_{factorial}(1) \stackrel{?}{\leq} c + c$$

Factorial

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Goal: Show that $T_{factorial}(N) \leq c \cdot N$ for some $c > 0$ $(N > N_0)$

$$T_{factorial}(2) \overset{?}{\leq} 2 \cdot c$$

$$c_2 + T_{factorial}(1) \overset{?}{\leq} 2 \cdot c$$

$$c_2 + T_{factorial}(1) \overset{?}{\leq} c + c$$

$$c_2 \overset{?}{\leq} c$$

Factorial

$$T_{factorial}(N) = \begin{cases} c_1 & \textbf{if } N \leq 1 \\ c_2 + T_{factorial}(N-1) & \textbf{otherwise} \end{cases}$$

Goal: Show that $T_{factorial}(N) \leq c \cdot N$ for some $c > 0$ $(N > N_0)$

$$T_{factorial}(2) \stackrel{?}{\leq} 2 \cdot c$$
$$c_2 + T_{factorial}(1) \stackrel{?}{\leq} 2 \cdot c$$
$$c_2 + T_{factorial}(1) \stackrel{?}{\leq} c + c$$
$$c_2 \stackrel{?}{\leq} c \checkmark$$

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Can't I automate the proof?

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1. Generate a hypothesis.
2. Prove the hypothesis for the base case.
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3. Prove the hypothesis inductively.
   - Prove that it holds for some specific case N
   - You can assume that you've already proved it for $N - 1$

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We showed there exists a c such that...

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4. $T(4) \leq 4 \cdot c$ ($\#3$ + Inductive Proof)
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7. $\ldots$

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The proof holds for any $N \geq 1 \rightarrow T(N) \in O(N)$ ✓

Factorial

What is the complexity of Factorial?

```
public long factorial(long N)
{
  if(N <= 1){ return 1; }
  else { return N * factorial(N-1); }
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Answer: $O(N)$

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Answer: $O(N)$[1]

How much memory does it use?

[1]Technically it's $\theta(N)$, but we haven't proven $T(N) \in \Omega(N)$

Stack Frames

Every time you call a function, it allocates some memory for local variables (e.g., `N`).

This chunk of memory is called a **Stack Frame**.

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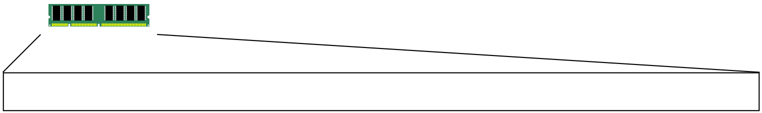
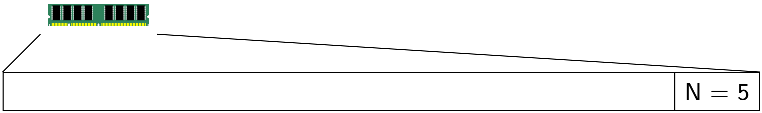
This is where the term `StackOverflowError` comes from.

Stack Frames



Stack Frames



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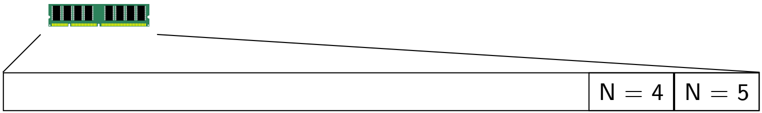

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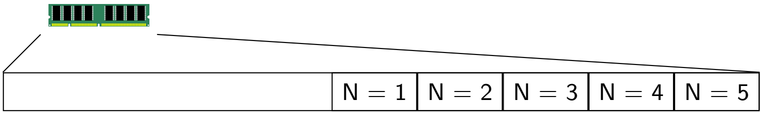

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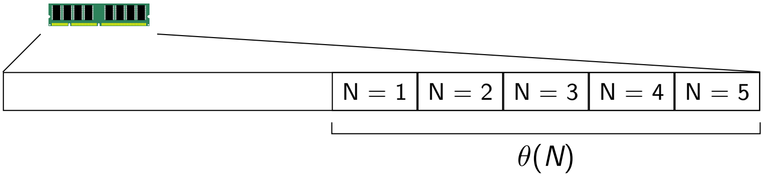

Factorial (as a loop)

```
public long factorial(long N)
{
  long total = 1;
  for(long i = N; i > 0; i--)
  {
    total *= i
  }
  return total
}
```

Factorial

Why does this work?

Factorial (as a loop)

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public long factorial(long N)
{
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```

Each call to factorial only makes **one** recursive call.

Factorial (as a loop)

- Is $\texttt{N} > 1$?
- Compute $\texttt{arg} = \texttt{N} - 1$
- Call `factorial(arg)`
- Compute $\texttt{N} \times$ result
- Return

Factorial (as a loop)

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- Compute $\mathtt{arg} = \mathtt{N} - 1$ ← Requires stack frame
- Call `factorial(arg)`
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- Return

Factorial (as a loop)

```
public long factorial(long N, long total)
{
  if(N <= 1){ return total; }
  else { return factorial(N-1, N * total); }
}
```

Factorial (as a loop)

- Is $\texttt{N} > 1$?
- Compute $\texttt{arg1} = \texttt{N} - 1$
- Compute $\texttt{arg2} = \texttt{N} \times$ total
- Call `factorial(arg1, arg2)`
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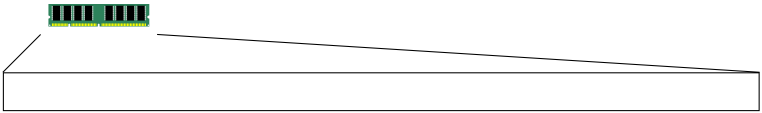
- Compute $\texttt{arg1} = \texttt{N} - 1$ ← Requires stack frame
- Compute $\texttt{arg2} = \texttt{N} \times$ total ← Requires stack frame
- Call `factorial(arg1, arg2)`
- Return ← Stack frame unnecessary

Stack Frames



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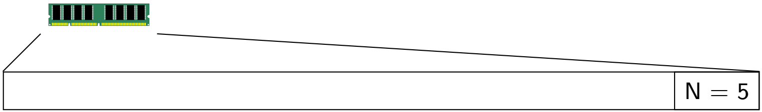

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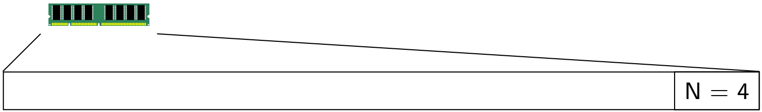

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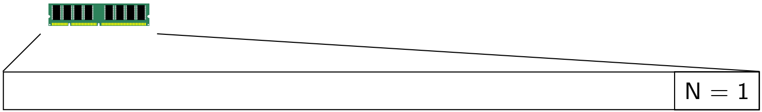

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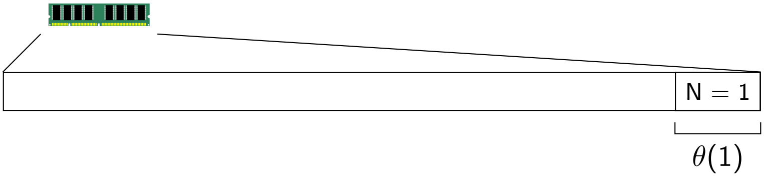

Tail Recursion

If the recursive call is the last operation before the return, most languages optimize the recursion away.

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[2]... but not Java

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If the recursive call is the last operation before the return, most languages optimize the recursion away[2].

This is called **Tail Recursion**

[2]... but not Java

Fibonacci

Time permitting...

Fibonacci

What's the complexity:

```
public long fib(long N)
{
  if(n <= 1){ return 1; }
  else { fib(n-1) + fib(n-2) }
}
```