

# CSE 250

## Data Structures

Dr. Eric Mikida  
epmikida@buffalo.edu  
208 Capen Hall

**Lec 07: Proving Bounds**

# Announcements and Feedback

- PA1 out now
  - Testing due Sunday (no late submissions, no grace days)
  - Implementation due next Sunday

# "Shape" of a Function

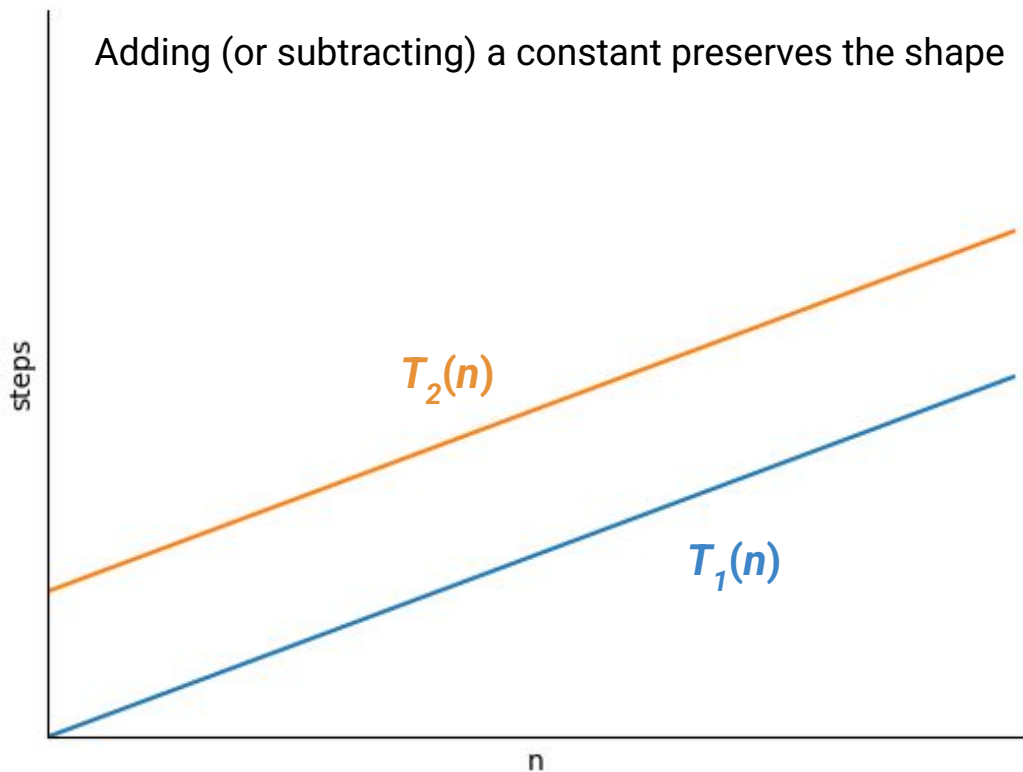
When do we consider two functions to have the same shape?

# Additive Factors

Consider two growth functions:

$$T_1(n) = 3n$$

$$T_2(n) = 3n + 3$$

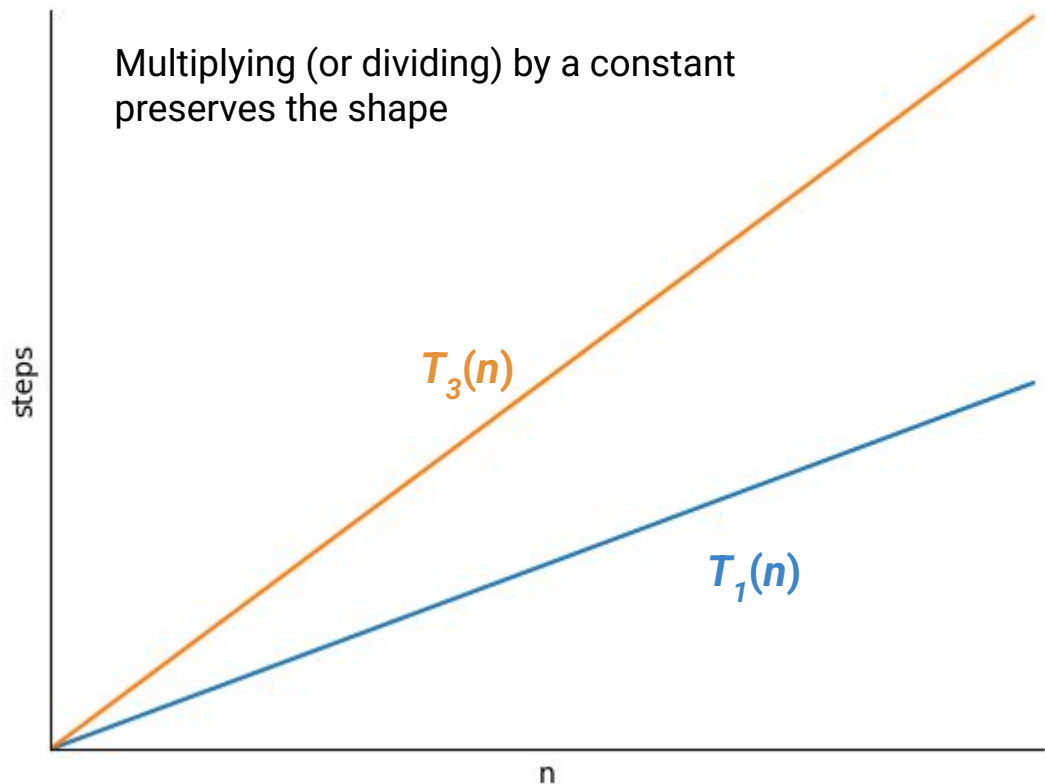


# Multiplicative Factors

Consider two growth functions:

$$T_1(n) = 3n$$

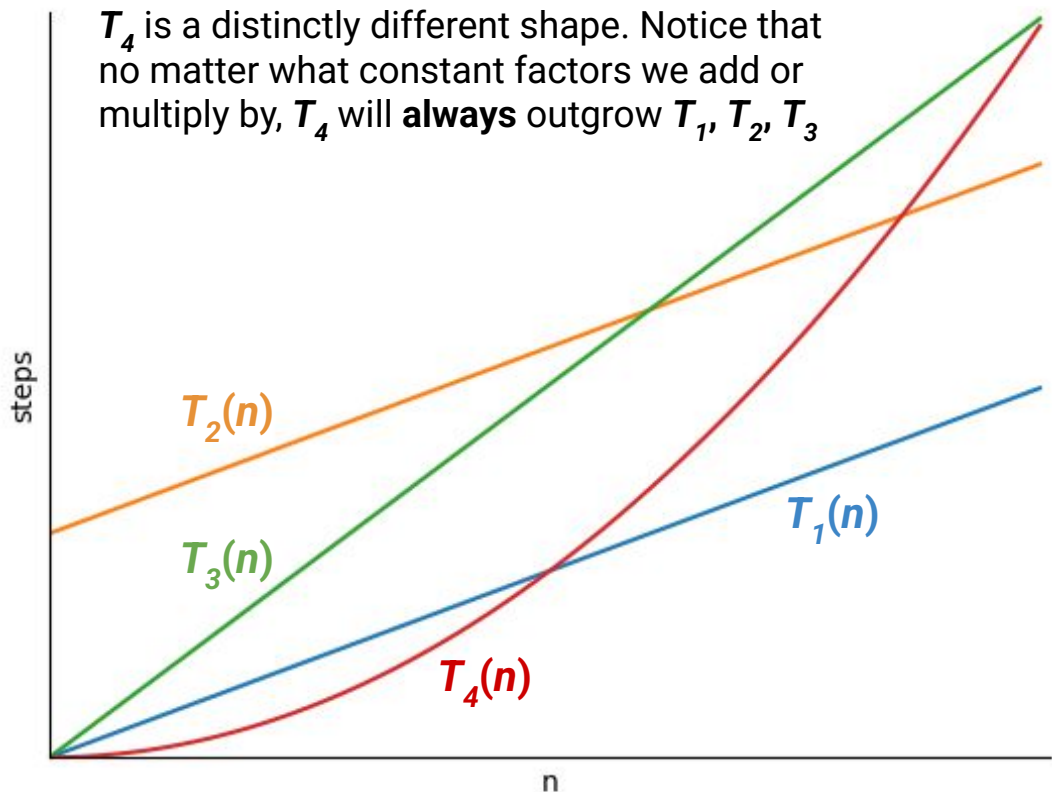
$$T_3(n) = 6n$$



# A Counter Example

Now consider:

$$T_4(n) = n^2$$



# Complexity Class

$f$  and  $g$  are in the same complexity class, denoted  $g(n) \in \Theta(f(n))$ , iff:

$g$  is bounded from above by something  $f$ -shaped

*and*

$g$  is bounded from below by something  $f$ -shaped

# Bounded from Above: Big O

$g(n)$  is bounded from above by  $f(n)$  if:

There exists a constant  $n_0 \geq 0$  and a constant  $c > 0$  such that:

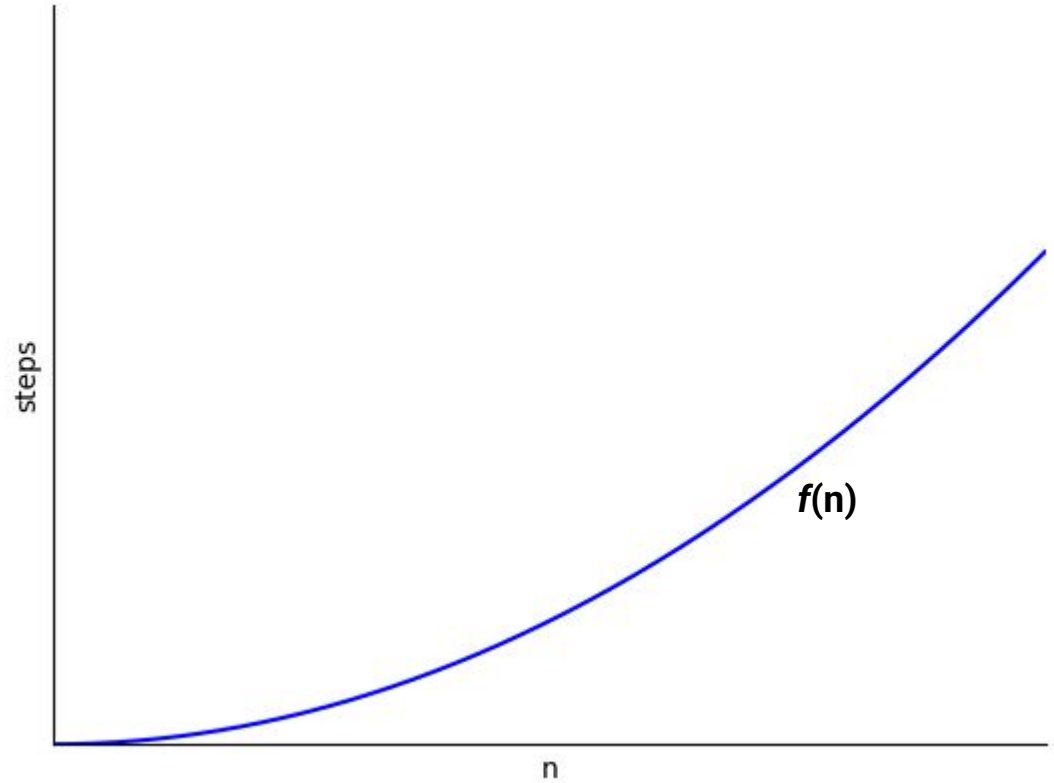
$$\text{For all } n \geq n_0, g(n) \leq c \cdot f(n)$$

In this case, we say that  $g(n) \in O(f(n))$

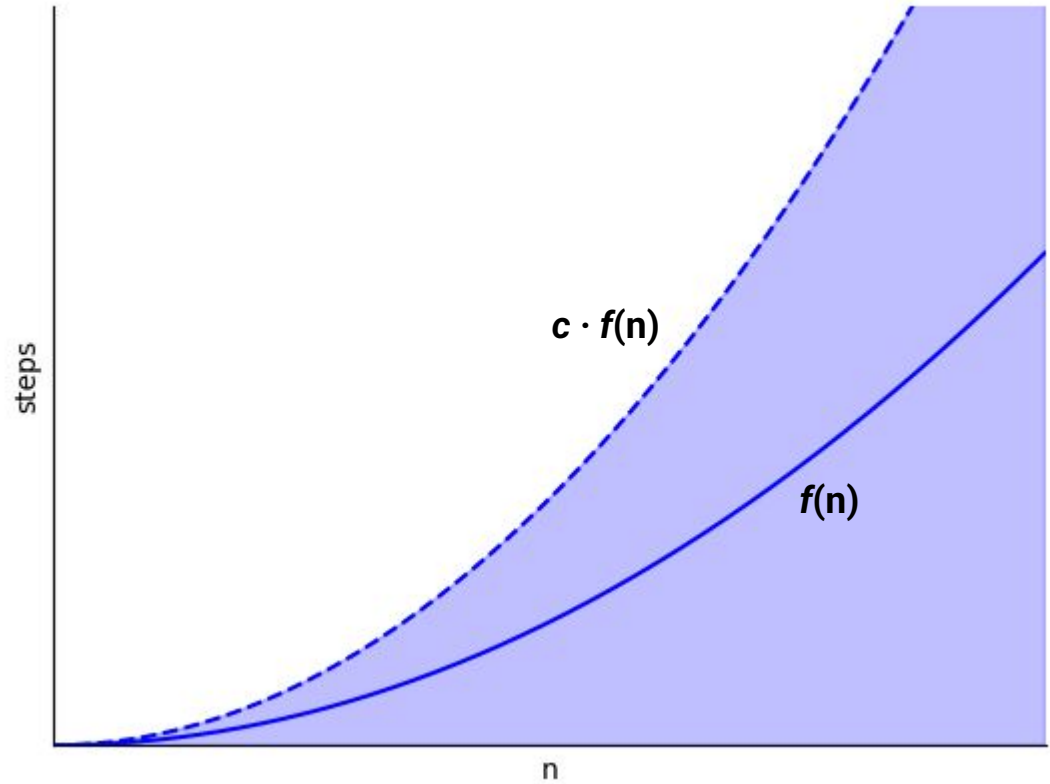
$O(f(n))$  is the set of all functions bounded from above by  $f(n)$



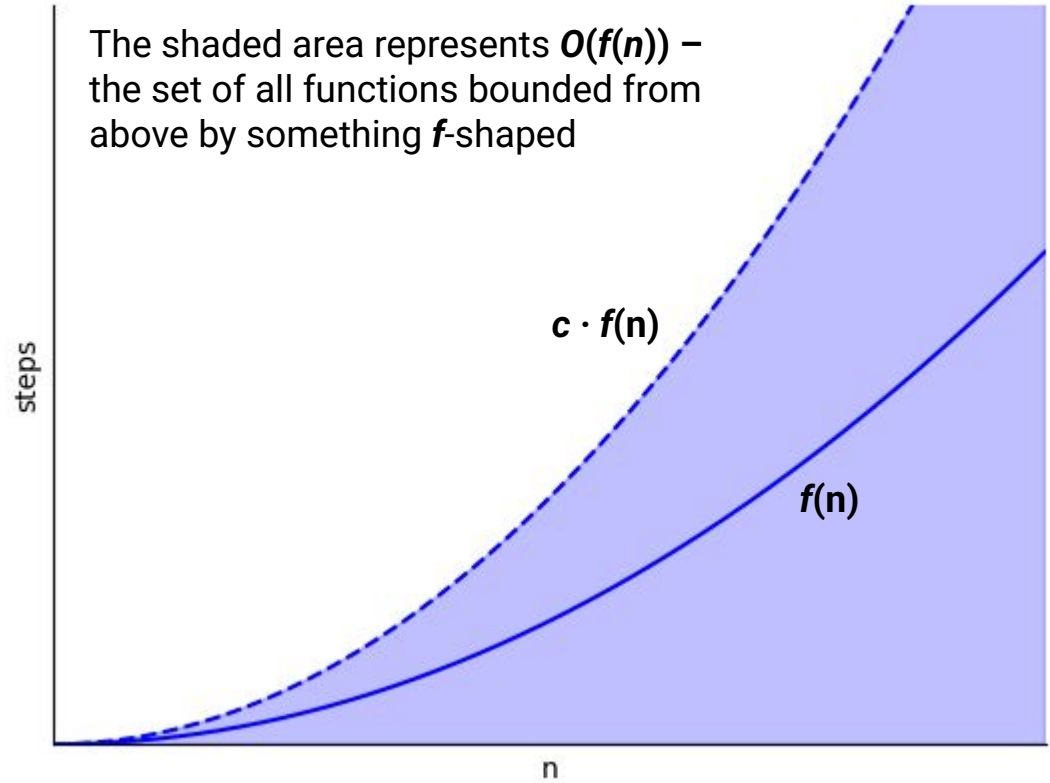
# Bounded from Above: Big O



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# Bounded from Above: Big O



# Bounded from Above: Big O

$$g(n) = \frac{n^2}{2} + 4n + 7$$

Prove that  $g(n) \in O(n^2)$

$$\frac{n^2}{2} + 4n + 7 \leq c \cdot n^2$$

# Inequality Tricks

1.  $f(n) \geq g(n)$  is true if  $f(n)/a \geq g(n)/a$  (for any  $a > 0$ )
2.  $f(n) \geq g(n)$  is true if  $f(n)*a \geq g(n)*a$  (for any  $a > 0$ )
3.  $x + a \geq y + b$  is true if  $x \geq y$  and  $a \geq b$  (for any  $a, b$ )
4.  $x \geq y$  is true if  $x \geq a$  and  $a \geq y$  (for any  $a$ )
5.  $1 \leq \log(n) \leq n \leq n^2 \leq n^k$  (for  $k \geq 2$ )  $\leq 2^n$

# Bounded from Above: Big O

$$\frac{n^2}{2} + 4n + 7 \leq c \cdot n^2$$

First show:

$$\frac{n^2}{2} \leq c_1 \cdot n^2$$

$$4n \leq c_2 \cdot n^2$$

$$7 \leq c_3 \cdot n^2$$

# Bounded from Above: Big O

$$\frac{n^2}{2} \leq c_1 \cdot n^2$$

# Bounded from Above: Big O

$$\frac{n^2}{2} \leq c_1 \cdot n^2$$

This is true for all  $n \geq 0$  if we set  $c_1$  to  $1/2$



# Bounded from Above: Big O

$$4n \leq c_2 \cdot n^2$$

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$$4n \leq c_2 \cdot n^2$$

This is true for all  $n \geq 1$  if we set  $c_2$  to 4

# Bounded from Above: Big O

$$7 \leq c_3 \cdot n^2$$

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This is true for all  $n \geq 1$  if we set  $c_3$  to **7**

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$$\frac{n^2}{2} + 4n + 7 \leq c \cdot n^2$$

First show:

$$\frac{n^2}{2} \leq \frac{1}{2} \cdot n^2$$

$$4n \leq 4 \cdot n^2$$

$$7 \leq 7 \cdot n^2$$

$$\frac{n^2}{2} + 4n + 7 \leq \frac{1}{2} \cdot n^2 + 4 \cdot n^2 + 7 \cdot n^2 = \left(\frac{1}{2} + 4 + 7\right) \cdot n^2$$

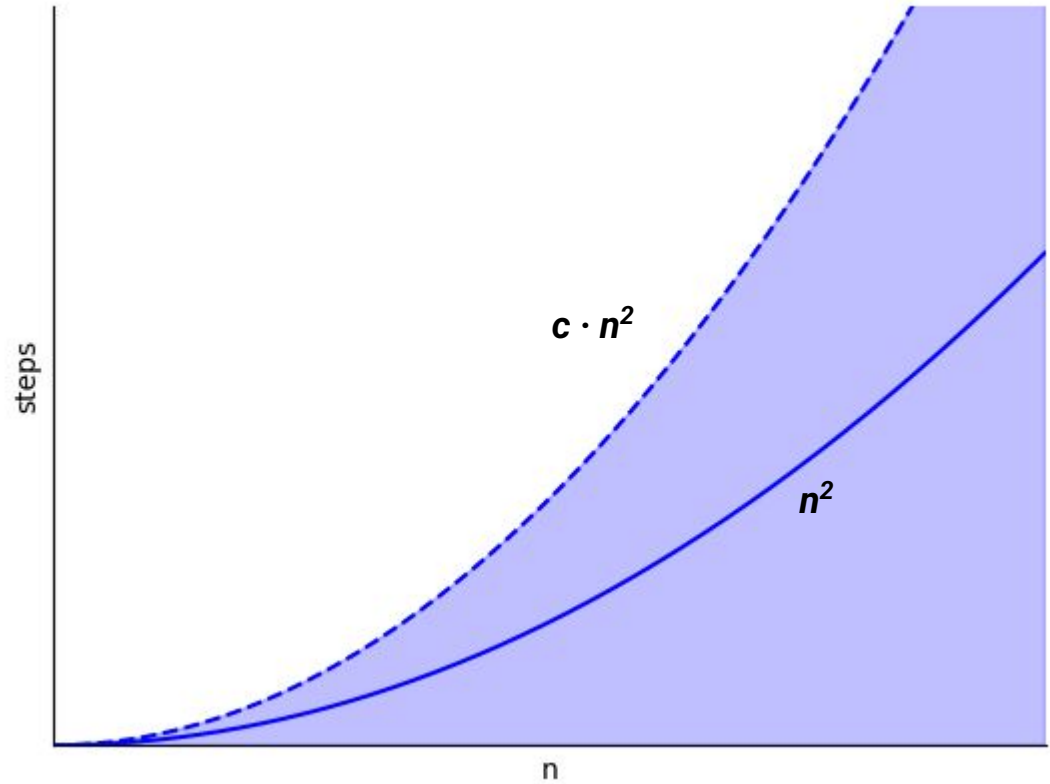
# Bounded from Above: Big O

$$\frac{n^2}{2} + 4n + 7 \leq c \cdot n^2$$

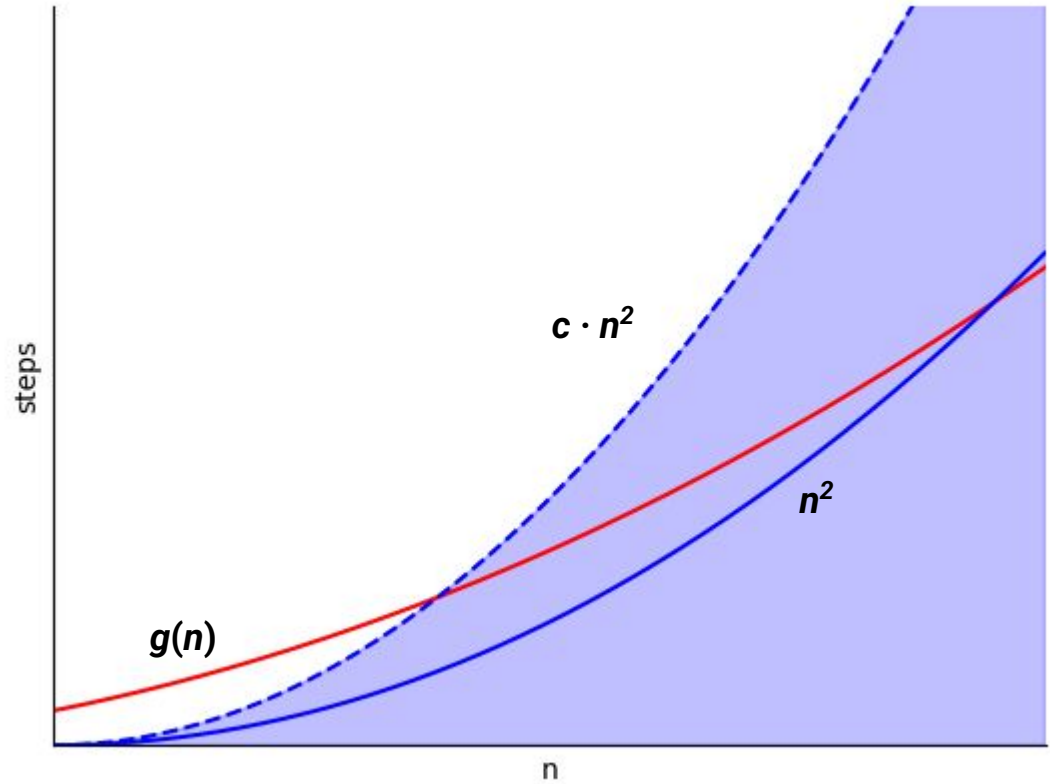
Therefore if we let  $c = 11.5$ , then for all  $n \geq 1$  the above holds true

Therefore  $g(n) \in O(n^2)$

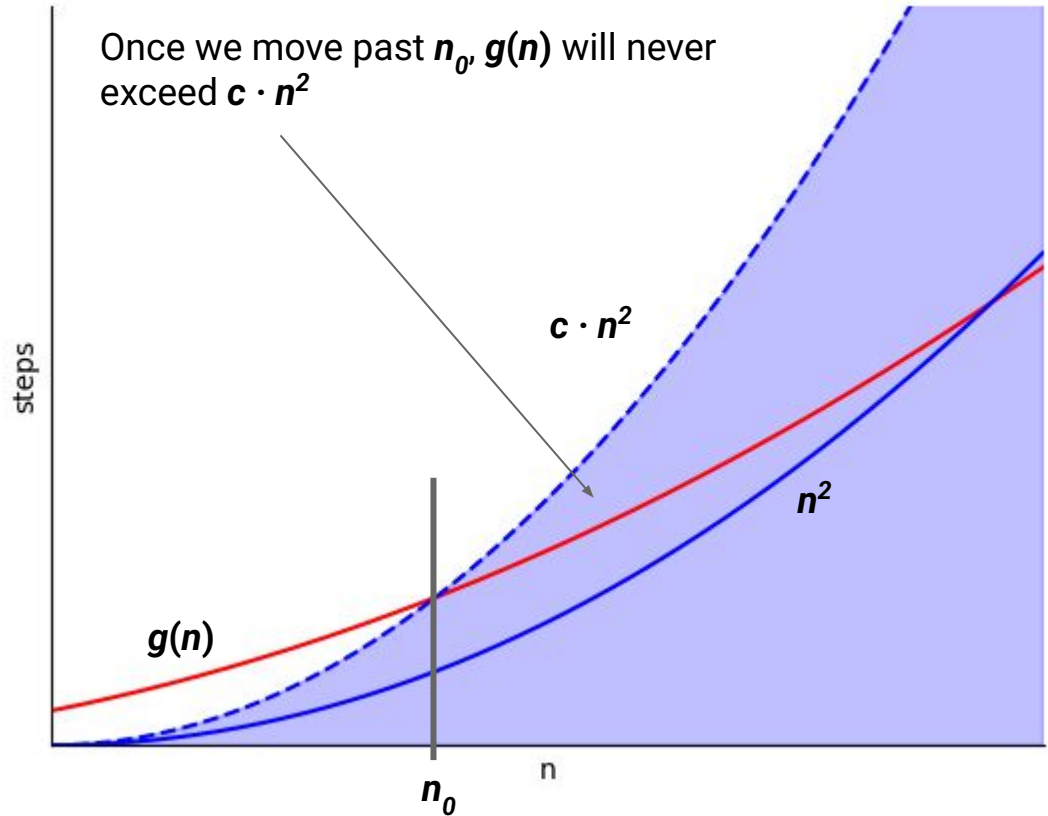
# Bounded from Above: Big O



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# Bounded from Above: Big O



# Bounded from Below: Big $\Omega$

$g(n)$  is bounded from below by  $f(n)$  if:

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$$\text{For all } n \geq n_0, g(n) \geq c \cdot f(n)$$

In this case, we say that  $g(n) \in \Omega(f(n))$

$\Omega(f(n))$  is the set of all functions bounded from below by  $f(n)$

# Bounded from Below: Big $\Omega$

$$g(n) = \frac{n^2}{2} + 4n + 7$$

Prove that  $g(n) \in \Omega(n^2)$

$$\frac{n^2}{2} + 4n + 7 \geq c \cdot n^2$$



# Bounded from Below: Big $\Omega$

$$\boxed{\frac{n^2}{2}} + 4n + 7 \geq c \cdot n^2$$

We'll start with a similar approach...

$$\frac{n^2}{2} \geq c_1 \cdot n^2$$

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$$\frac{n^2}{2} \geq c_1 \cdot n^2$$

This is true for all  $n \geq 0$  if we set  $c_1$  to  $1/2$

# Bounded from Below: Big $\Omega$

$$\frac{n^2}{2} + 4n + 7 \geq c \cdot n^2$$

Now that we've shown this...what else do we need to show for the overall equation to be true?

$$\frac{n^2}{2} \geq c_1 \cdot n^2$$

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$$\frac{n^2}{2} + 4n + 7 \geq c \cdot n^2$$

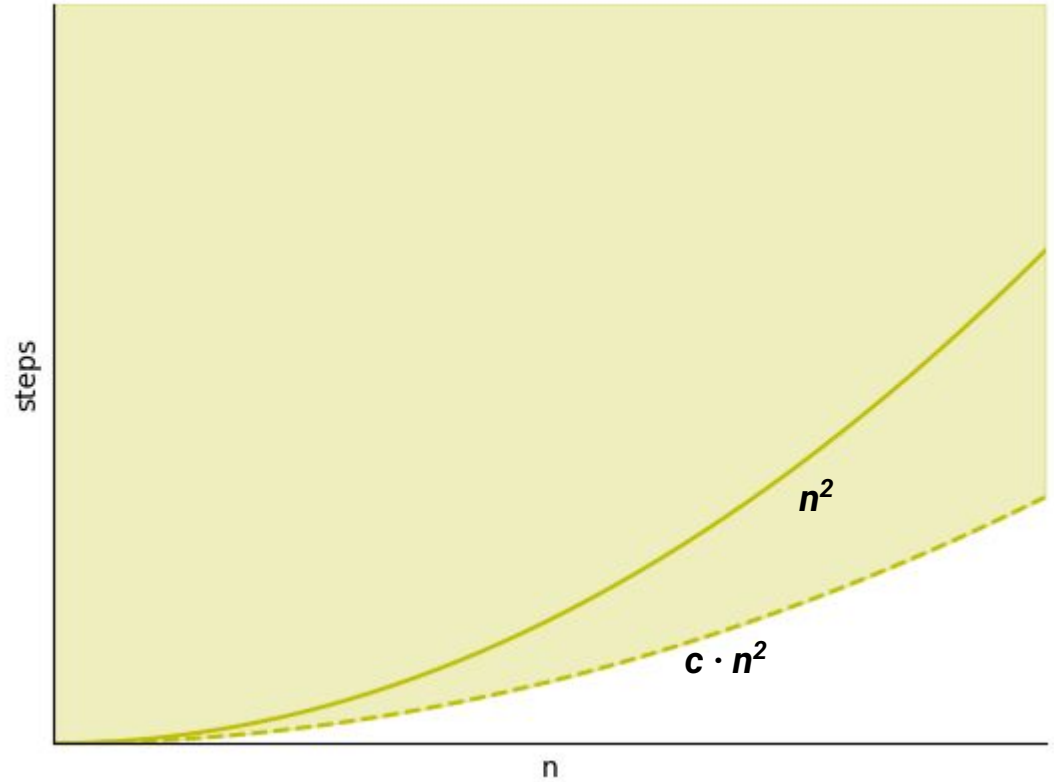
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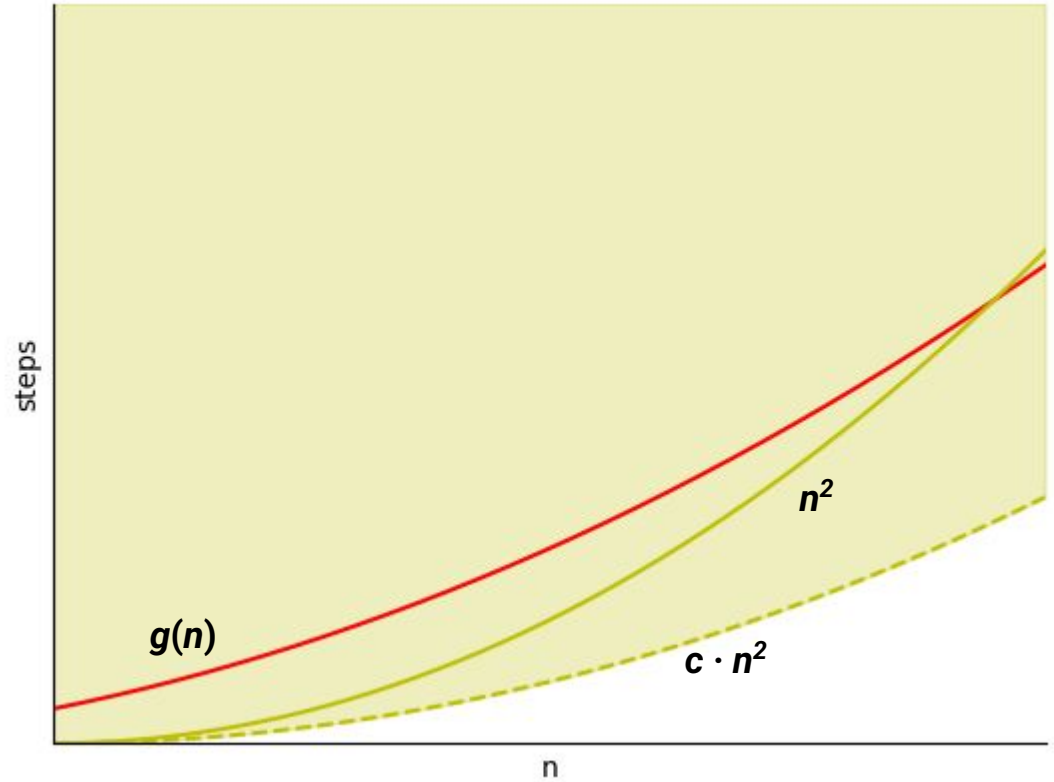
**Just need to show that  $4n$  and  $7$  are  $\geq 0$**

By adding non-negative things to the first term we can only make it bigger!

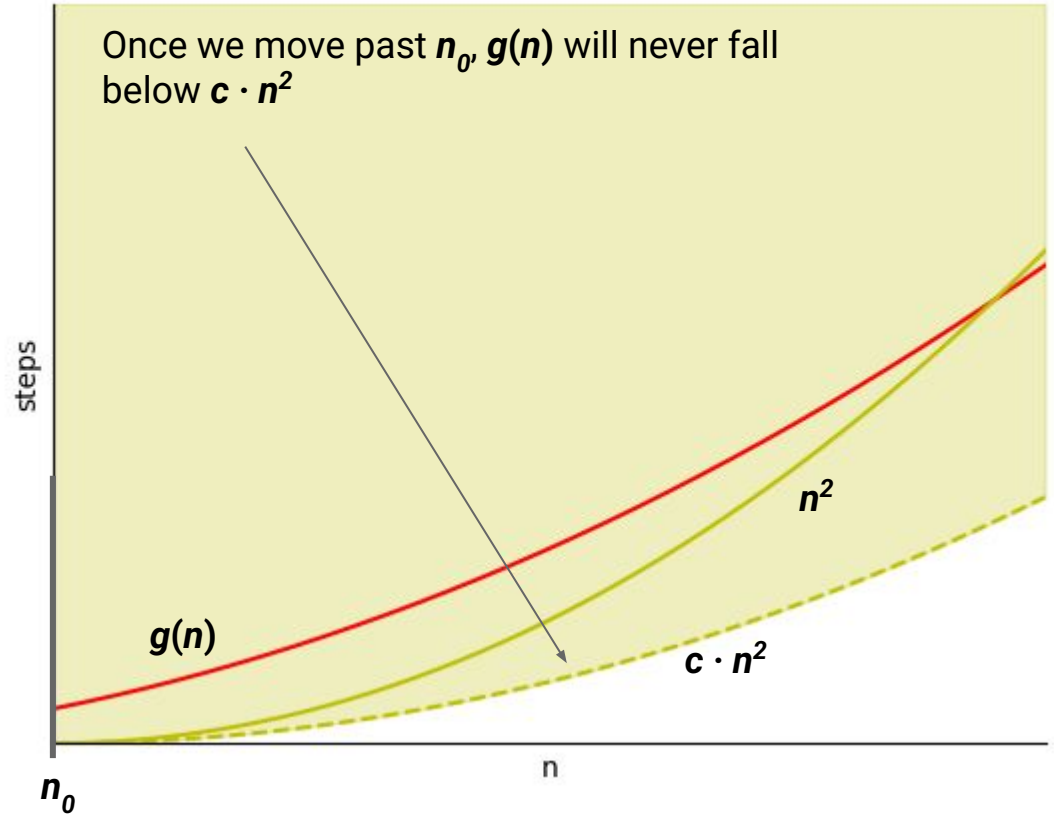
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# Complexity Class: Big $\Theta$

$f$  and  $g$  are in the same complexity class, denoted  $g(n) \in \Theta(f(n))$ , iff:

$g$  is bounded from above by something  $f$ -shaped

*and*

$g$  is bounded from below by something  $f$ -shaped

# Complexity Class: Big $\Theta$

$f$  and  $g$  are in the same complexity class, denoted  $g(n) \in \Theta(f(n))$ , iff:

$$g(n) \in O(f(n))$$

*and*

$$g(n) \in \Omega(f(n))$$

# Bounded from Below: Big $\Theta$

$$g(n) = \frac{n^2}{2} + 4n + 7$$

Prove that  $g(n) \in \Theta(n^2)$

# Bounded from Below: Big $\Theta$

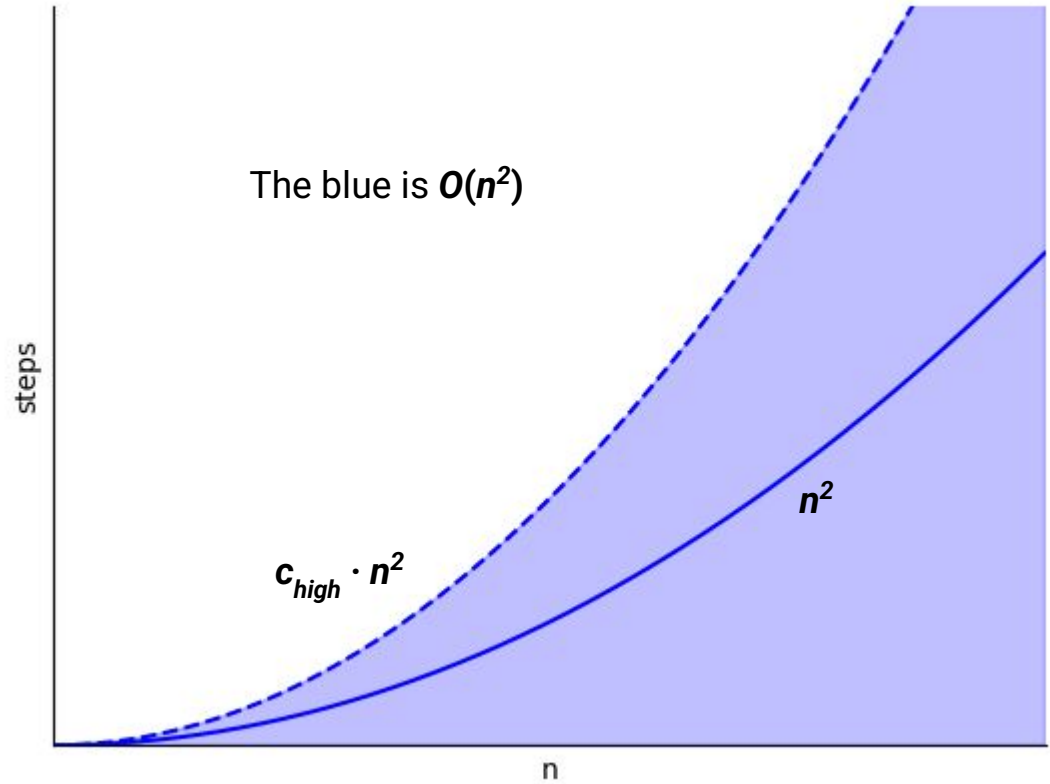
$$g(n) = \frac{n^2}{2} + 4n + 7$$

Prove that  $g(n) \in \Theta(n^2)$

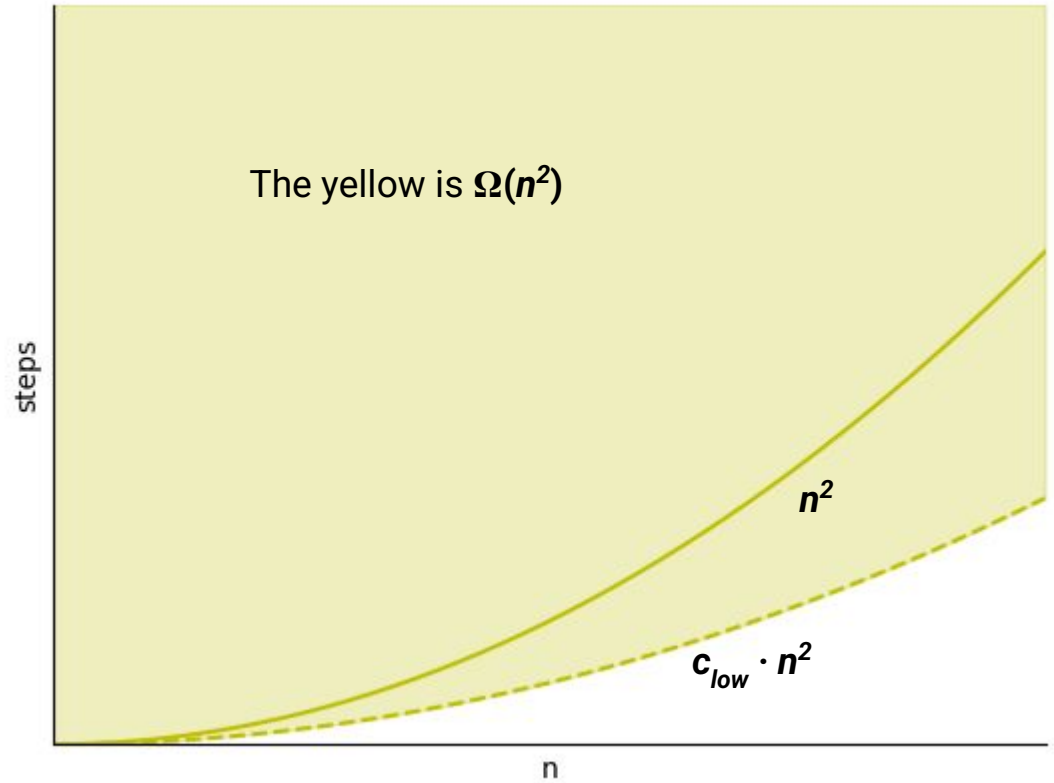
We just proved that  $g(n) \in O(n^2)$  and  $g(n) \in \Omega(n^2)$

Therefore we have proven that  $g(n) \in \Theta(n^2)$

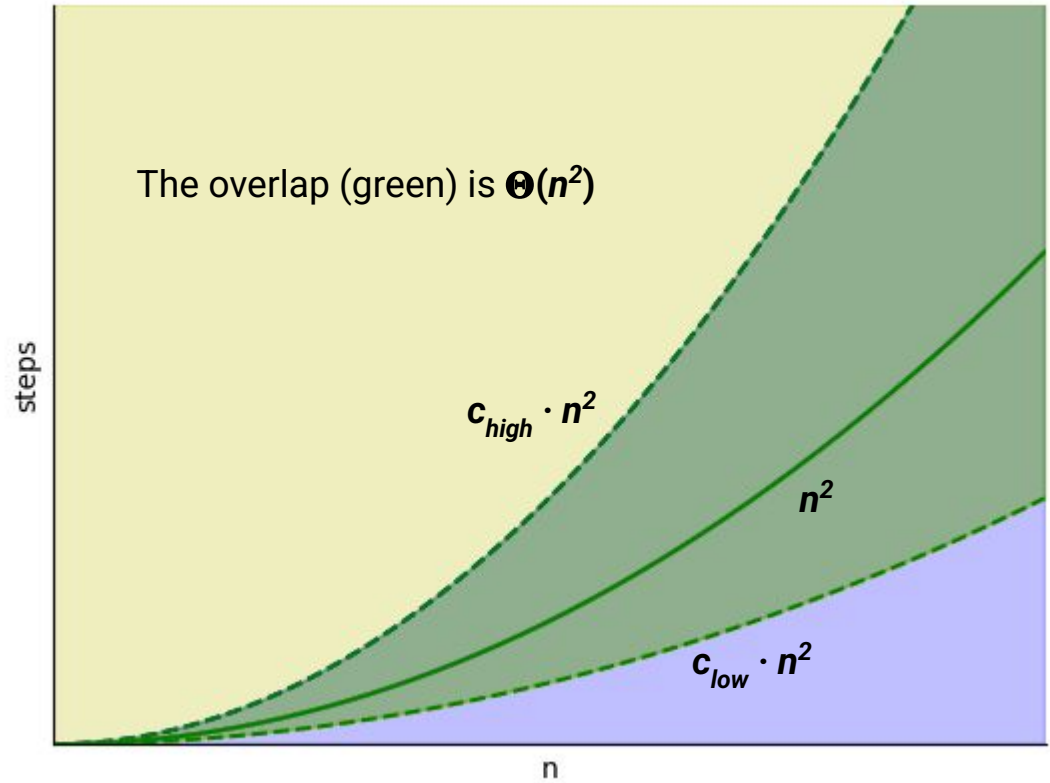
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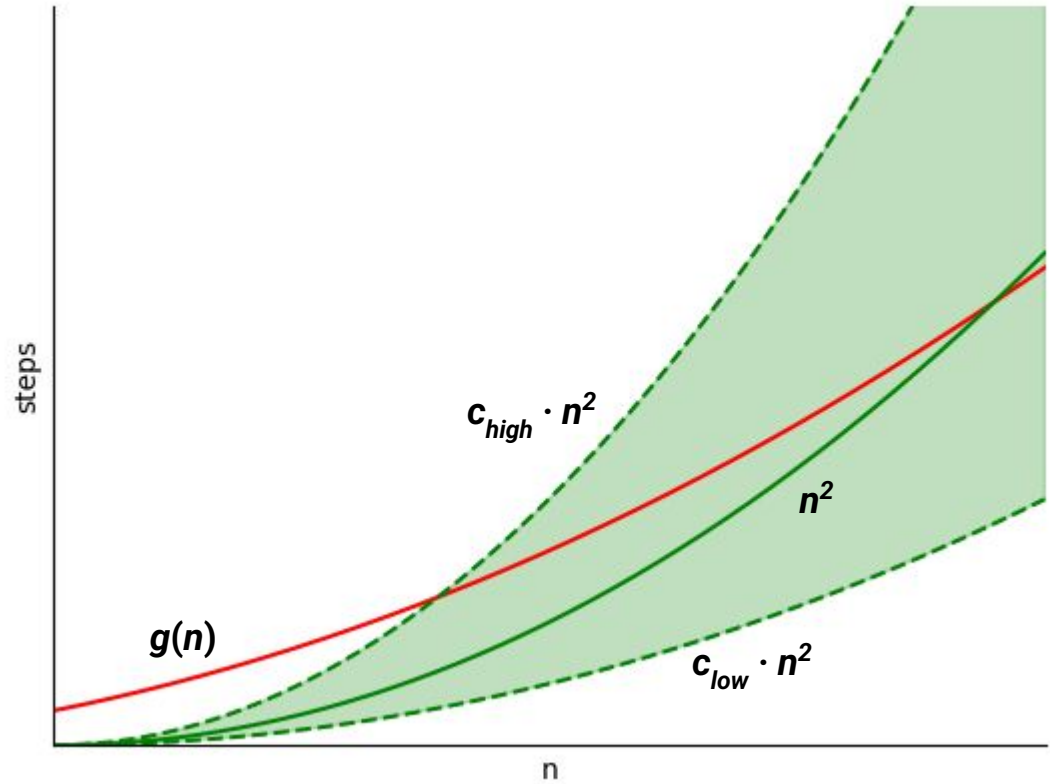
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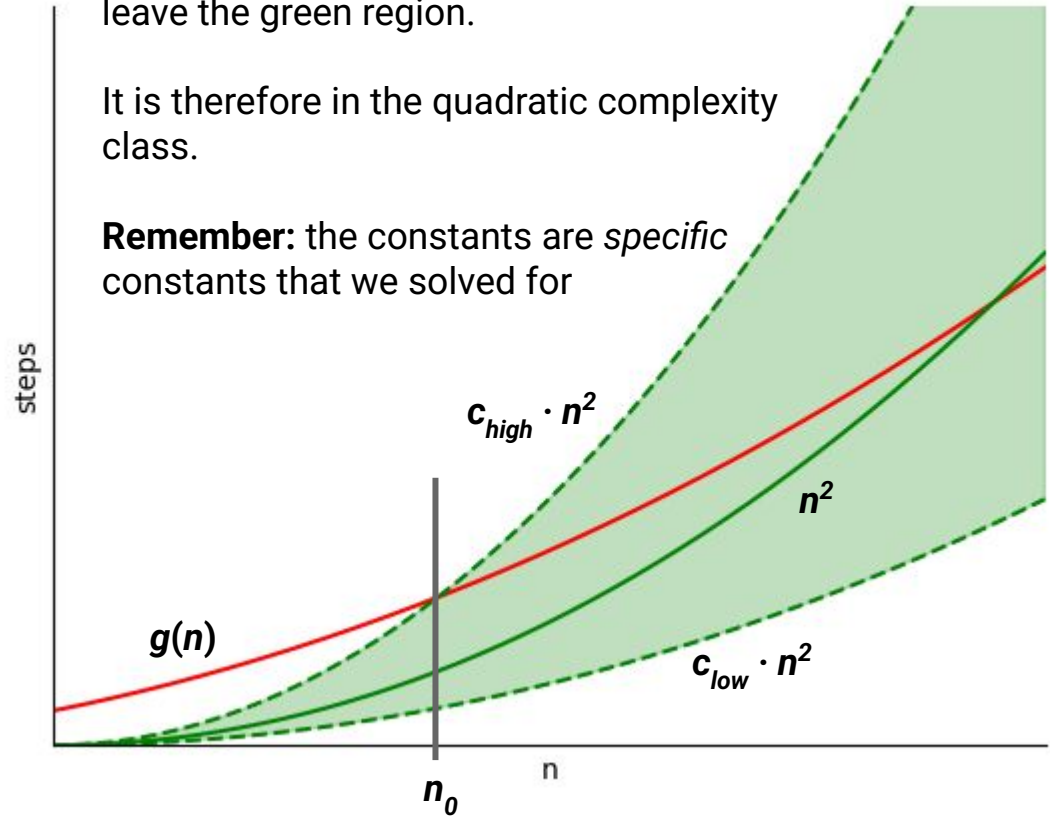


# Complexity Class: Big $\Theta$

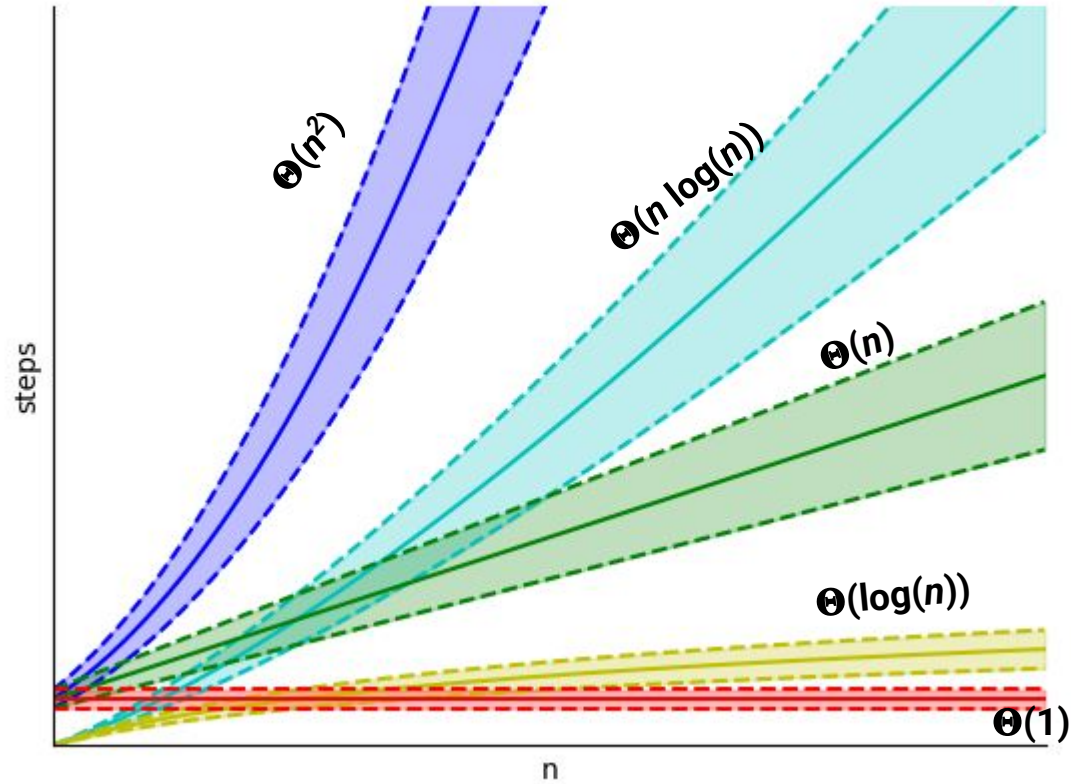
Once we move past  $n_0$ ,  $g(n)$  will never leave the green region.

It is therefore in the quadratic complexity class.

**Remember:** the constants are *specific* constants that we solved for

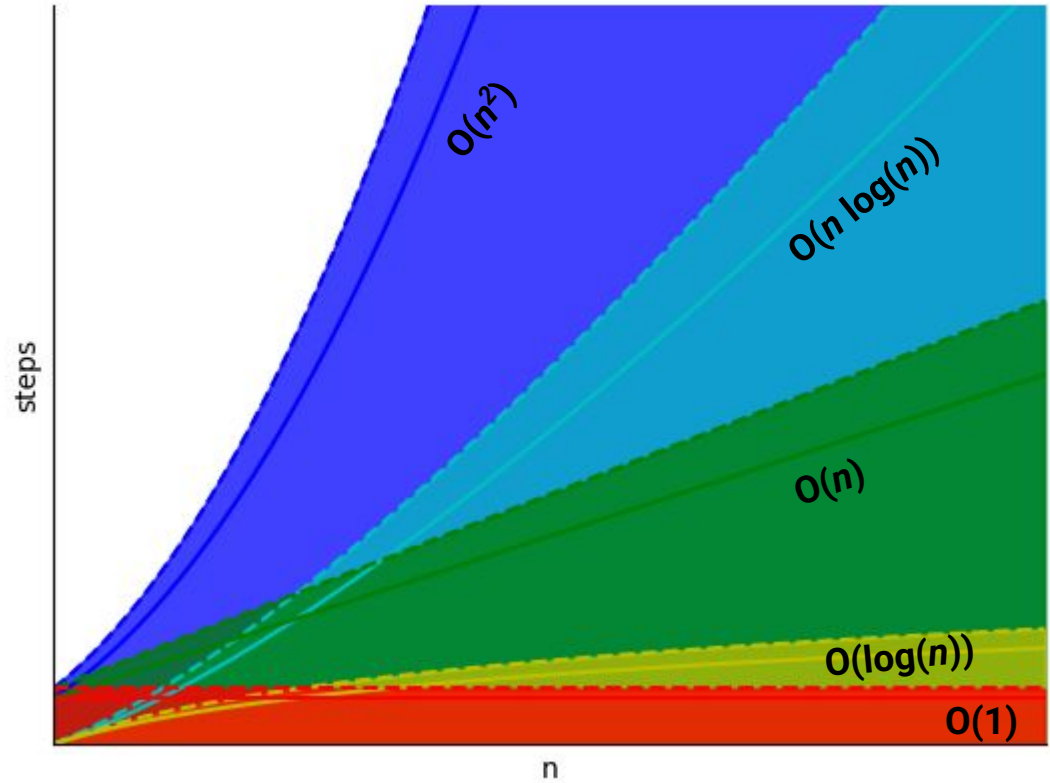


# Complexity Class Ranking



$$\Theta(1) < \Theta(\log(n)) < \Theta(n) < \Theta(n \log(n)) < \Theta(n^2) < \Theta(n^3) < \Theta(2^n)$$

# Big O Subsets



$$O(1) \subset O(\log(n)) \subset O(n) \subset O(n \log(n)) \subset O(n^2) \subset O(n^3) \subset O(2^n)$$

# Shortcut

What complexity class do each of the following belong to:

$$f(n) = 4n + n^2$$

$$g(n) = 2^n + 4n$$

$$h(n) = 100 n \log(n) + 73n$$

# Shortcut

What complexity class do each of the following belong to:

$$f(n) = 4n + n^2 \in \Theta(n^2)$$

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**Shortcut:** Just consider the complexity of the most dominant term

# Why Focus on Dominating Terms?

$f(n)$	10	20	50	100	1000
$\log(\log(n))$	0.43 ns	0.52 ns	0.62 ns	0.68 ns	0.82 ns
$\log(n)$	0.83 ns	1.01 ns	1.41 ns	1.66 ns	2.49 ns
$n$	2.5 ns	5 ns	12.5 ns	25 ns	0.25 $\mu$ s
$n \log(n)$	8.3 ns	22 ns	71 ns	0.17 $\mu$ s	2.49 $\mu$ s
$n^2$	25 ns	0.1 $\mu$ s	0.63 $\mu$ s	2.5 $\mu$ s	0.25 ms
$n^5$	25 $\mu$ s	0.8 ms	78 ms	2.5 s	<b>2.9 days</b>
$2^n$	0.25 $\mu$ s	0.26 ms	<b>3.26 days</b>	<b><math>10^{13}</math> years</b>	<b><math>10^{284}</math> years</b>
$n!$	0.91 ms	<b>19 years</b>	<b><math>10^{47}</math> years</b>	<b><math>10^{141}</math> years</b>	 55

# Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$ , therefore  $f(n) = 4n + n^2 \in O(n^2)$  ✓



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Is  $f(n)$  in  $O(n^3)$ ?

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Is  $f(n)$  in  $O(n)$ ? ✗

$n^2$ ,  $n^3$ , and  $2^n$  all bound  $f(n)$  from above  
 $n^2$  is a tight upper bound of  $f(n)$   
(there is no smaller upper bound for  $f(n)$ )

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Is  $f(n)$  in  $\Omega(n^3)$ ?

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$n^2$ ,  $n$ , and  $\log(n)$  all bound  $f(n)$  from below  
 $n^2$  is a tight lower bound of  $f(n)$   
(there is no larger lower bound for  $f(n)$ )

# Tight Bounds

If  $g(n) \in \Theta(f(n))$ , then:

- $g(n) \in O(f(n))$  is a tight upper bound
- $g(n) \in \Omega(f(n))$  is a tight lower bound



# Tight Bounds

If  $g(n) \in \Theta(f(n))$ , then:

- $g(n) \in O(f(n))$  is a tight upper bound
- $g(n) \in \Omega(f(n))$  is a tight lower bound

But what if the tight upper bound and tight lower bound are not the same?

# Multi-class Functions

$$T(n) = \begin{cases} n^2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

What is the tight upper bound of this function?

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What is the tight lower bound of this function?

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What is the tight lower bound of this function?  $T(n) \in \Omega(n)$

What is the complexity class of this function?

# Multi-class Functions

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What is the tight lower bound of this function?  $T(n) \in \Omega(n)$

What is the complexity class of this function? It does not have one!

# Multi-class Functions

$$T(n) = \begin{cases} n^2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

It is not bounded from above by  $n$ ,  
therefore it cannot be in  $\Theta(n)$

It is not bounded from below by  $n^2$ ,  
therefore it cannot be in  $\Theta(n^2)$

What is the tight upper bound of this function?  $T(n) \in O(n^2)$

What is the tight lower bound of this function?  $T(n) \in \Omega(n)$

What is the complexity class of this function? It does not have one!