CSE 250: Induction

Lecture 13

Sept 25, 2024

Reminders

- WA2 due Sun, Sept 29 at 11:59 PM
- Midterm 1 in class on Fri, Oct 04.
  - Covers: Asymptotics, Sequences/Lists, Arrays, Linked Lists, Recursion
  - Bounds: Tight Upper/Lower, Unqualified vs Amortized

Sorted Lists

Sorting an Array

- **Bubble Sort**: $O(N^2)$
- **Merge Sort**: $O(N \log(N))$

Sorted Lists

Sorting an Array

- **Bubble Sort**: $O(N^2)$
- **Merge Sort**: $O(N \log(N))$ (probably?)

Induction

1. Generate a hypothesis.
2. Prove the hypothesis for the base case.
3. Prove the hypothesis inductively.
   - Assume that the hypothesis is true for a small case (e.g., $N-1$).
   - Prove that the hypothesis is true for a bigger case (e.g., N).

Induction

We showed there exists a c such that...

- $T_f(1) \leq 1 \cdot c$ (Base Case)
- if $T_f(N-1) \leq (N-1) \cdot c$ then $T_f(N) \leq N \cdot c$ (Inductive Proof)

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1. $T_f(1) \leq 1 \cdot c$ (Base Case)
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4. $T_f(4) \leq 4 \cdot c$ (#3 + Inductive Proof)
5. $T_f(5) \leq 5 \cdot c$ (#4 + Inductive Proof)
6. $T_f(6) \leq 6 \cdot c$ (#5 + Inductive Proof)
7. $\ldots$

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7. ...

The proof holds for any $N \geq 1$

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5. $T_f(5) \leq 5 \cdot c$ (#4 + Inductive Proof)
6. $T_f(6) \leq 6 \cdot c$ (#5 + Inductive Proof)
7. $\ldots$

The proof holds for any $N \geq 1 \rightarrow T(N) \in O(N)$ ✓

Factorial

What is the complexity of Factorial?

```
public long factorial(long N)
{
  if(N <= 1){ return 1; }
  else { return N * factorial(N-1); }
}
```

Answer: $O(N)$

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```

Answer: $O(N)$[1]

How much memory does it use?

[1] Technically it's $\theta(N)$, but we haven't proven $T(N) \in \Omega(N)$

Stack Frames

Every time you call a function, it allocates some memory for local variables (e.g., `N`).

This chunk of memory is called a **Stack Frame**.

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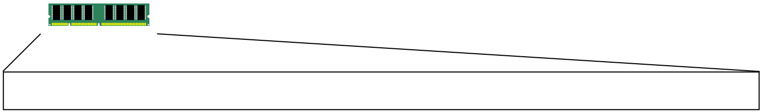
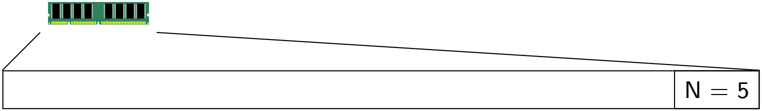
This is where the term `StackOverflowError` comes from.

Stack Frames



Stack Frames



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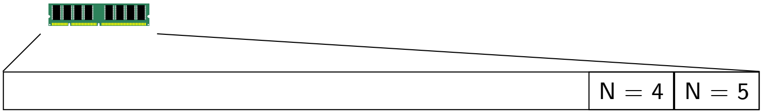

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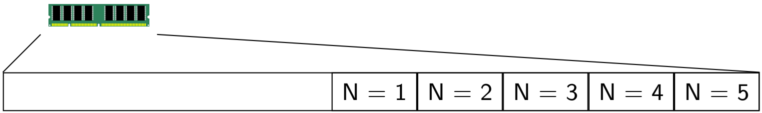
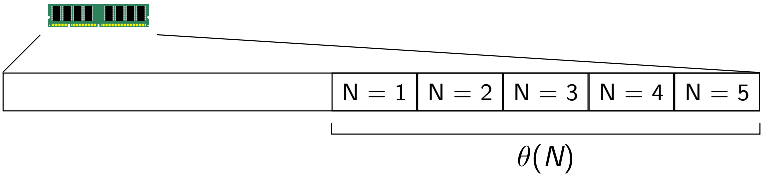


Stack Frames



Factorial (as a loop)

```
public long factorial(long N)
{
  long total = 1;
  for(long i = N; i > 0; i--)
  {
    total *= i
  }
  return total
}
```

Factorial

Why does this work?

Factorial (as a loop)

```
    public long factorial(long N)
    {
      if(N <= 1){ return 1; }
      else { return N * factorial(N-1); }
    }
```

Factorial (as a loop)

```
public long factorial(long N)
{
  if(N <= 1){ return 1; }
  else { return N * factorial(N-1); }
}
```

Each call to factorial only makes **one** recursive call.

Factorial (as a loop)

- Is $\texttt{N} > 1$?
- Compute $\texttt{arg} = \texttt{N} - 1$
- Call `factorial(arg)`
- Compute $\texttt{N} \times$ result
- Return

Factorial (as a loop)

- Is $\texttt{N} > 1$? ← Requires stack frame
- Compute $\texttt{arg} = \texttt{N} - 1$ ← Requires stack frame
- Call `factorial(arg)`
- Compute $\texttt{N} \times$ result ← Requires stack frame
- Return

Factorial (as a loop)

```
public long factorial(long N, long total)
{
  if(N <= 1){ return total; }
  else { return factorial(N-1, N * total); }
}
```

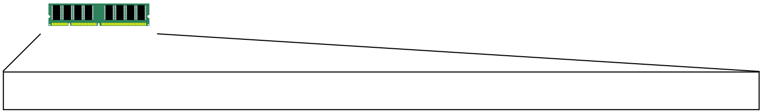
Factorial (as a loop)

- Is $\mathtt{N} > 1$?
- Compute $\mathtt{arg1} = \mathtt{N} - 1$
- Compute $\mathtt{arg2} = \mathtt{N} \times$ total
- Call `factorial(arg1, arg2)`
- Return

Factorial (as a loop)

- Is $\texttt{N} > 1$? ← Requires stack frame
- Compute $\texttt{arg1} = \texttt{N} - 1$ ← Requires stack frame
- Compute $\texttt{arg2} = \texttt{N} \times$ total ← Requires stack frame
- Call `factorial(arg1, arg2)`
- Return ← Stack frame unnecessary

Stack Frames



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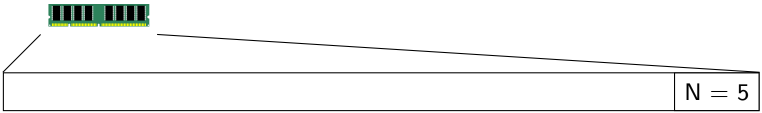

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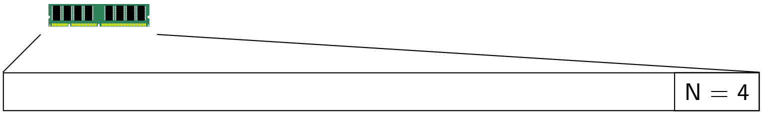

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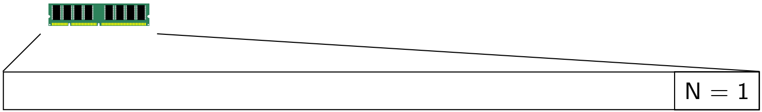

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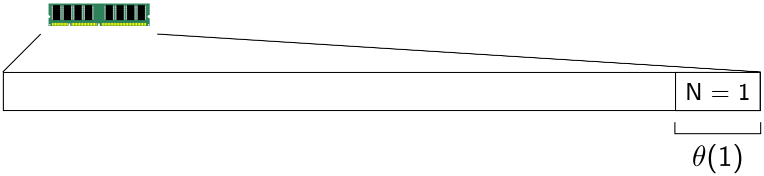

Tail Recursion

If the recursive call is the last operation before the return, most languages optimize the recursion away.

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If the recursive call is the last operation before the return, most languages optimize the recursion away[2].

[2]... but not Java

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If the recursive call is the last operation before the return, most languages optimize the recursion away[2].

This is called **Tail Recursion**

[2]... but not Java

Merge Sort

```
public ArrayList<E> mergeSort<E>(ArrayList<E> list)
{
  if(list.size() > 2){
    int splitIndex = input.size()/2;
    ArrayList<E> left =
      mergeSort(list.subList(0, splitIndex));
    ArrayList<E> right =
      mergeSort(list.subList(splitIndex, list.size));
    return merge(left, right);
  } else {
    if((list.size() == 2) && (list.get(0) > list.get(1))){
      E tmp = list.get(0);
      list.set(0, list.get(1));
      list.set(1, tmp);
    }
    return list;
  }
}
```

Merge Sort

- If $N > 2$
  - Split
  - $2\times$ `mergeSort(`$\frac{N}{2}$`)`
  - `merge(`N`)`
- Otherwise
  - Sort 2 elements

Merge Sort

- If $N > 2$
  - Split $O(1)$
  - $2\times$ `mergeSort(`$\frac{N}{2}$`)`
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- Otherwise
  - Sort 2 elements

Merge Sort

- If $N > 2$
  - Split $O(1)$
  - $2\times$ `mergeSort`$(\frac{N}{2})$ $2 \cdot T(\frac{N}{2})$
  - `merge`(N)
- Otherwise
  - Sort 2 elements

Merge Sort

- If $N > 2$
  - Split $O(1)$
  - $2\times$ `mergeSort`$(\frac{N}{2})$ $2 \cdot T(\frac{N}{2})$
  - `merge`(N) $O(N)$
- Otherwise
  - Sort 2 elements

Merge Sort

- If $N > 2$
  - Split — $O(1)$
  - $2\times$ `mergeSort`$(\frac{N}{2})$ — $2 \cdot T(\frac{N}{2})$
  - `merge`(N) — $O(N)$
- Otherwise
  - Sort 2 elements — $O(1)$

Merge Sort

- If $N > 2$
  - Split $O(1)$
  - $2\times$ `mergeSort(`$\frac{N}{2}$`)` $2 \cdot T(\frac{N}{2})$
  - `merge(`N`)` $O(N)$
- Otherwise
  - Sort 2 elements $O(1)$

$$T_{merge}(N) = \begin{cases} O(N) + 2 \cdot T_{merge}(\frac{N}{2}) & \textbf{if } N > 2 \\ O(1) & \textbf{otherwise} \end{cases}$$

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Induction

- **Hypothesis:**

Merge Sort

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Induction

- **Hypothesis:** $T_{merge}(N) \in O(N \cdot \log(N))$

Merge Sort

$$T_{merge}(N) = \begin{cases} O(N) + 2 \cdot T_{merge}(\frac{N}{2}) & \textbf{if } N > 2 \\ O(1) & \textbf{otherwise} \end{cases}$$

Induction

- **Hypothesis:** $T_{merge}(N) \in O(N \cdot \log(N))$
  - $T_{merge}(N) \leq c \cdot N \log(N)$

Merge Sort

$$T_{merge}(N) = \begin{cases} O(N) + 2 \cdot T_{merge}(\frac{N}{2}) & \textbf{if } N > 2 \\ O(1) & \textbf{otherwise} \end{cases}$$

Induction

- **Hypothesis:** $T_{merge}(N) \in O(N \cdot \log(N))$
  - $T_{merge}(N) \leq c \cdot N \log(N)$
- **Base Step:**

Merge Sort

$$T_{merge}(N) = \begin{cases} O(N) + 2 \cdot T_{merge}(\frac{N}{2}) & \textbf{if } N > 2 \\ O(1) & \textbf{otherwise} \end{cases}$$

Induction

- **Hypothesis:** $T_{merge}(N) \in O(N \cdot \log(N))$
  - $T_{merge}(N) \leq c \cdot N \log(N)$
- **Base Step:**
  - Show $T_{merge}(2) \leq c \cdot 2 \cdot \log(2) = c \cdot 2$

Merge Sort

$$T_{merge}(N) = \begin{cases} O(N) + 2 \cdot T_{merge}(\frac{N}{2}) & \textbf{if } N > 2 \\ O(1) & \textbf{otherwise} \end{cases}$$

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  - $T_{merge}(N) \leq c \cdot N \log(N)$
- **Base Step:**
  - Show $T_{merge}(2) \leq c \cdot 2 \cdot \log(2) = c \cdot 2$
- **Inductive Step:**
  - Assume $T_{merge}(\frac{N}{2}) \leq c \cdot \frac{N}{2} \cdot \log\left(\frac{N}{2}\right)$

Merge Sort

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  - Show $T_{merge}(N) \leq c \cdot N \cdot \log(N)$

Base Step

$$T_{merge}(N) = \begin{cases} O(N) + 2 \cdot T_{merge}(\frac{N}{2}) & \textbf{if } N > 2 \\ O(1) & \textbf{otherwise} \end{cases}$$

$$T_{merge}(2) \overset{?}{\leq} c \cdot 2$$

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$$T_{merge}(2) \stackrel{?}{\leq} c \cdot 2$$

$$O(1) \leq 2c$$

Inductive Step

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Assume: $T_{merge}(\frac{N}{2}) \leq c \cdot \frac{N}{2} \cdot \log\left(\frac{N}{2}\right)$

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Assume: $T_{merge}(\frac{N}{2}) \leq c \cdot \frac{N}{2} \cdot \log\left(\frac{N}{2}\right)$

$$T_{merge}(N) \overset{?}{\leq} c \cdot N \log(N)$$

$$O(N) + 2 \cdot T_{merge}\left(\frac{N}{2}\right) \overset{?}{\leq} c \cdot N \log(N)$$

Inductive Step

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Assume: $T_{merge}(\frac{N}{2}) \leq c \cdot \frac{N}{2} \cdot \log\left(\frac{N}{2}\right)$

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$$O(N) + 2 \cdot c \cdot \frac{N}{2} \cdot \log\left(\frac{N}{2}\right) \overset{?}{\leq} c \cdot N \log(N)$$

Inductive Step

$$O(N) + 2 \cdot c \cdot \frac{N}{2} \cdot \log\left(\frac{N}{2}\right) \stackrel{?}{\leq} \cdot N \log(N)$$

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$$O(N) + c \cdot N \cdot \log\left(\frac{N}{2}\right) \stackrel{?}{\leq} c \cdot N \log(N)$$

$$O(N) + c \cdot N \cdot (\log(N) - \log(2)) \stackrel{?}{\leq} c \cdot N \log(N)$$

Inductive Step

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$$O(N) + c \cdot N \cdot \log(N) \stackrel{?}{\leq} c \cdot N \log(N) + c \cdot N$$

Inductive Step

$$O(N) + 2 \cdot c \cdot \frac{N}{2} \cdot \log\left(\frac{N}{2}\right) \stackrel{?}{\leq} \cdot N \log(N)$$

$$O(N) + c \cdot N \cdot \log\left(\frac{N}{2}\right) \stackrel{?}{\leq} c \cdot N \log(N)$$

$$O(N) + c \cdot N \cdot (\log(N) - \log(2)) \stackrel{?}{\leq} c \cdot N \log(N)$$

$$O(N) + c \cdot N \cdot \log(N) \stackrel{?}{\leq} c \cdot N \log(N) + c \cdot N$$

$$O(N) \leq c \cdot N$$

Merge Sort

So... $T_{merge}(N) \in O(N \cdot \log(N))$

Quicksort

Merge Sort

- Split up the problem ($O(1)$)
- Merge the sub-solutions ($O(N)$)

Quicksort

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Quicksort

- Split up the problem ($O(N)$)
- Merge the sub-solutions ($O(1)$)

Quicksort

- If the list has only 1 element, it's sorted. Return.
- Find the **median** value ('pivot').
- Create a list of elements $<$ **medium**.
- Create a list of elements $>$ **medium**.
- Sort each of the two lists
- Concatenate the lists

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Quicksort

- If the list has only 1 element, it's sorted. Return. $O(1)$
- Find the **median** value ('pivot'). $T_{pivot}(N)$
- Create a list of elements $<$ **medium**. $O(N)$
- Create a list of elements $>$ **medium**. $O(N)$
- Sort each of the two lists $2 \cdot T(\frac{N}{2})$
- Concatenate the lists

Quicksort

- If the list has only 1 element, it's sorted. Return. — $O(1)$
- Find the **median** value ('pivot'). — $T_{pivot}(N)$
- Create a list of elements $<$ **medium**. — $O(N)$
- Create a list of elements $>$ **medium**. — $O(N)$
- Sort each of the two lists — $2 \cdot T(\frac{N}{2})$
- Concatenate the lists — $O(N)$ or $O(1)$

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- If the list has only 1 element, it's sorted. Return. — $O(1)$
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- Sort each of the two lists — $2 \cdot T(\frac{N}{2})$
- Concatenate the lists — $O(N)$ or $O(1)$

$$T_{qs}(N) = \begin{cases} 1 & \textbf{if } N \leq 1 \\ O(N) + 2T_{qs}(\frac{N}{2}) + T_{pivot}(N) & \textbf{otherwise} \end{cases}$$

Quicksort

- If the list has only 1 element, it's sorted. Return. $O(1)$
- Find the **median** value ('pivot'). $O(N\log(N))$
- Create a list of elements $<$ **medium**. $O(N)$
- Create a list of elements $>$ **medium**. $O(N)$
- Sort each of the two lists $2 \cdot T(\frac{N}{2})$
- Concatenate the lists $O(N)$ or $O(1)$

$$T_{qs}(N) = \begin{cases} 1 & \textbf{if } N \leq 1 \\ O(N) + 2T_{qs}(\frac{N}{2}) + N\log(N) & \textbf{otherwise} \end{cases}$$

Quicksort

Idea: If we pick a pivot value at random,
in expectation, half of the values will be lower.

The Worst-Case Pivot

What is the worst case runtime?

The Worst-Case Pivot

What if we always pick the worst pivot?

1. `[ 8, 7, 6, 5, 4, 3, 2, 1 ]`

The Worst-Case Pivot

What if we always pick the worst pivot?

1. [8, 7, 6, 5, 4, 3, 2, 1]
2. [7, 6, 5, 4, 3, 2, 1], 8, []

The Worst-Case Pivot

What if we always pick the worst pivot?

1. [8, 7, 6, 5, 4, 3, 2, 1]
2. [7, 6, 5, 4, 3, 2, 1], 8, []
3. [6, 5, 4, 3, 2, 1], 7, [], 8, []

The Worst-Case Pivot

What if we always pick the worst pivot?

1. [8, 7, 6, 5, 4, 3, 2, 1]
2. [7, 6, 5, 4, 3, 2, 1], 8, []
3. [6, 5, 4, 3, 2, 1], 7, [], 8, []
4. [5, 4, 3, 2, 1], 6, [], 7, [], 8, []

The Worst-Case Pivot

What if we always pick the worst pivot?

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...

The Worst-Case Pivot

What if we always pick the worst pivot?

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4. [5, 4, 3, 2, 1], 6, [], 7, [], 8, []

...

- For each level, $O(N) work$
- At worst, $O(N)$ levels

Total:

The Worst-Case Pivot

What if we always pick the worst pivot?

1. [8, 7, 6, 5, 4, 3, 2, 1]
2. [7, 6, 5, 4, 3, 2, 1], 8, []
3. [6, 5, 4, 3, 2, 1], 7, [], 8, []
4. [5, 4, 3, 2, 1], 6, [], 7, [], 8, []

...

- For each level, $O(N)$ *work*
- At worst, $O(N)$ levels

Total: $T_{quicksort}(N) \in O(N^2)$

Quicksort

Expected Runtime

Is the worst case runtime representative?

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No! (in typical cases, it will be faster)

Expected Runtime

Is the worst case runtime representative?

No! (in typical cases, it will be faster)

Is there something we <u>can</u> say about the runtime?

Quicksort

If we pick the X'th largest element as a pivot, what is $T_{qs}(N)$?

Quicksort

If we pick the X'th largest element as a pivot, what is $T_{qs}(N)$?

- $X = 1$:

Quicksort

If we pick the X'th largest element as a pivot, what is $T_{qs}(N)$?

- $X = 1$: $\theta(N)$

Quicksort

If we pick the X'th largest element as a pivot, what is $T_{qs}(N)$?

- $X = 1$: $\theta(N) + T_{qs}(0)$

Quicksort

If we pick the X'th largest element as a pivot, what is $T_{qs}(N)$?

- $X = 1$: $\theta(N) + T_{qs}(0) + T_{qs}(N-1)$

Quicksort

If we pick the X'th largest element as a pivot, what is $T_{qs}(N)$?

- $X = 1$: $\theta(N) + T_{qs}(0) + T_{qs}(N-1)$
- $X = 2$: $\theta(N) + T_{qs}(1) + T_{qs}(N-2)$

Quicksort

If we pick the X'th largest element as a pivot, what is $T_{qs}(N)$?

- $X = 1$: $\theta(N) + T_{qs}(0) + T_{qs}(N-1)$
- $X = 2$: $\theta(N) + T_{qs}(1) + T_{qs}(N-2)$
- $X = 3$: $\theta(N) + T_{qs}(2) + T_{qs}(N-3)$
- $X = 4$: $\theta(N) + T_{qs}(3) + T_{qs}(N-4)$
- ...
- $X = N-1$: $\theta(N) + T_{qs}(N-2) + T_{qs}(1)$
- $X = N$: $\theta(N) + T_{qs}(N-1) + T_{qs}(0)$

Quicksort

If we pick the X'th largest element as a pivot, what is $T_{qs}(N)$?

- $X = 1$: $\theta(N) + T_{qs}(0) + T_{qs}(N-1)$
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- $X = 3$: $\theta(N) + T_{qs}(2) + T_{qs}(N-3)$
- $X = 4$: $\theta(N) + T_{qs}(3) + T_{qs}(N-4)$
- ...
- $X = N-1$: $\theta(N) + T_{qs}(N-2) + T_{qs}(1)$
- $X = N$: $\theta(N) + T_{qs}(N-1) + T_{qs}(0)$

$$T_{qs}(N) = \theta(N) + T_{qs}(X-1) + T_{qs}(N-X)$$

Quicksort

If we pick the X'th largest element as a pivot, what is $T_{qs}(N)$?

- $X = 1$: $\theta(N) + T_{qs}(0) + T_{qs}(N-1)$
- $X = 2$: $\theta(N) + T_{qs}(1) + T_{qs}(N-2)$
- $X = 3$: $\theta(N) + T_{qs}(2) + T_{qs}(N-3)$
- $X = 4$: $\theta(N) + T_{qs}(3) + T_{qs}(N-4)$
- ...
- $X = N-1$: $\theta(N) + T_{qs}(N-2) + T_{qs}(1)$
- $X = N$: $\theta(N) + T_{qs}(N-1) + T_{qs}(0)$

$$T_{qs}(N) = \theta(N) + T_{qs}(X-1) + T_{qs}(N-X) \qquad (\text{for } X \in [1, N])$$

Probabilities

What is the chance we (randomly) pick $X = 1$?

Probabilities

What is the chance we (randomly) pick $X = 1$? $P[X = 1] = \frac{1}{N}$

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What is the chance we pick $X = \left\lfloor \frac{N}{2} \right\rfloor$?

Probabilities

What is the chance we (randomly) pick $X = 1$? $P[X = 1] = \frac{1}{N}$

What is the chance we pick $X = \left\lfloor \frac{N}{2} \right\rfloor$? $P\left[X = \left\lfloor \frac{N}{2} \right\rfloor\right] = \frac{1}{N}$

Probabilities

What is the chance we (randomly) pick $X = 1$? $P[X = 1] = \frac{1}{N}$

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What is the chance we pick $X = N$?

Probabilities

What is the chance we (randomly) pick $X = 1$? $P[X = 1] = \frac{1}{N}$

What is the chance we pick $X = \left\lfloor \frac{N}{2} \right\rfloor$? $P\left[X = \left\lfloor \frac{N}{2} \right\rfloor\right] = \frac{1}{N}$

What is the chance we pick $X = N$? $P[X = N] = \frac{1}{N}$

Probability

If I roll a d6 (a 6-sided die 🎲) k times,
what is the average over all possible outcomes?

k=1

If I roll a d6 (a 6-sided die 🎲) 1 time...

Roll	Probability	Contribution
1		

k=1

If I roll a d6 (a 6-sided die 🎲) 1 time...

Roll	Probability	Contribution
1	$\frac{1}{6}$	

k=1

If I roll a d6 (a 6-sided die 🎲) 1 time...

Roll	Probability	Contribution
1	$\frac{1}{6}$	1

k=1

If I roll a d6 (a 6-sided die 🎲) 1 time...

Roll	Probability	Contribution
1	$\frac{1}{6}$	1
2	$\frac{1}{6}$	2
3	$\frac{1}{6}$	3
4	$\frac{1}{6}$	4
5	$\frac{1}{6}$	5
6	$\frac{1}{6}$	6

Expectation

$$\frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6$$

Expectation

$$\frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 = 3.5$$

Expectation

$$\frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 = 3.5$$

$$= \sum_i \textbf{Probability}_i \cdot \textbf{Contribution}_i$$

Expectation

$$\frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 = 3.5$$

$$= \sum_i \textbf{Probability}_i \cdot \textbf{Contribution}_i$$

If X is a variable representing the outcome of the roll, we call this number the **expectation** of X, or $E[X]$.

Expectation

$$\frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 = 3.5$$

$$= \sum_i \textbf{Probability}_i \cdot \textbf{Contribution}_i$$

If X is a variable representing the outcome of the roll, we call this number the **expectation** of X, or $E[X]$.

$$E[X] = \sum_i P_i \cdot X_i$$

$k = 2$

If I roll a d6 (a 6-sided die 🎲) 2 times...
Does one roll affect the outcome of the other?

$k = 2$

If I roll a d6 (a 6-sided die 🎲) 2 times...
Does one roll affect the outcome of the other?

No: Each roll is a independent event.

Independent Events

If X and Y are random variables representing the outcome of each roll (i.e., independent random variables):

$$E[X + Y] = E[X] + E[Y]$$

Independent Events

If X and Y are random variables representing the outcome of each roll (i.e., independent random variables):

$$E[X+Y] = E[X] + E[Y]$$

$$= 3.5 + 3.5 = 7$$

Back to Quicksort

$$T_{qs}(N) = \begin{cases} \theta(1) & \textbf{if } N \leq 1 \\ T_{qs}(X-1) + T_{qs}(N-X) + \theta(N) & \textbf{otherwise} \end{cases}$$

Back to Quicksort

$$T_{qs}(N) = \begin{cases} \theta(1) & \textbf{if } N \leq 1 \\ T_{qs}(X-1) + T_{qs}(N-X) + \theta(N) & \textbf{otherwise} \end{cases}$$

... but X is random!

Expected Runtimes

$$E[T_{qs}(N)] = \begin{cases} \theta(1) & \textbf{if } N \leq 1 \\ E[T_{qs}(X-1) + T_{qs}(N-X) + \theta(N)] & \textbf{otherwise} \end{cases}$$

Expected Runtimes

$$E[T_{qs}(N)] = \begin{cases} \theta(1) & \textbf{if } N \leq 1 \\ E[T_{qs}(X-1)] + E[T_{qs}(N-X)] + E[\theta(N)] & \textbf{otherwise} \end{cases}$$

Expected Runtimes

$$E[T_{qs}(N)] = \begin{cases} \theta(1) & \textbf{if } N \leq 1 \\ E[T_{qs}(X-1)] + E[T_{qs}(N-X)] + \theta(N) & \textbf{otherwise} \end{cases}$$

Expected Runtimes

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Expected Runtimes

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$$= \sum_{i=1}^{N} \frac{1}{N} \cdot T(n-i) \qquad (T(n-1) \text{ down to } T(0))$$

Expected Runtimes

$$E[T(X-1)]$$

$$= \sum_{i=1}^{N} P_i \cdot T(X_i - 1)$$

$$= \sum_{i=1}^{N} \frac{1}{N} \cdot T(i-1) \qquad (T(0) \text{ up to } T(n-1))$$

$$= \sum_{i=1}^{N} \frac{1}{N} \cdot T(n-i) \qquad (T(n-1) \text{ down to } T(0))$$

$$= E[T(N-X)]$$

Expected Runtimes

$$E[T_{qs}(N)] = \begin{cases} \theta(1) & \textbf{if } N \leq 1 \\ E[T_{qs}(X-1)] + E[T_{qs}(N-X)] + \theta(N) & \textbf{otherwise} \end{cases}$$

Expected Runtimes

$$E[T_{qs}(N)] = \begin{cases} \theta(1) & \textbf{if } N \leq 1 \\ 2 \cdot E[T_{qs}(X-1)] + \theta(N) & \textbf{otherwise} \end{cases}$$

Expected Runtimes

$$E[T_{qs}(N)] = \begin{cases} \theta(1) & \textbf{if } N \leq 1 \\ 2 \cdot E[T_{qs}(X-1)] + \theta(N) & \textbf{otherwise} \end{cases}$$

The X we pick at each step is independent.

Expected Runtimes

$$E[T_{qs}(N)] = \begin{cases} \theta(1) & \textbf{if } N \leq 1 \\ 2 \cdot \left(\sum_{i=1}^{N} \frac{1}{N} E[T_{qs}(i-1)] \right) + \theta(N) & \textbf{otherwise} \end{cases}$$

Expected Runtimes

$$E[T_{qs}(N)] = \begin{cases} \theta(1) & \textbf{if } N \leq 1 \\ 2 \cdot \left(\sum_{i=0}^{N-1} \frac{1}{N} E[T_{qs}(i)]\right) + \theta(N) & \textbf{otherwise} \end{cases}$$

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Back to Induction!

Expected Runtimes

$$E[T_{qs}(N)] = \begin{cases} \theta(1) & \textbf{if } N \leq 1 \\ 2 \cdot \left(\sum_{i=0}^{N-1} \frac{1}{N} E[T_{qs}(i)]\right) + \theta(N) & \textbf{otherwise} \end{cases}$$

Back to Induction! **Inductive Hypothesis:**
$E[T_{qs}(N)] \in O(N \log(N))$

Expected Runtimes

$$E[T_{qs}(N)] = \begin{cases} \theta(1) & \textbf{if } N \leq 1 \\ 2 \cdot \left(\sum_{i=0}^{N-1} \frac{1}{N} E[T_{qs}(i)]\right) + \theta(N) & \textbf{otherwise} \end{cases}$$

Back to Induction! **Inductive Hypothesis:**
$E[T_{qs}(N)] \in O(N \log(N))$

$E[T_{qs}(N)] \leq c \cdot N \cdot \log(N)$

Quicksort's Runtime

$$E[T_{qs}(N)] = \begin{cases} c_1 & \textbf{if } N \leq 1 \\ 2 \cdot \left(\sum_{i=0}^{N-1} \frac{1}{N} E[T_{qs}(i)]\right) + c_2 N & \textbf{otherwise} \end{cases}$$

Quicksort's Runtime

$$E[T_{qs}(N)] = \begin{cases} c_1 & \textbf{if } N \leq 1 \\ 2 \cdot \left(\sum_{i=0}^{N-1} \frac{1}{N} E[T_{qs}(i)]\right) + c_2 N & \textbf{otherwise} \end{cases}$$

Base Case

$$E[T_{qs}(1)] \overset{?}{\leq} c \cdot 1 \cdot \log(1)$$

Quicksort's Runtime

$$E[T_{qs}(N)] = \begin{cases} c_1 & \textbf{if } N \leq 1 \\ 2 \cdot \left(\sum_{i=0}^{N-1} \frac{1}{N} E[T_{qs}(i)]\right) + c_2 N & \textbf{otherwise} \end{cases}$$

Base Case

$$E[T_{qs}(1)] \overset{?}{\leq} c \cdot 1 \cdot \log(1)$$

$$E[T_{qs}(1)] \overset{?}{\leq} c \cdot 1 \cdot 0$$

Quicksort's Runtime

$$E[T_{qs}(N)] = \begin{cases} c_1 & \textbf{if } N \leq 1 \\ 2 \cdot \left(\sum_{i=0}^{N-1} \frac{1}{N} E[T_{qs}(i)] \right) + c_2 N & \textbf{otherwise} \end{cases}$$

Base Case

$$E[T_{qs}(1)] \stackrel{?}{\leq} c \cdot 1 \cdot \log(1)$$

$$E[T_{qs}(1)] \stackrel{?}{\leq} c \cdot 1 \cdot 0$$

$$E[T_{qs}(1)] \not\leq 0$$

Quicksort's Runtime

$$E[T_{qs}(N)] = \begin{cases} c_1 & \textbf{if } N \leq 1 \\ 2 \cdot \left(\sum_{i=0}^{N-1} \frac{1}{N} E[T_{qs}(i)]\right) + c_2 N & \textbf{otherwise} \end{cases}$$

Base Case

$$E[T_{qs}(1)] \overset{?}{\leq} c \cdot 1 \cdot \log(1)$$

$$E[T_{qs}(1)] \overset{?}{\leq} c \cdot 1 \cdot 0$$

$$E[T_{qs}(1)] \not\leq 0 \qquad \ldots \text{ oops}$$

Quicksort's Expected Runtime

$$E[T_{qs}(N)] = \begin{cases} c_1 & \textbf{if } N \leq 1 \\ 2 \cdot \left(\sum_{i=0}^{N-1} \frac{1}{N} E[T_{qs}(i)]\right) + c_2 N & \textbf{otherwise} \end{cases}$$

Quicksort's Expected Runtime

$$E[T_{qs}(N)] = \begin{cases} c_1 & \textbf{if } N \leq 1 \\ 2 \cdot \left(\sum_{i=0}^{N-1} \frac{1}{N} E[T_{qs}(i)]\right) + c_2 N & \textbf{otherwise} \end{cases}$$

Base Case (with $N_0 = 2$)

$$E[T_{qs}(2)] \overset{?}{\leq} c \cdot 2 \cdot \log(2)$$

Quicksort's Expected Runtime

$$E[T_{qs}(N)] = \begin{cases} c_1 & \textbf{if } N \leq 1 \\ 2 \cdot \left(\sum_{i=0}^{N-1} \frac{1}{N} E[T_{qs}(i)]\right) + c_2 N & \textbf{otherwise} \end{cases}$$

Base Case (with $N_0 = 2$)

$$E[T_{qs}(2)] \overset{?}{\leq} c \cdot 2 \cdot \log(2)$$

$$2 \cdot \left(\sum_{i=0}^{1} \frac{1}{2} E[T_{qs}(i)]\right) + 2c_2 \overset{?}{\leq} c \cdot 2 \cdot \log(2)$$

Quicksort's Expected Runtime

$$E[T_{qs}(N)] = \begin{cases} c_1 & \textbf{if } N \leq 1 \\ 2 \cdot \left(\sum_{i=0}^{N-1} \frac{1}{N} E[T_{qs}(i)]\right) + c_2 N & \textbf{otherwise} \end{cases}$$

Base Case (with $N_0 = 2$)

$$E[T_{qs}(2)] \overset{?}{\leq} c \cdot 2 \cdot \log(2)$$

$$2 \cdot \left(\sum_{i=0}^{1} \frac{1}{2} E[T_{qs}(i)]\right) + 2c_2 \overset{?}{\leq} c \cdot 2 \cdot \log(2)$$

$$\left(\sum_{i=0}^{1} \frac{1}{2} E[T_{qs}(i)]\right) + c_2 \overset{?}{\leq} c$$

Quicksort's Expected Runtime

$$\left(\sum_{i=0}^{1} \frac{1}{2} E[T_{qs}(i)]\right) + c_2 \stackrel{?}{\leq} c$$

Quicksort's Expected Runtime

$$\left(\sum_{i=0}^{1} \frac{1}{2} E[T_{qs}(i)]\right) + c_2 \stackrel{?}{\leq} c$$

$$\frac{1}{2}(c_1 + c_1) + c_2 \stackrel{?}{\leq} c$$

Quicksort's Expected Runtime

$$\left(\sum_{i=0}^{1} \frac{1}{2} E[T_{qs}(i)]\right) + c_2 \stackrel{?}{\leq} c$$

$$\frac{1}{2}(c_1 + c_1) + c_2 \stackrel{?}{\leq} c$$

$$c_1 + c_2 \leq c$$

(true if we set $c \geq c_1 + c_2$)

Quicksort's Expected Runtime

$$E[T_{qs}(N)] = \begin{cases} c_1 & \textbf{if } N \leq 1 \\ 2 \cdot \left(\sum_{i=0}^{N-1} \frac{1}{N} E[T_{qs}(i)]\right) + c_2 N & \textbf{otherwise} \end{cases}$$

Inductive Step

Assume: $E[T_{qs}(N')] \leq c \cdot N' \log(N')$ for all $2 \leq N' \leq N$

Show: $E[T_{qs}(N)] \leq c \cdot N \log(N)$

$$2 \cdot \left(\sum_{i=0}^{N-1} \frac{1}{N} E[T_{qs}(i)]\right) + c_2 \overset{?}{\leq} c \cdot N \log(N)$$

$$2 \cdot \frac{1}{N} \left(\sum_{i=0}^{N-1} E[T_{qs}(i)]\right) + c_2 \overset{?}{\leq} c \cdot N \log(N)$$

Quicksort's Expected Runtime

$$2 \cdot \frac{1}{N}\left(\sum_{i=0}^{N-1} E[T_{qs}(i)]\right) + c_2 \overset{?}{\leq} c \cdot N \log(N)$$

Quicksort's Expected Runtime

$$2 \cdot \frac{1}{N}\left(\sum_{i=0}^{N-1} E[T_{qs}(i)]\right) + c_2 \overset{?}{\leq} c \cdot N\log(N)$$

$$2 \cdot \frac{1}{N}\left(E[T_{qs}(0)] + E[T_{qs}(1)] + \sum_{i=2}^{N-1} E[T_{qs}(i)]\right) + c_2 \overset{?}{\leq} c \cdot N\log(N)$$

Quicksort's Expected Runtime

$$2 \cdot \frac{1}{N}\left(\sum_{i=0}^{N-1} E[T_{qs}(i)]\right) + c_2 \overset{?}{\leq} c \cdot N\log(N)$$

$$2 \cdot \frac{1}{N}\left(E[T_{qs}(0)] + E[T_{qs}(1)] + \sum_{i=2}^{N-1} E[T_{qs}(i)]\right) + c_2 \overset{?}{\leq} c \cdot N\log(N)$$

$$2 \cdot \frac{1}{N}\left(2 \cdot c_1 + \sum_{i=2}^{N-1} E[T_{qs}(i)]\right) + c_2 \overset{?}{\leq} c \cdot N\log(N)$$

Quicksort's Expected Runtime

$$2 \cdot \frac{1}{N}\left(\sum_{i=0}^{N-1} E[T_{qs}(i)]\right) + c_2 \overset{?}{\leq} c \cdot N\log(N)$$

$$2 \cdot \frac{1}{N}\left(E[T_{qs}(0)] + E[T_{qs}(1)] + \sum_{i=2}^{N-1} E[T_{qs}(i)]\right) + c_2 \overset{?}{\leq} c \cdot N\log(N)$$

$$2 \cdot \frac{1}{N}\left(2 \cdot c_1 + \sum_{i=2}^{N-1} E[T_{qs}(i)]\right) + c_2 \overset{?}{\leq} c \cdot N\log(N)$$

$$\frac{2}{N}\left(\sum_{i=2}^{N-1} E[T_{qs}(i)]\right) + \frac{2 \cdot c_1}{N} + c_2 \overset{?}{\leq} c \cdot N\log(N)$$

Quicksort's Expected Runtime

$$\frac{2}{N}\left(\sum_{i=2}^{N-1} E[T_{qs}(i)]\right) + \frac{2 \cdot c_1}{N} + c_2 \overset{?}{\leq} c \cdot N \log(N)$$

Quicksort's Expected Runtime

$$\frac{2}{N}\left(\sum_{i=2}^{N-1} E[T_{qs}(i)]\right) + \frac{2 \cdot c_1}{N} + c_2 \overset{?}{\leq} c \cdot N \log(N)$$

$$\frac{2}{N}\left(\sum_{i=2}^{N-1} c \cdot i \log(i)\right) + \frac{2 \cdot c_1}{N} + c_2 \overset{?}{\leq} c \cdot N \log(N)$$

Quicksort's Expected Runtime

$$\frac{2}{N}\left(\sum_{i=2}^{N-1} E[T_{qs}(i)]\right) + \frac{2 \cdot c_1}{N} + c_2 \overset{?}{\leq} c \cdot N \log(N)$$

$$\frac{2}{N}\left(\sum_{i=2}^{N-1} c \cdot i \log(i)\right) + \frac{2 \cdot c_1}{N} + c_2 \overset{?}{\leq} c \cdot N \log(N)$$

$$\frac{2c}{N}\left(\sum_{i=2}^{N-1} i \log(i)\right) + \frac{2 \cdot c_1}{N} + c_2 \overset{?}{\leq} c \cdot N \log(N)$$

Quicksort's Expected Runtime

$$\frac{2}{N}\left(\sum_{i=2}^{N-1} E[T_{qs}(i)]\right) + \frac{2 \cdot c_1}{N} + c_2 \overset{?}{\leq} c \cdot N \log(N)$$

$$\frac{2}{N}\left(\sum_{i=2}^{N-1} c \cdot i \log(i)\right) + \frac{2 \cdot c_1}{N} + c_2 \overset{?}{\leq} c \cdot N \log(N)$$

$$\frac{2c}{N}\left(\sum_{i=2}^{N-1} i \log(i)\right) + \frac{2 \cdot c_1}{N} + c_2 \overset{?}{\leq} c \cdot N \log(N)$$

The following left-hand-side is strictly bigger than the preceding.

$$\frac{2c}{N}\left(\sum_{i=1}^{N-1} i \log(N)\right) + \frac{2 \cdot c_1}{N} + c_2 \overset{?}{\leq} c \cdot N \log(N)$$

Quicksort's Expected Runtime

$$\frac{2c}{N}\left(\sum_{i=1}^{N-1} i\log(N)\right) + \frac{2\cdot c_1}{N} + c_2 \overset{?}{\leq} c\cdot N\log(N)$$

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$$\frac{2c\log(N)}{N}\left(\sum_{i=1}^{N-1} i\right) + \frac{2 \cdot c_1}{N} + c_2 \overset{?}{\leq} c \cdot N\log(N)$$

Quicksort's Expected Runtime

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$$\frac{2c\log(N)}{N}\left(\frac{(N-1)\cdot N}{2}\right) + \frac{2\cdot c_1}{N} + c_2 \overset{?}{\leq} c\cdot N\log(N)$$

Quicksort's Expected Runtime

$$\frac{2c}{N}\left(\sum_{i=1}^{N-1} i\log(N)\right) + \frac{2\cdot c_1}{N} + c_2 \overset{?}{\leq} c\cdot N\log(N)$$

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$$c\cdot N\log(N) - c\cdot\log(N) + \frac{2\cdot c_1}{N} + c_2 \overset{?}{\leq} c\cdot N\log(N)$$

Quicksort's Expected Runtime

$$\frac{2c}{N}\left(\sum_{i=1}^{N-1} i\log(N)\right) + \frac{2\cdot c_1}{N} + c_2 \overset{?}{\leq} c\cdot N\log(N)$$

$$\frac{2c\log(N)}{N}\left(\sum_{i=1}^{N-1} i\right) + \frac{2\cdot c_1}{N} + c_2 \overset{?}{\leq} c\cdot N\log(N)$$

$$\frac{2c\log(N)}{N}\left(\frac{(N-1)\cdot N}{2}\right) + \frac{2\cdot c_1}{N} + c_2 \overset{?}{\leq} c\cdot N\log(N)$$

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Quicksort's Expected Runtime

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$$2 \cdot c_1 + c_2 \overset{?}{\leq} c \cdot \log(N)$$

True for any $N \geq 2$ if $c \geq 2 \cdot c_1 + c_2$

Quicksort's Expected Runtime

$$\frac{2 \cdot c_1}{N} + c_2 \overset{?}{\leq} c \cdot \log(N)$$

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Quicksort's Expected Runtime

$$E[T_{qs}(N)] \in O(N \log(N))$$

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So is Quicksort $O(N \log(N))$? **No!**

Quicksort's Expected Runtime

$$E[T_{qs}(N)] \in O(N \log(N))$$

So is Quicksort $O(N \log(N))$? **No!**

Quicksort's **Expected** runtime is $O(N \log(N))$

Bound Guarantees

- $f(N)$ is a Worst-Case Bound $(T(N) \in O(f(N)))$
  The algorithm **always** runs in at most $c \cdot f(N)$ steps.

- $f(N)$ is an Amortized Worst-Case Bound
  N **invocations** of the algorithm **always** run in at most $N \cdot c \cdot f(N)$ steps.

- $f(N)$ is an Expected Worst-Case Bound $(E[T(N)] \in O(f(N)))$
  The algorithm is **statistically likely** to run in at most $c \cdot f(N)$ steps.