CSE 250
Data Structures

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Lec 10: ArrayList and Amortized Runtime

Announcements

- PA1 Implementation due Sunday @ 11:59PM
- WA2 will be released after the PA1 deadline

The List ADT

```
1  public interface List<E>
2      extends Sequence<E> { // Everything a sequence has, and...
3    /** Extend the sequence with a new element at the end */
4    public void add(E value);
5
6    /** Extend the sequence by inserting a new element */
7    public void add(int idx, E value);
8
9    /** Remove the element at a given index */
10   public void remove(int idx);
11 }
```

List Runtimes (so far…)

	ArrayList	Linked List (by index)	Linked List (by reference)
`get(...)`	$\Theta(1)$	$\Theta(idx)$ or $O(n)$	$\Theta(1)$
`set(...)`	$\Theta(1)$	$\Theta(idx)$ or $O(n)$	$\Theta(1)$
`size()`	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
`add(...)`	TBD	$\Theta(idx)$ or $O(n)$	$\Theta(1)$
`remove(...)`	TBD	$\Theta(idx)$ or $O(n)$	$\Theta(1)$

ArrayLists

Question: How can we implement **`add(e)`** on an **`ArrayList`**?

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Problem: Arrays have a fixed size!

ArrayLists - Attempt #1

1. Allocate a new array of size $n + 1$ — $O(1)$
2. Copy all n elements to the new array — $O(n)$
3. Insert the new item at position n — $O(1)$

Total: $O(n)$

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Can we do better?

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Total: $O(n)$

Can we do better?

Idea: Keep extra space and track how many elements you have

ArrayList Representation

```
1  class ArrayList<T> extends List<T> {
2    private int used;
3    private Optional<T>[] data;
4
5    public int size() { return used; }
6
7    /* ... */
8  }
```

ArrayList Representation

```
1  public T get(int i) {
2    if (i < 0 || i >= used) { throw new IndexOutOfBoundsException(i); }
3    return data[i].get()
4  }
5
6  public void set(int i, T value) {
7    if (i < 0 || i >= used) { throw new IndexOutOfBoundsException(i); }
8    data[i] = Optional.of(value);
9  }
```

ArrayList Representation

The methods from the Sequence ADT are all still $\Theta(1)$

What about `remove`?

ArrayList - remove(i)

```
1  public void remove(int i) {
2    /* Sanity-check inputs */
3    if (i < 0 || i >= used) { throw new IndexOutOfBoundsException(i); }
4
5    /* Shift elements left */
6    for (int j = i; j < used - 1; j++) {
7      data[j] = data[j+1];
8    }
9    data[used-1] = Optional.empty();
10   used--;
   }
```

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   }
```

Have to shift over right-most elements to fill the hole created by the removed element!

Analysis of `remove(i)`

$$T_{\text{remove}}(n) = \begin{cases} 1 & \text{if i} = \text{used - 1} \\ 2 & \text{if i} = \text{used - 2} \\ 3 & \text{if i} = \text{used - 3} \\ \dots & \dots \\ n-1 & \text{if i} = 0 \end{cases}$$

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$$T_{\text{remove}}(n) \in O(n), \Omega(1)$$

ArrayList - `add(elem)`

Basic Idea:

1. If we are out of space, first create a new array of a larger size and copy the old elements over
2. Then, set the element at index **`used`** to **`elem`**

ArrayList - add(elem)

```
1  public void add(T elem) {
2    if(used == data.length) { /* Sad case 😕 */
3      int newLength = ???
4      Optional<T>[] newData = new Optional<T>[newLength];
5      System.arrayCopy(data, 0, newData, 0, data.length);
6      data = newData;
7    }
8    /* Happy case 😀 */
9    data[used] = Optional.of(elem)
10   used++;
11 }
```

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```

How we choose the new length will be important!

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6      data = newData;
7    }
8    /* Happy case 😀 */
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10   used++;
11 }
```

This is the expensive part!

Analysis of `add(elem)`

$$T_{\text{add}}(n) = \begin{cases} 1 & \text{if used} < \text{data.length} \\ n & \text{if used} = \text{data.length} \end{cases}$$

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How often do our **add** calls require ***O*(*n*)** time?

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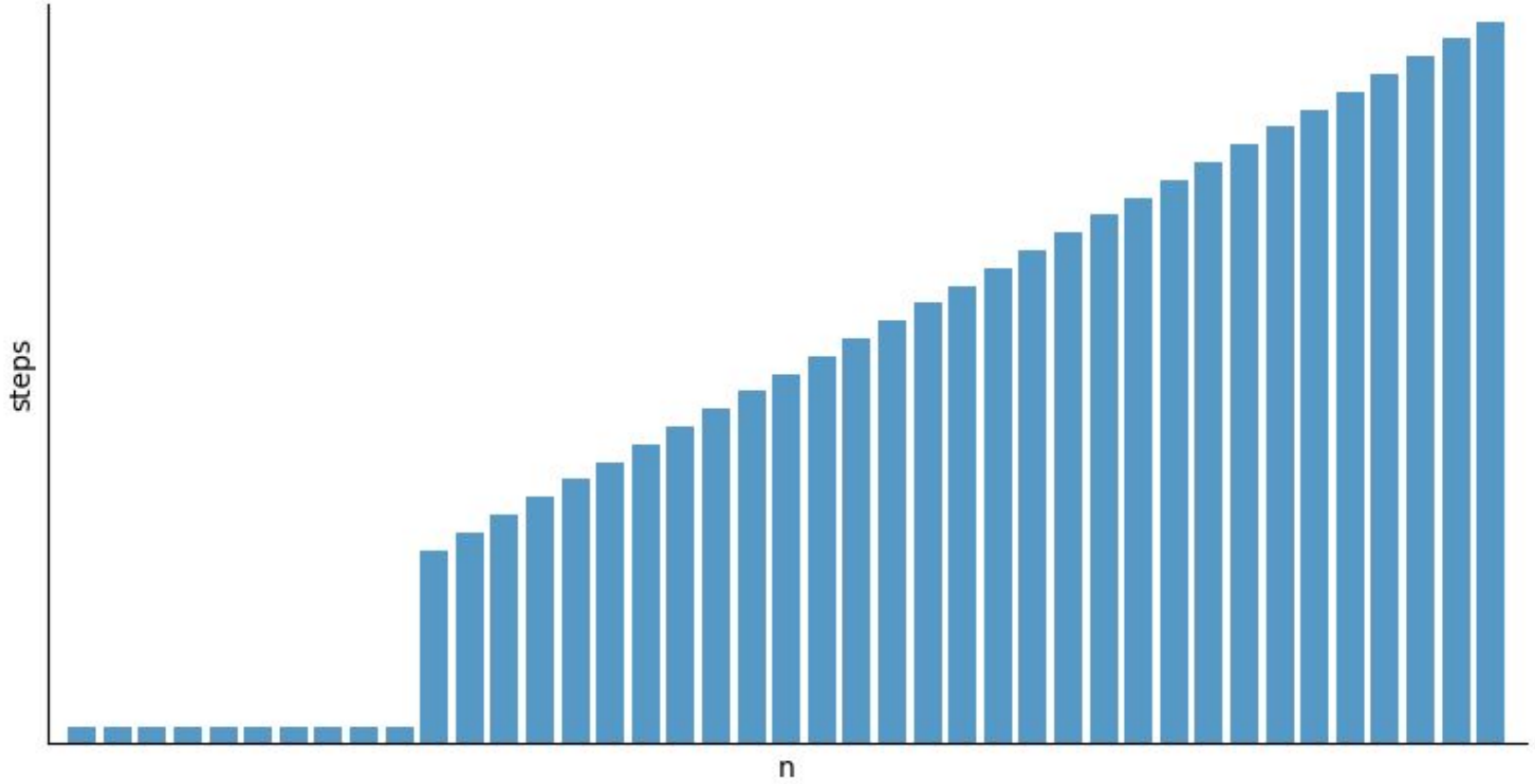
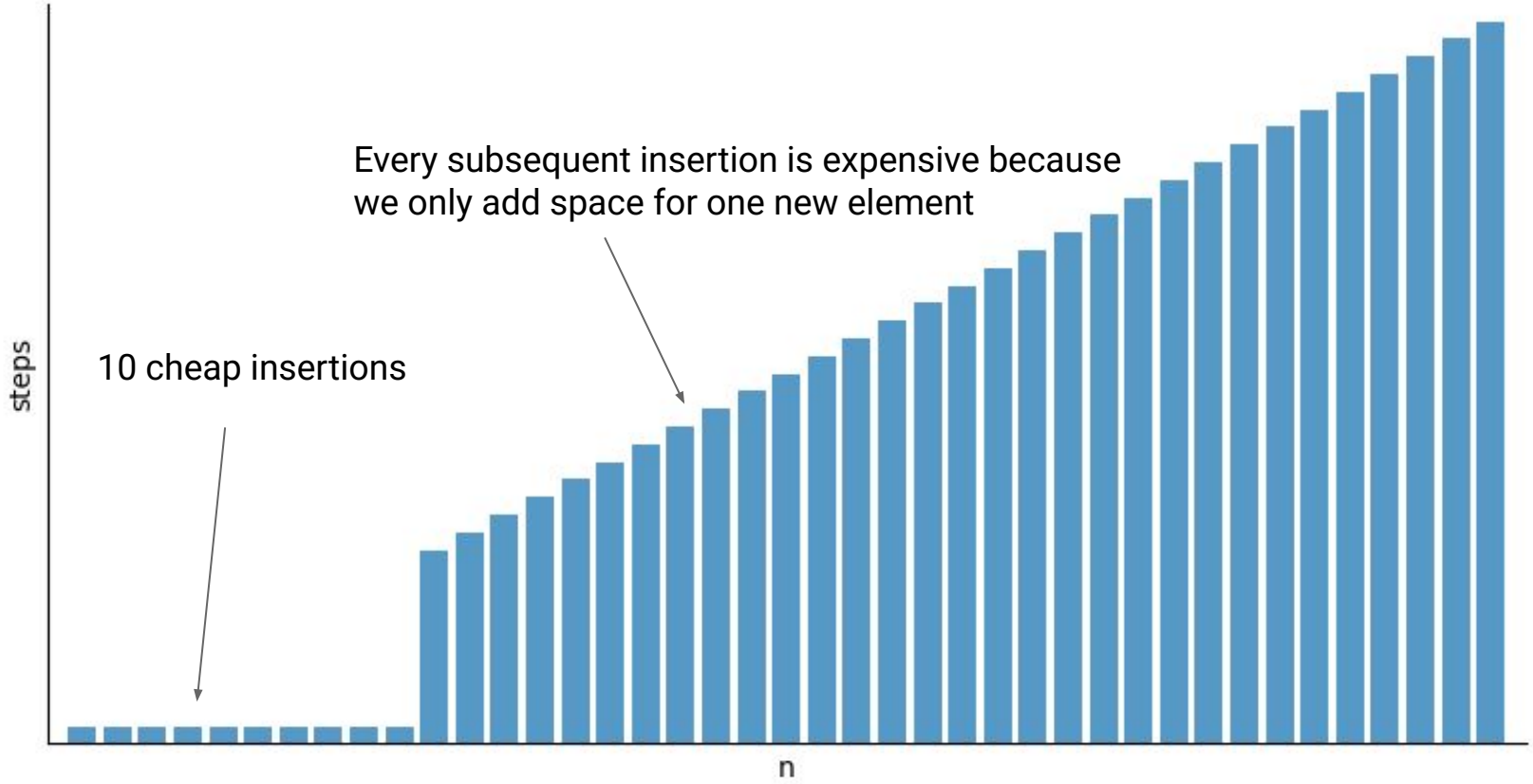
How often do our **`add`** calls require ***O*(*n*)** time?

It depends on how we calculate **`newLength`**



Initial size = 10, newLength = data.length + 1



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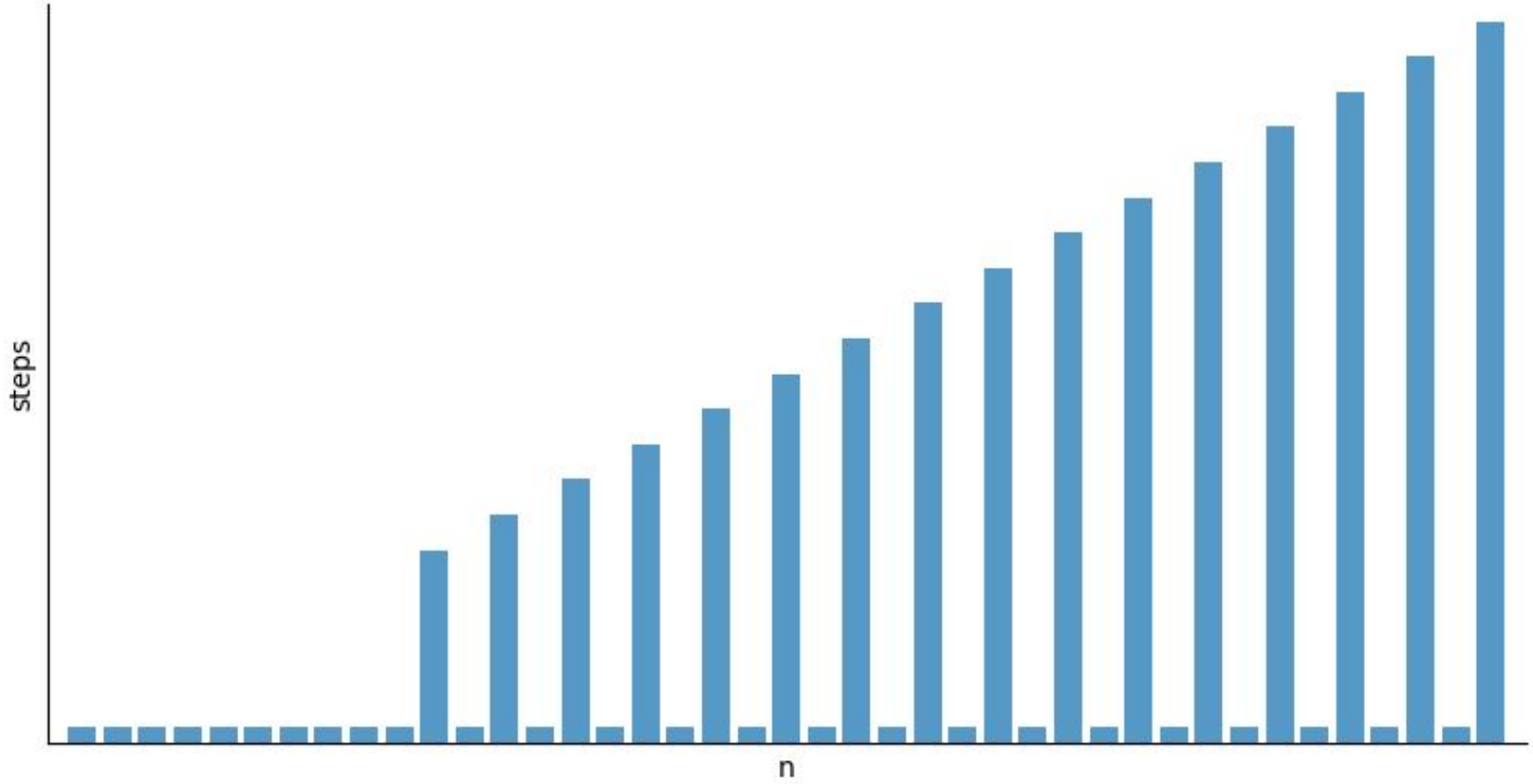

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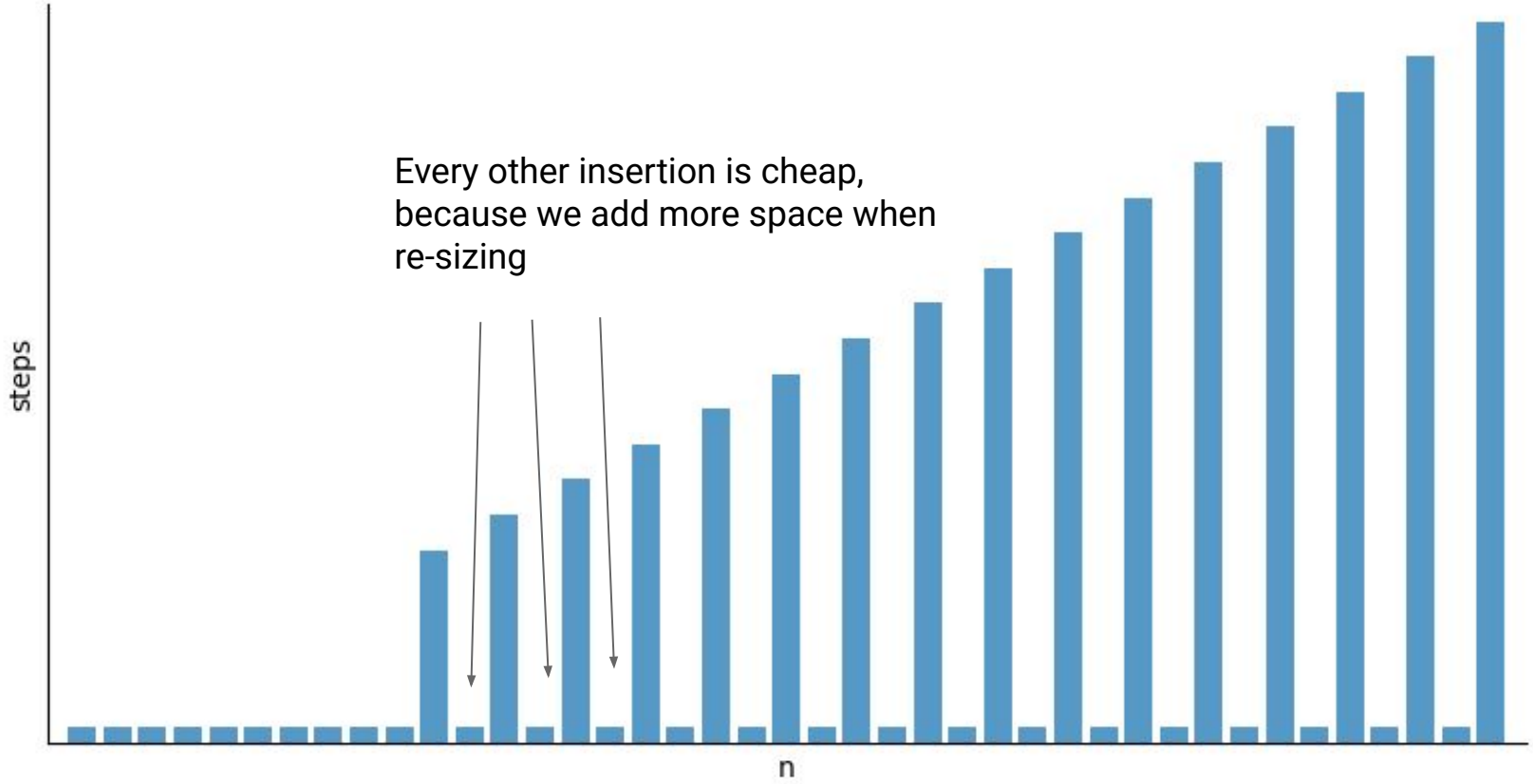

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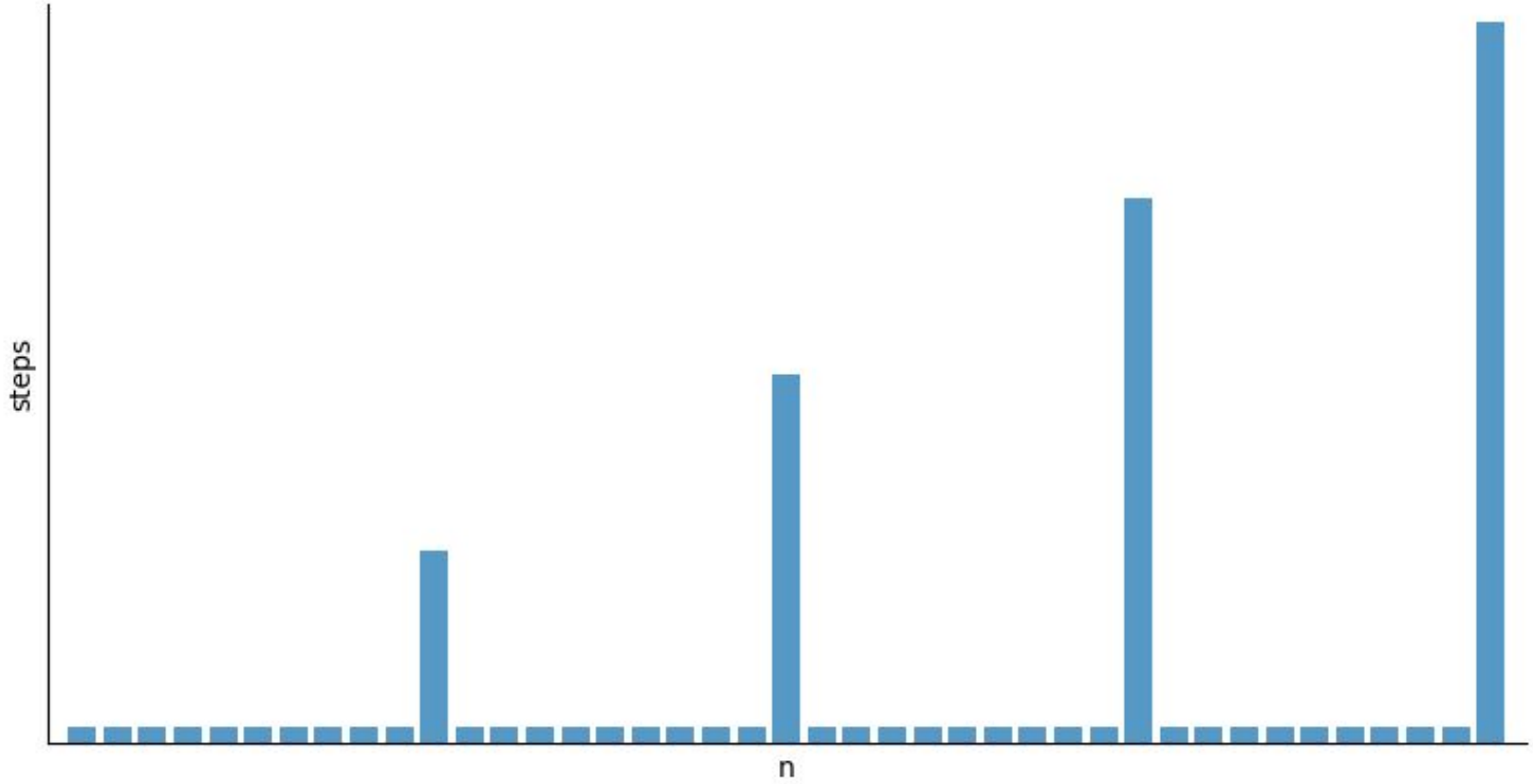

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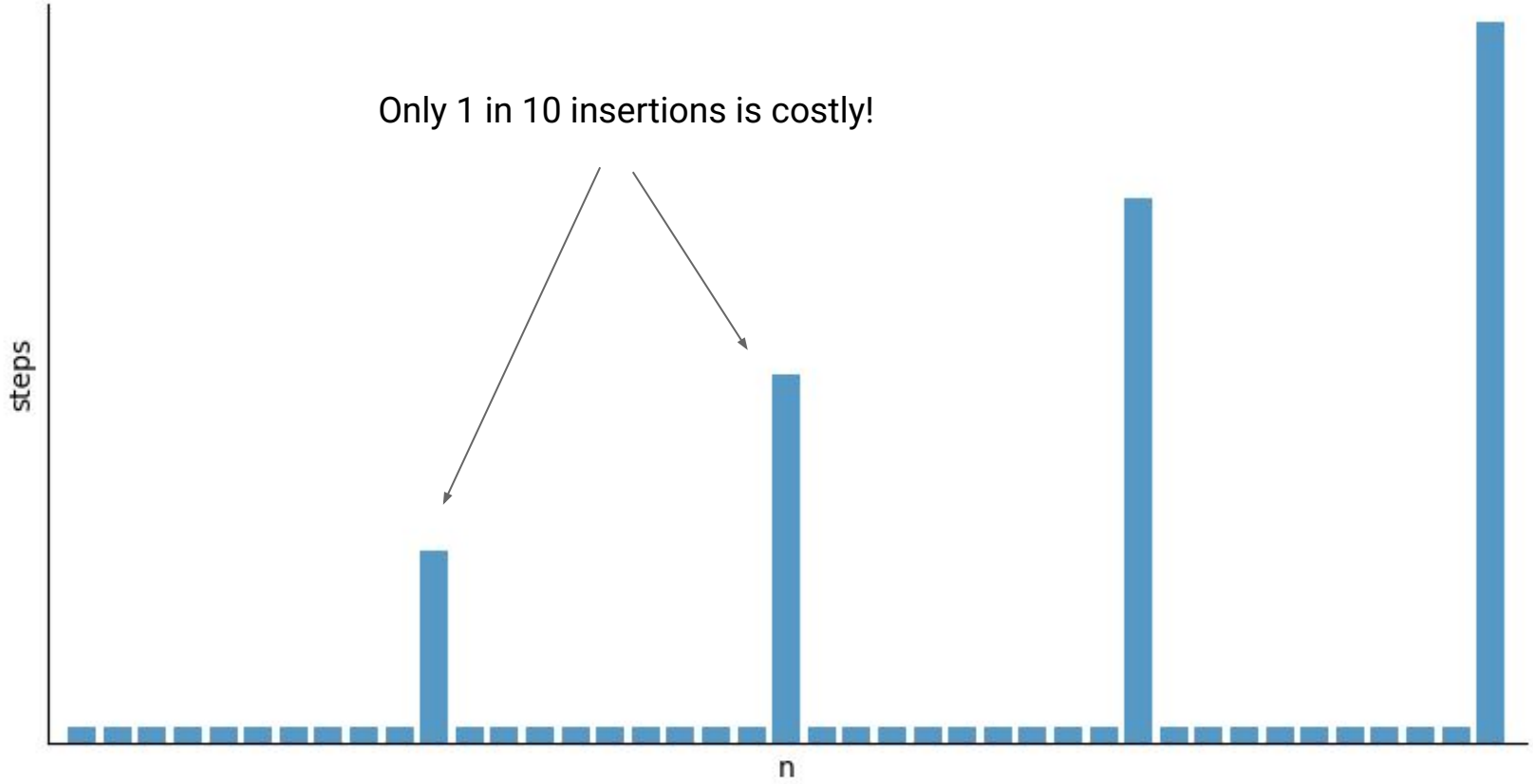

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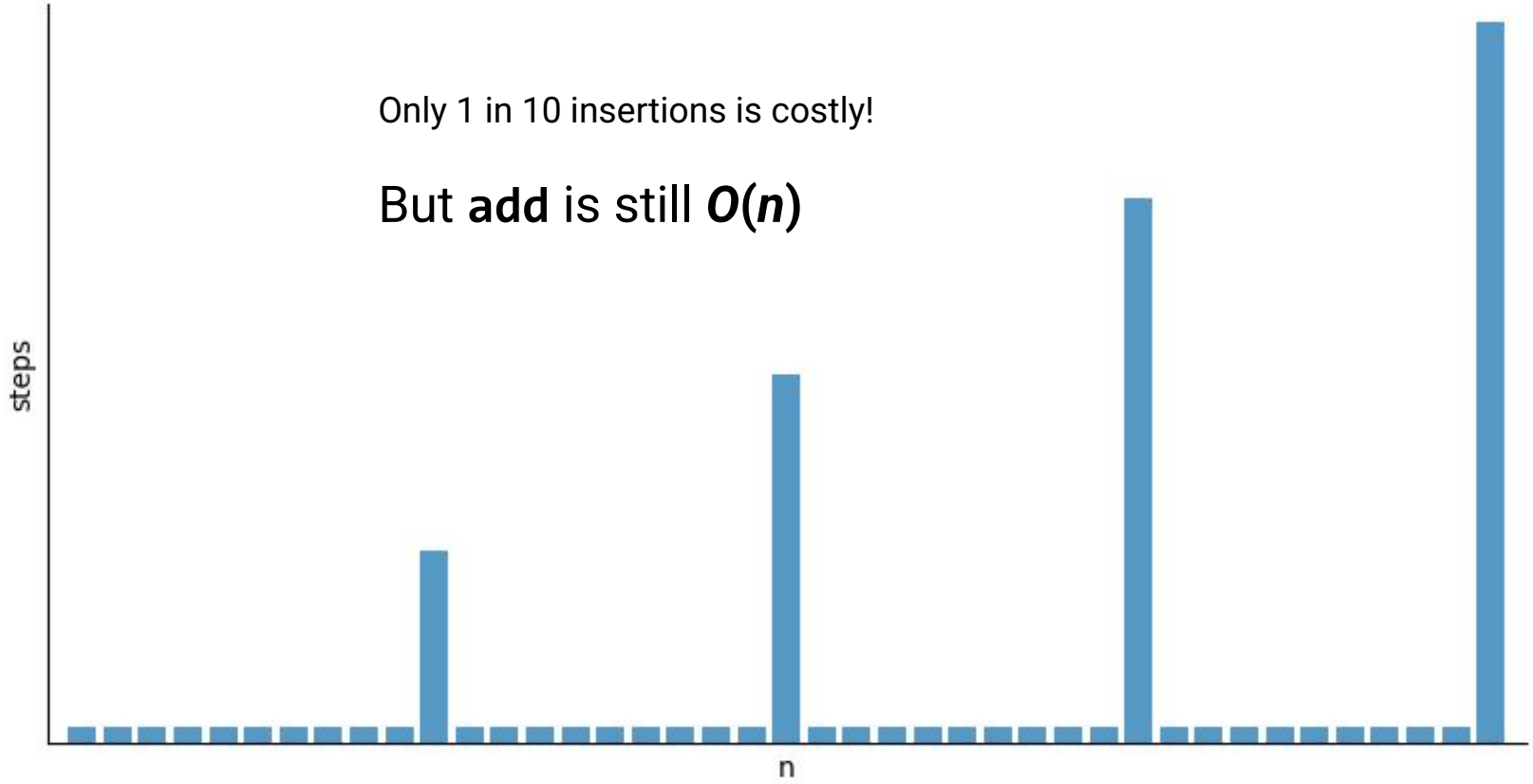

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A Note on Runtime Complexity

So far, when we've discussed runtime bounds we have done so without taking any extra information/context into account.

For example, the worst-case runtime of **`ArrayList.add`** is $O(n)$

Our analysis doesn't capture the fact that oftentimes it is faster than $O(n)$

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We refer to this as the unqualified runtime...it is the runtime without any extra qualifications, caveats, etc

But, sometimes the extra context can be relevant...
how can we include this in our analysis?

Common Pattern: Repeated Calls

Oftentimes we will want to call a function many times in a row

- ie read through a CSV file and add all records to a List

What can we say about the runtime in this case?

Adding *n* Elements to a LinkedList

```
1 List<Integer> list = new LinkedList<Integer>();
2 for (int i = 0; i < n; i++) {
3   list.add(i);
4 }
```

Adding *n* Elements to a LinkedList

```
1  Θ(1)
2  for (int i = 0; i < n; i++) {
3    Θ(1)
4  }
```

Adding *n* Elements to a LinkedList

```
1  Θ(1)
2  Θ(n)
```

Total Complexity: Θ(n)

Adding *n* Elements to an ArrayList

```
1 List<Integer> list = new ArrayList<Integer>();
2 for (int i = 0; i < n; i++) {
3   list.add(i);
4 }
```

Adding *n* Elements to an ArrayList

```
1  O(1)
2  for (int i = 0; i < n; i++) {
3    O(n)
4  }
```

Adding *n* Elements to an ArrayList

1	$O(1)$
2	$O(n^2)$

Upper Bound: $O(n^2)$

But is this a tight upper bound?

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2	$O(n^2)$

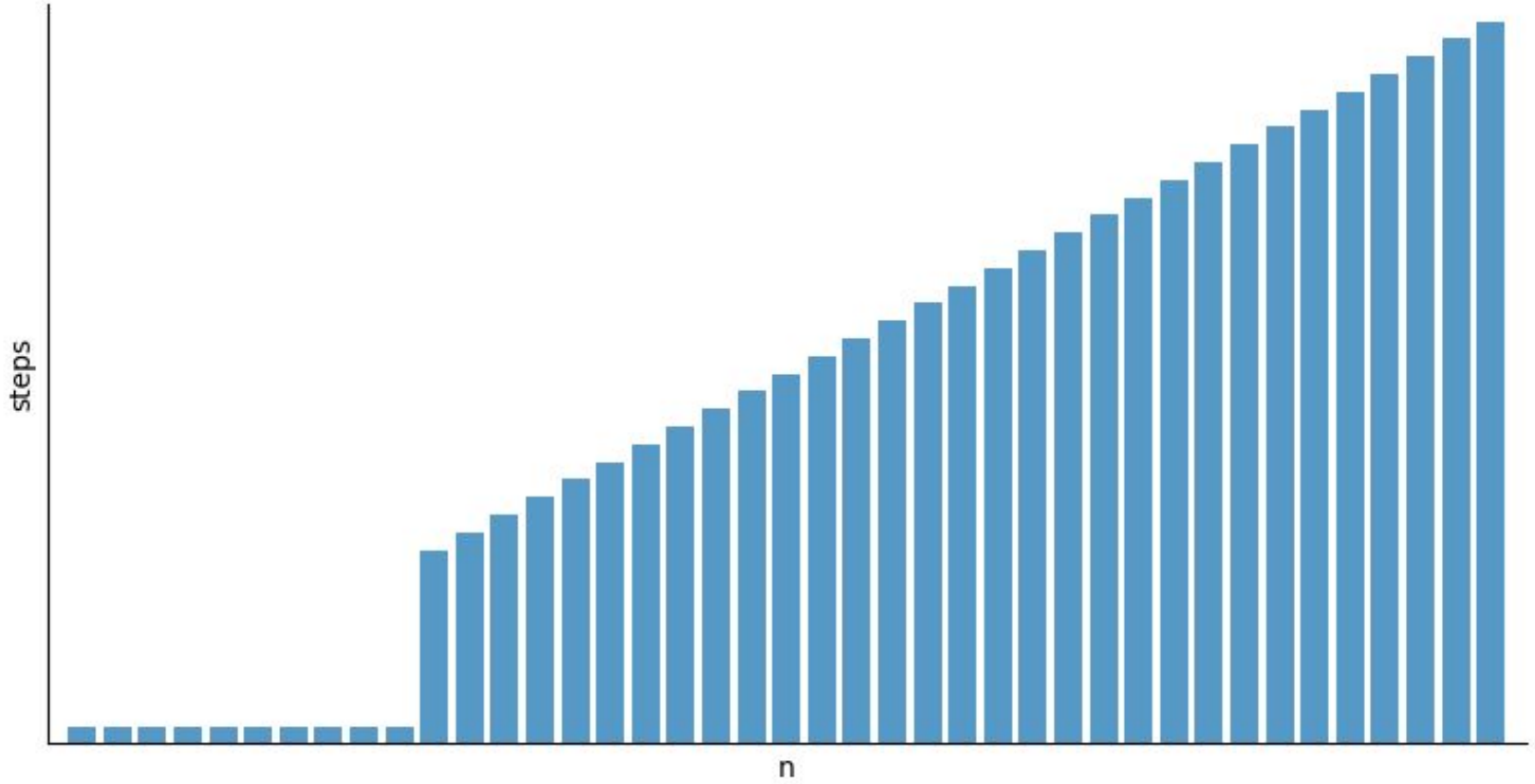
Upper Bound: $O(n^2)$

But is this a tight upper bound? Let's do a more detailed analysis

Adding *n* Elements to an ArrayList

```
1  List<Integer> list = new ArrayList<Integer>();
2  for (int i = 0; i < n; i++) {
3    list.add(i);
4  }
```

How many steps does this loop take IN TOTAL?



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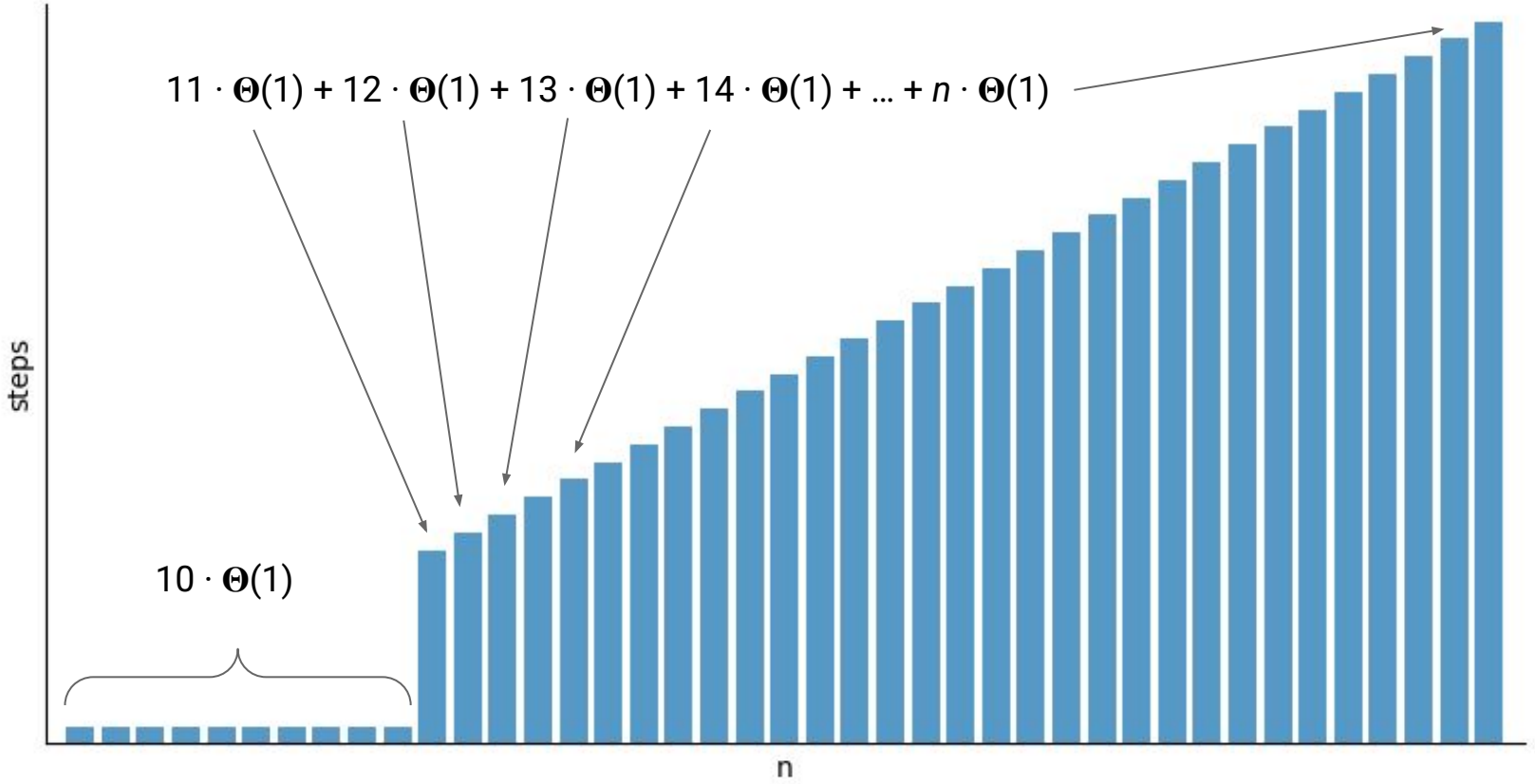

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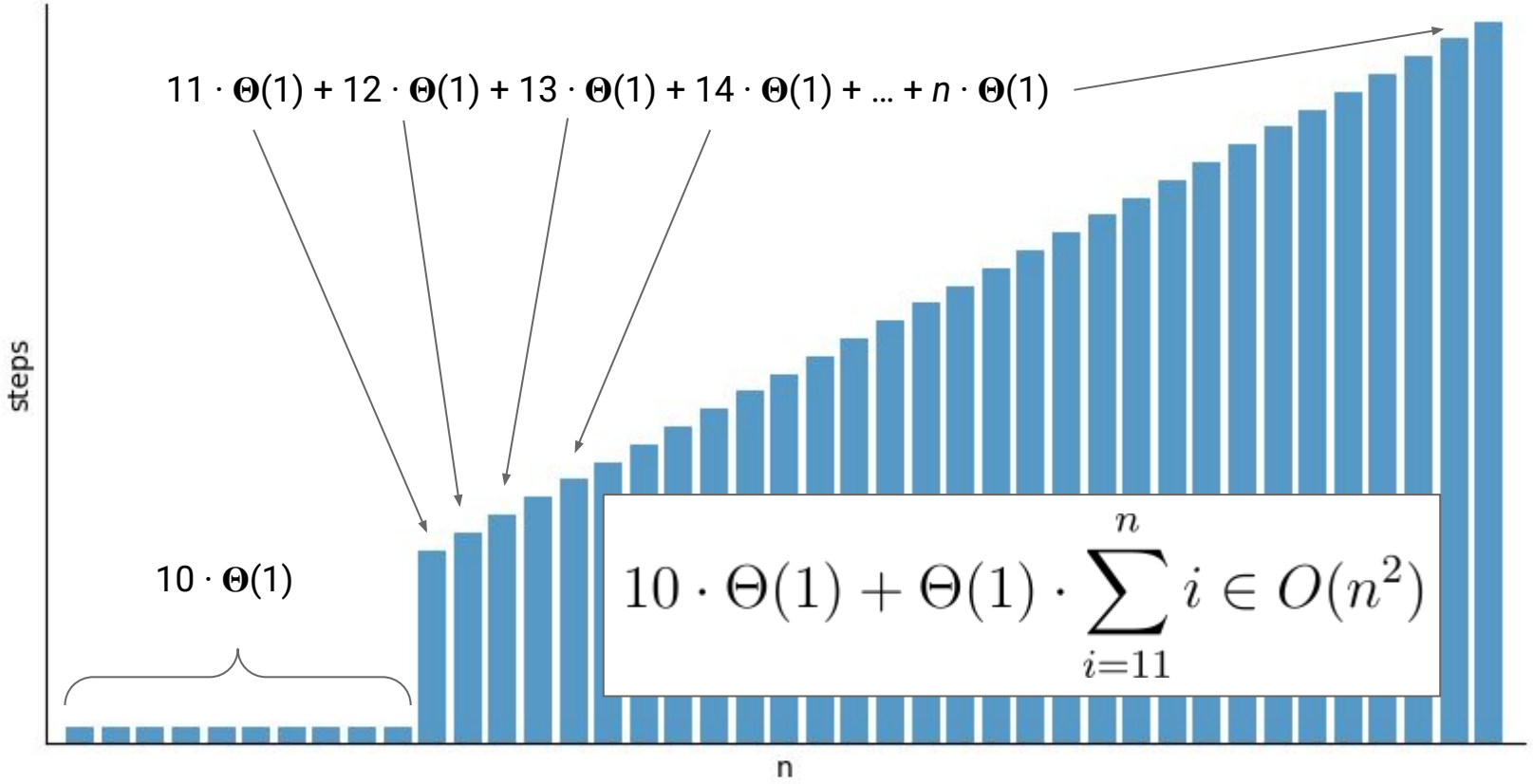

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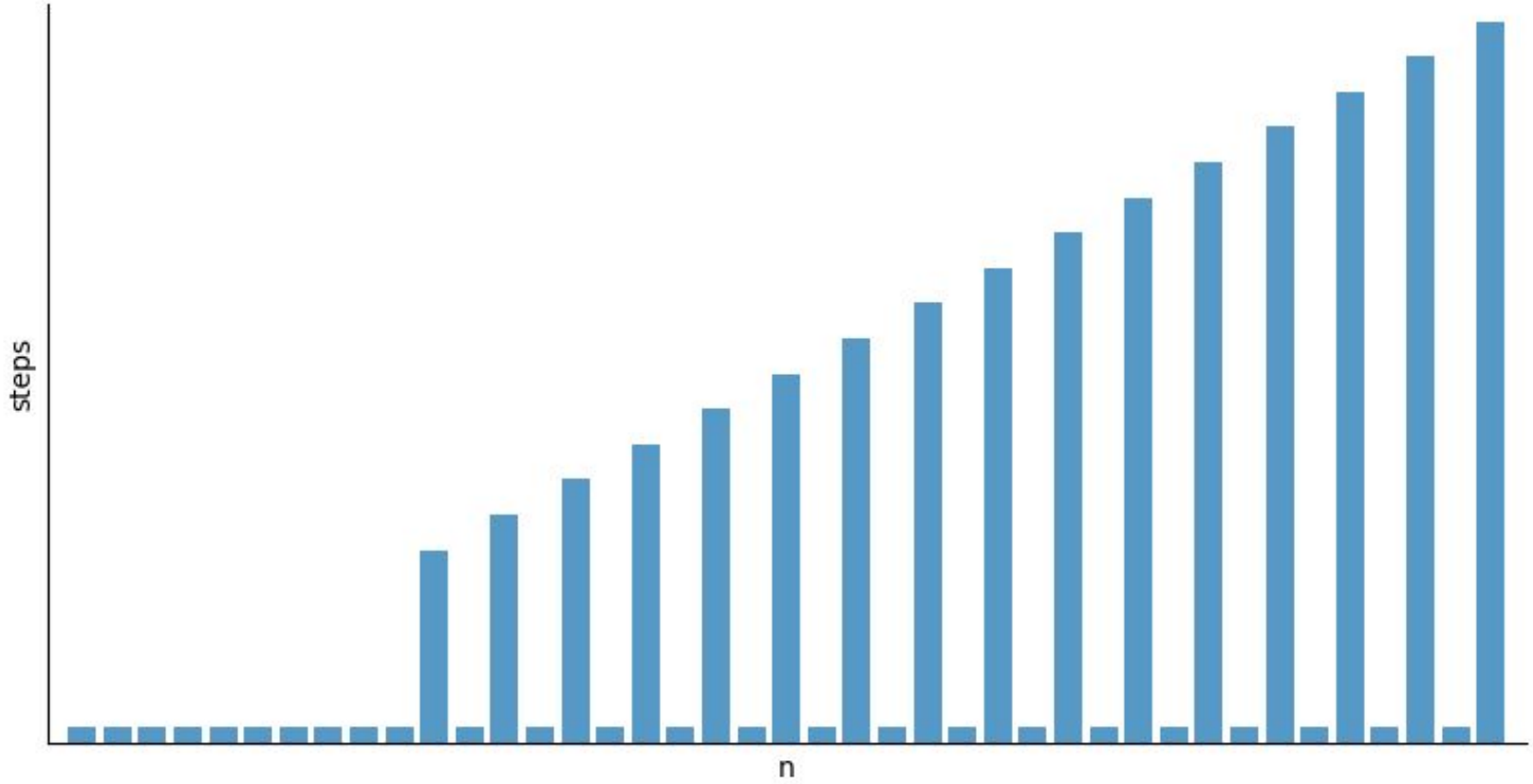
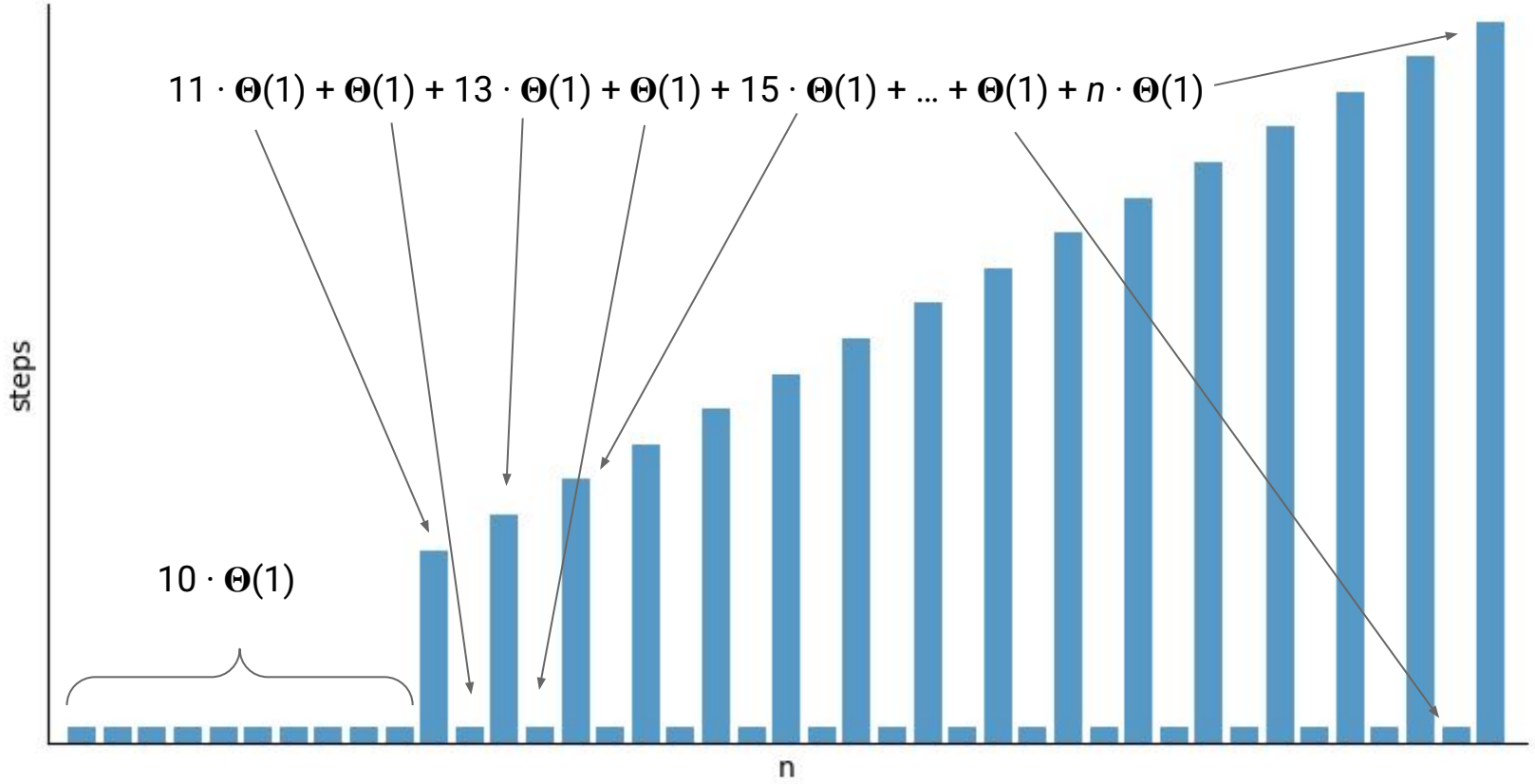


Initial size = 10, newLength = data.length + 2



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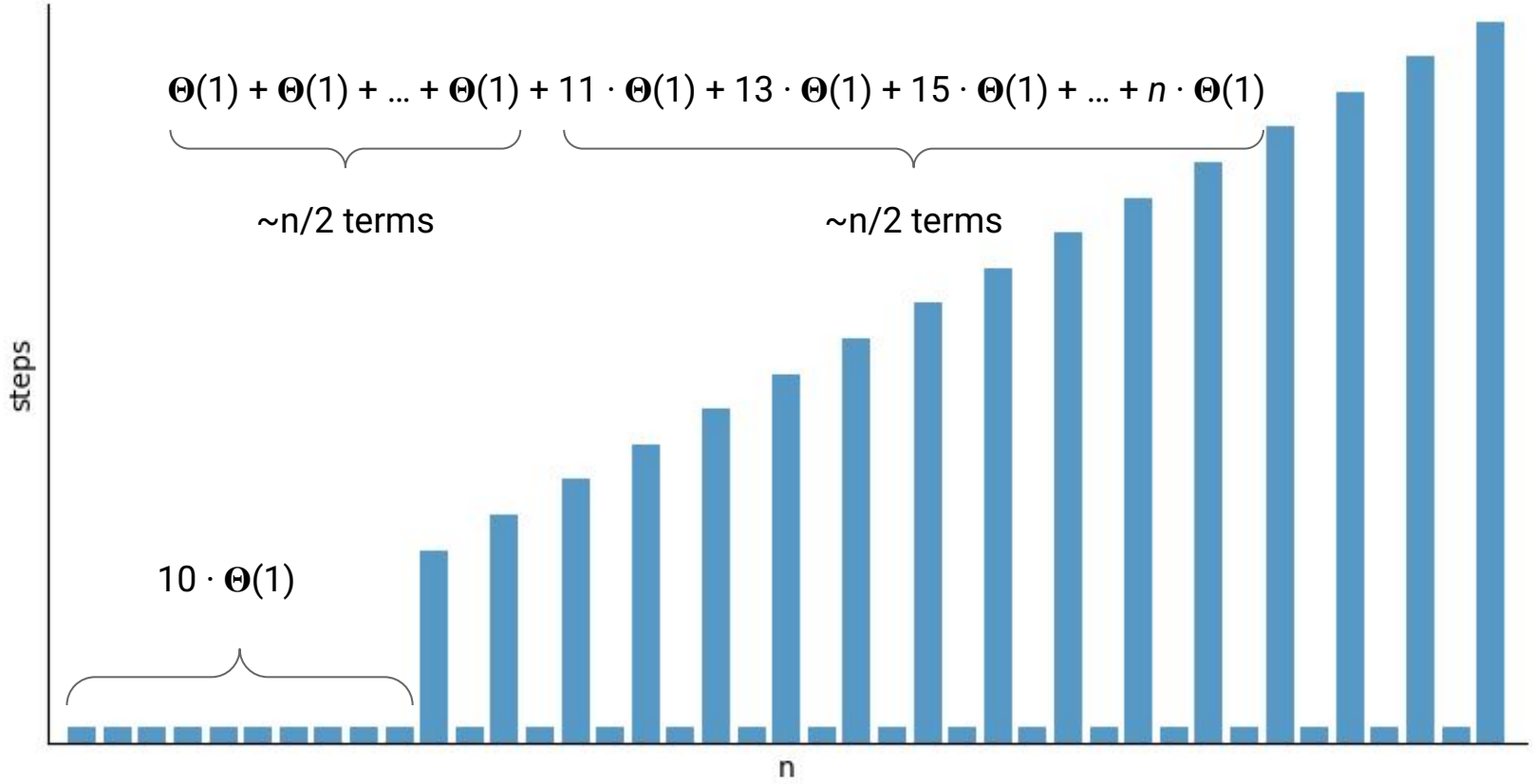

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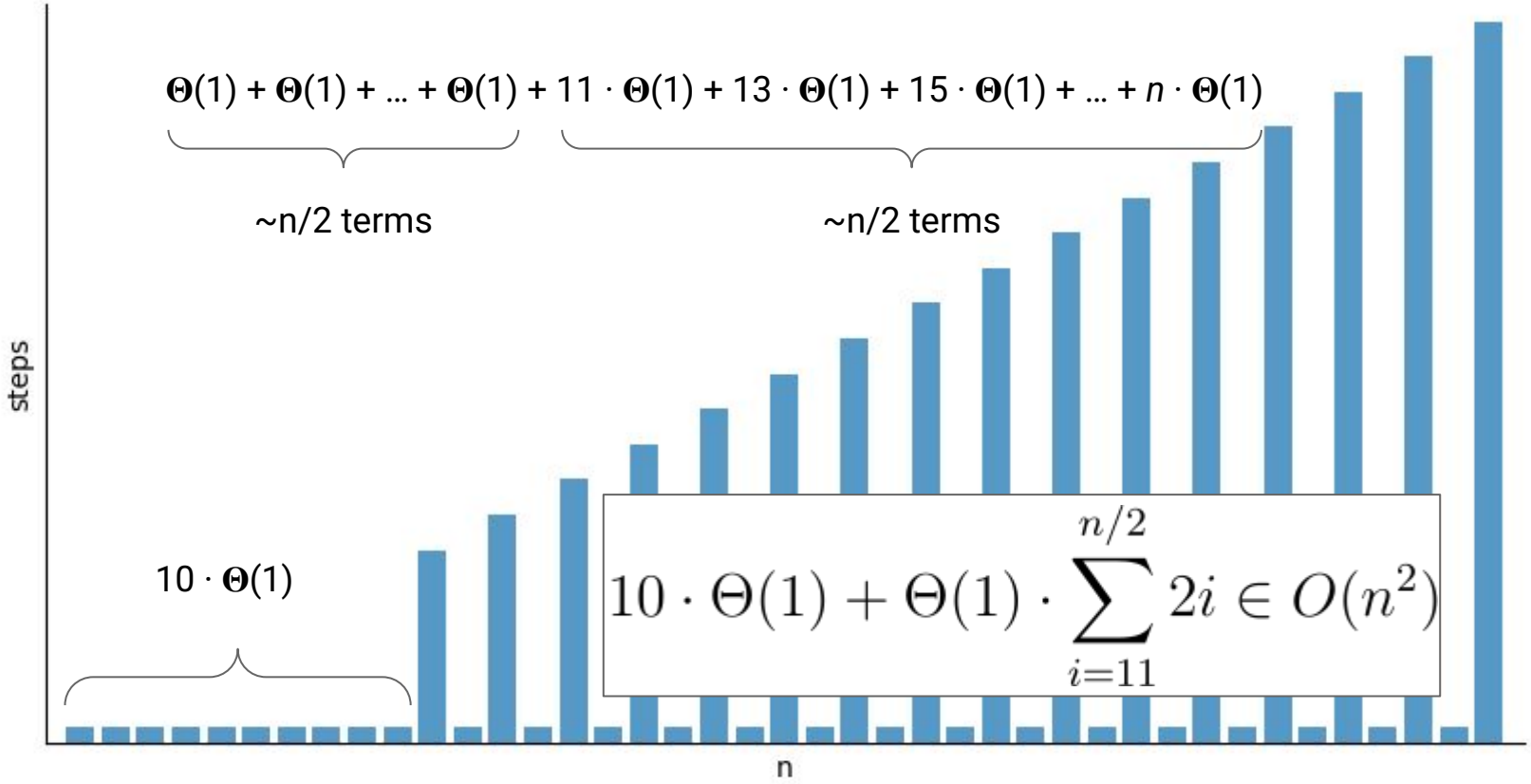

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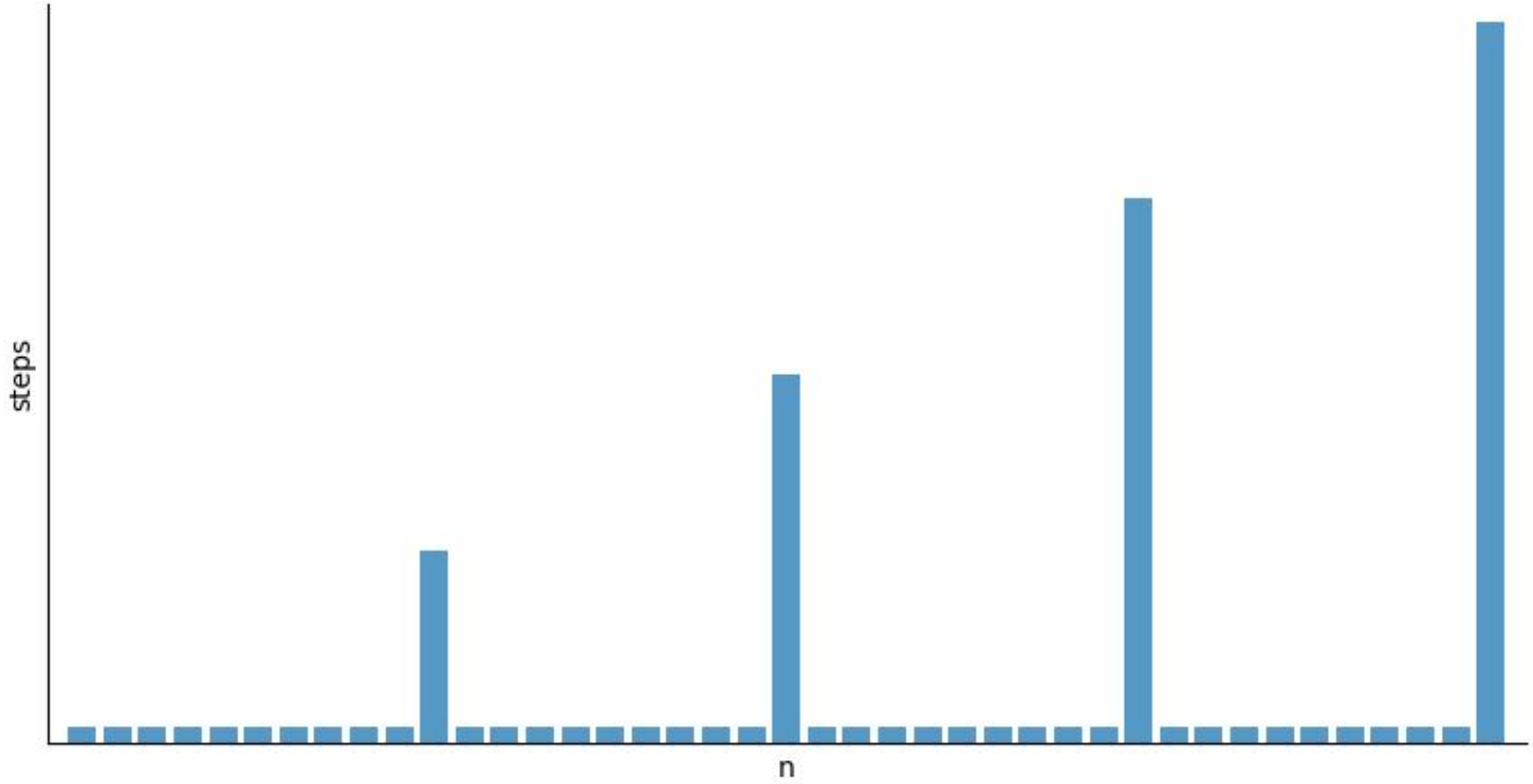

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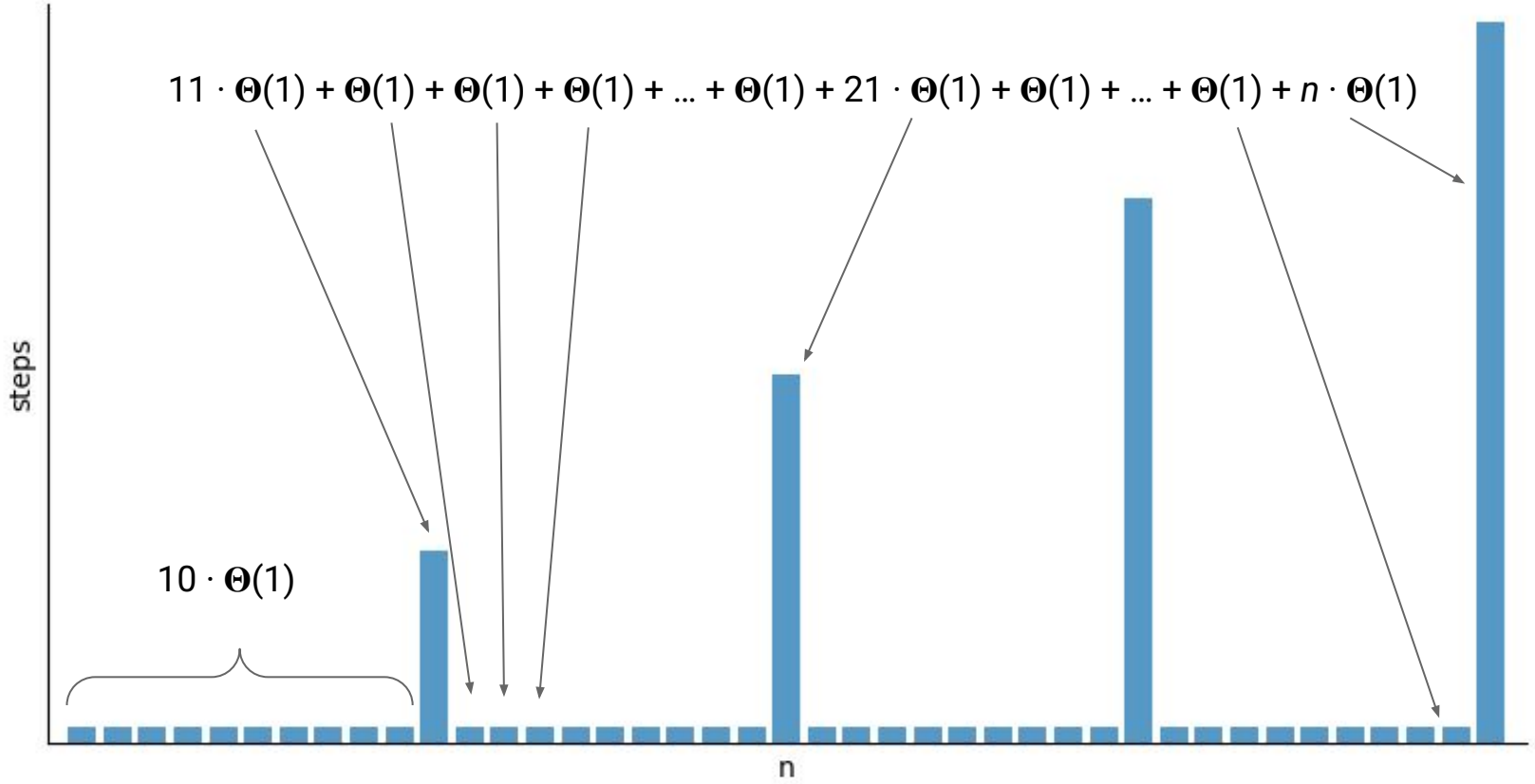

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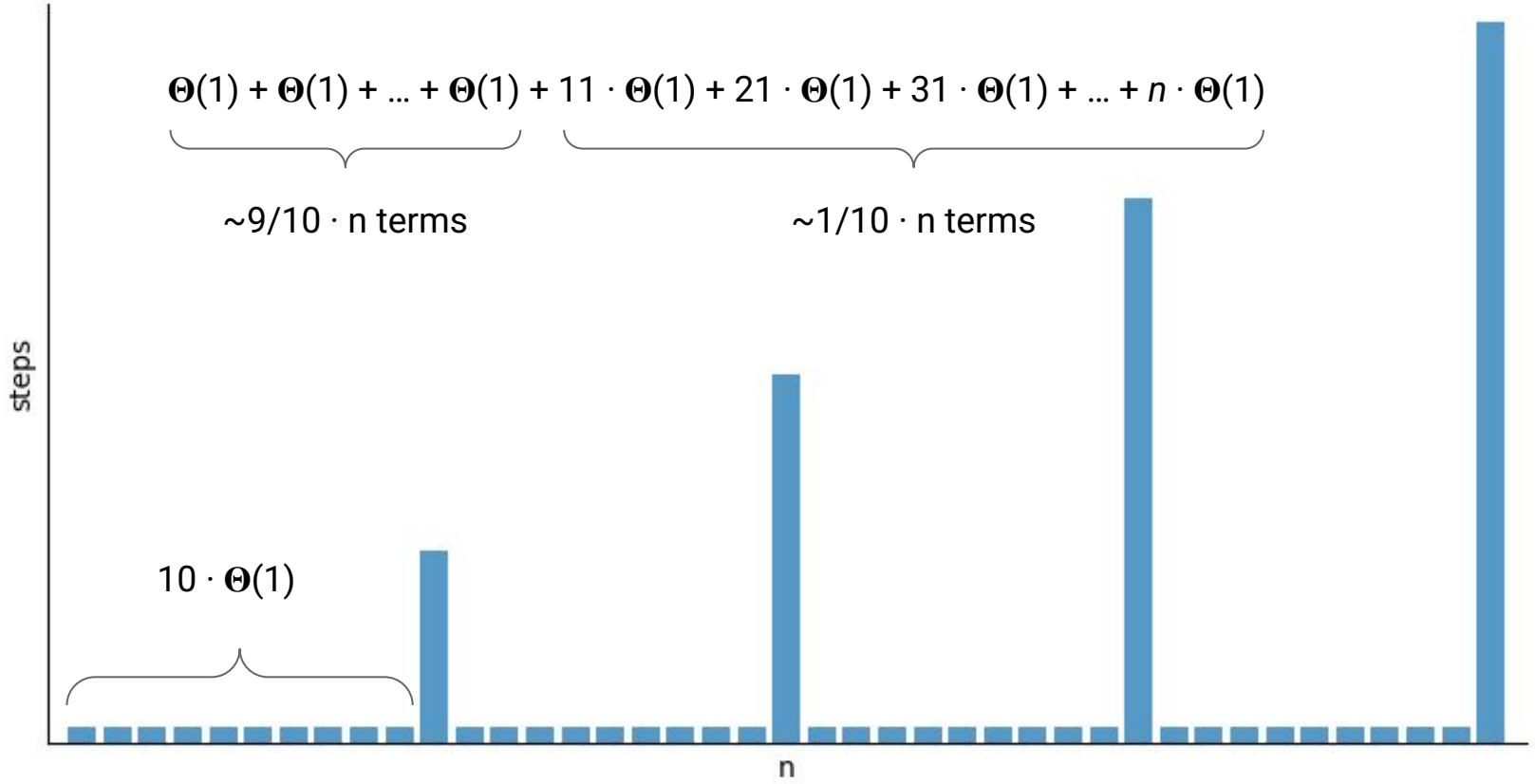

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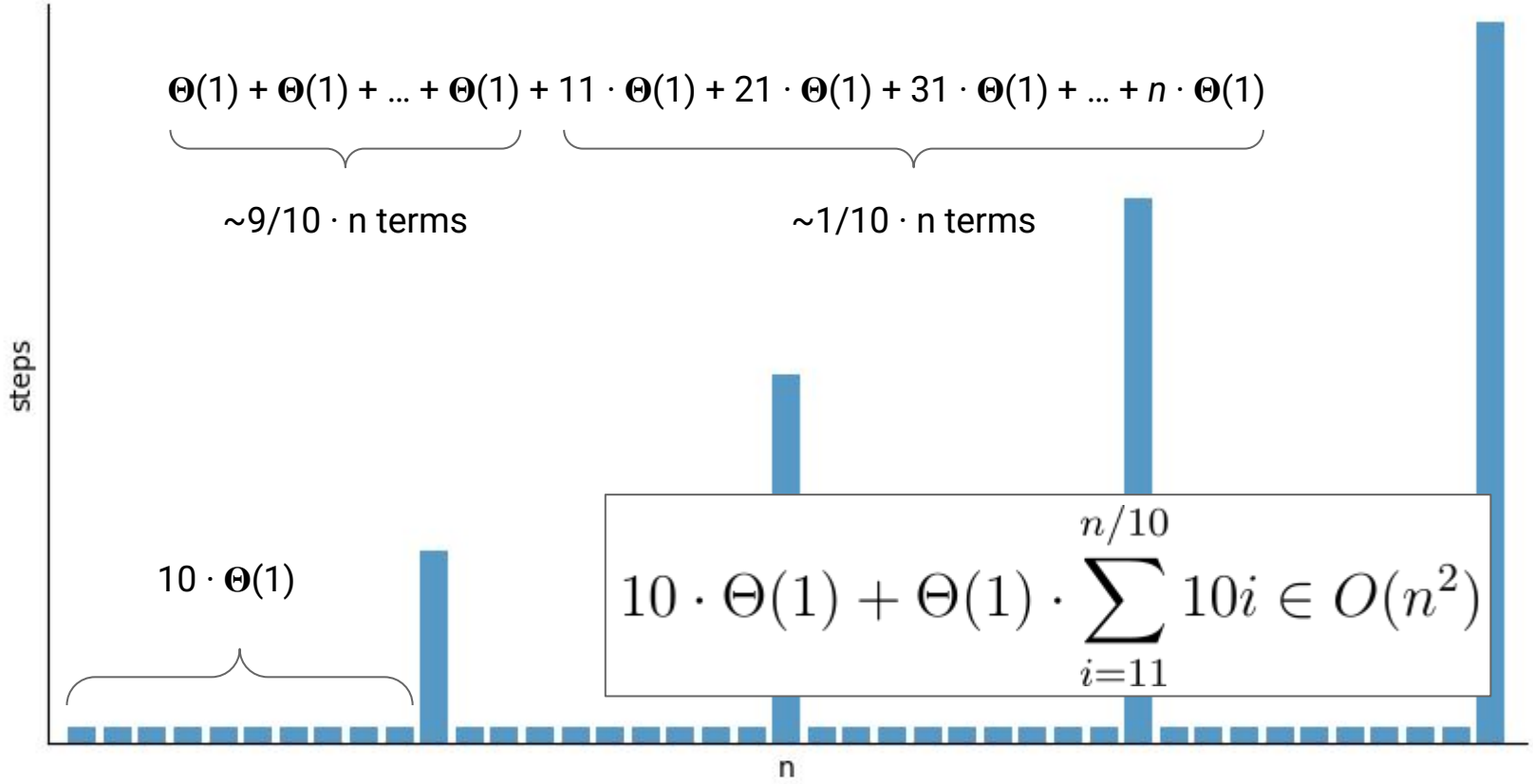

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A New Type of Bounds

Problem: If we increase the size by a constant amount, doing ***n*** adds still costs a total of n^2

How else could we increase the size?

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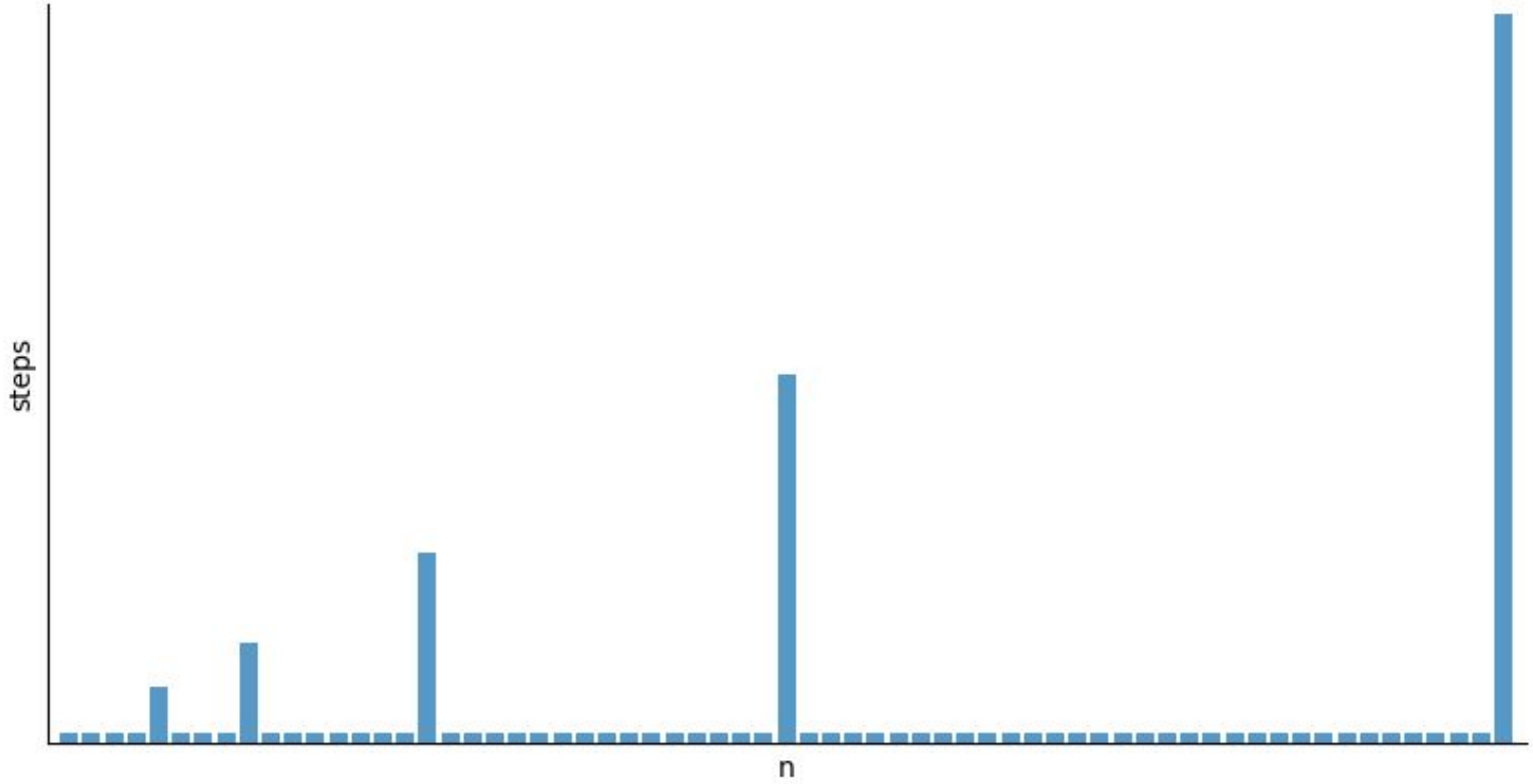
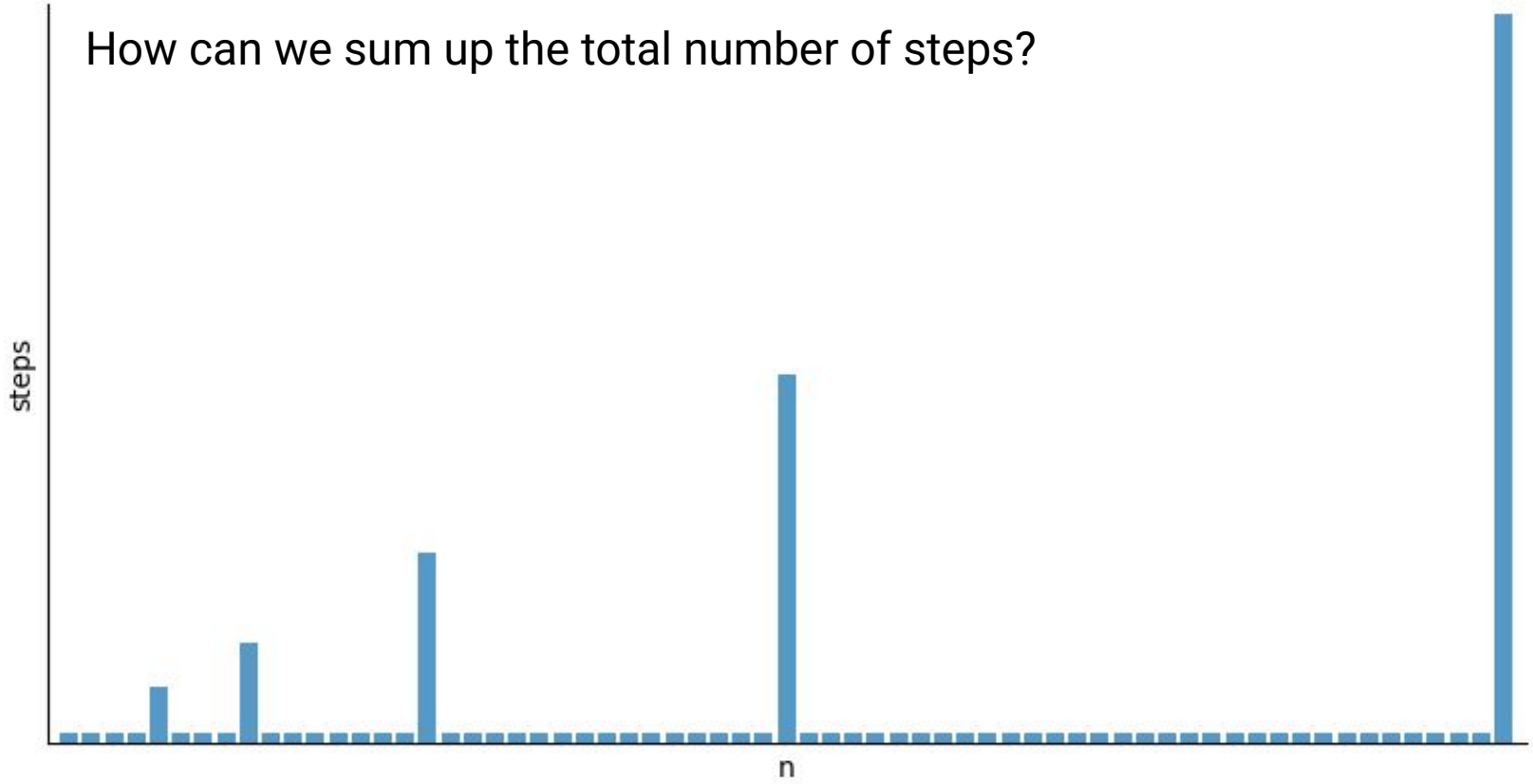
How else could we increase the size? **Double It!**



Initial size = 4, newLength = data.length × 2

How can we sum up the total number of steps?



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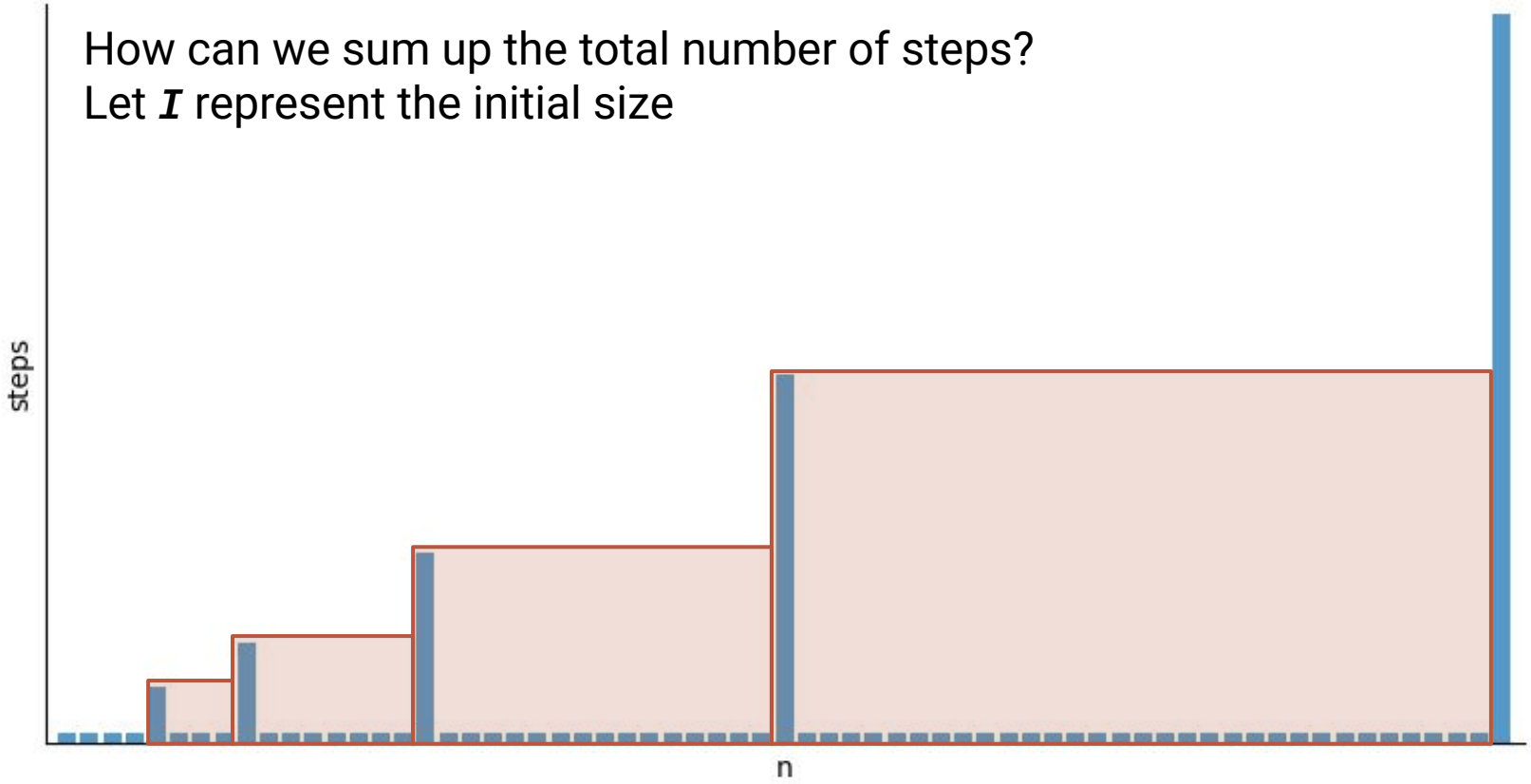
How can we sum up the total number of steps?
Let I represent the initial size



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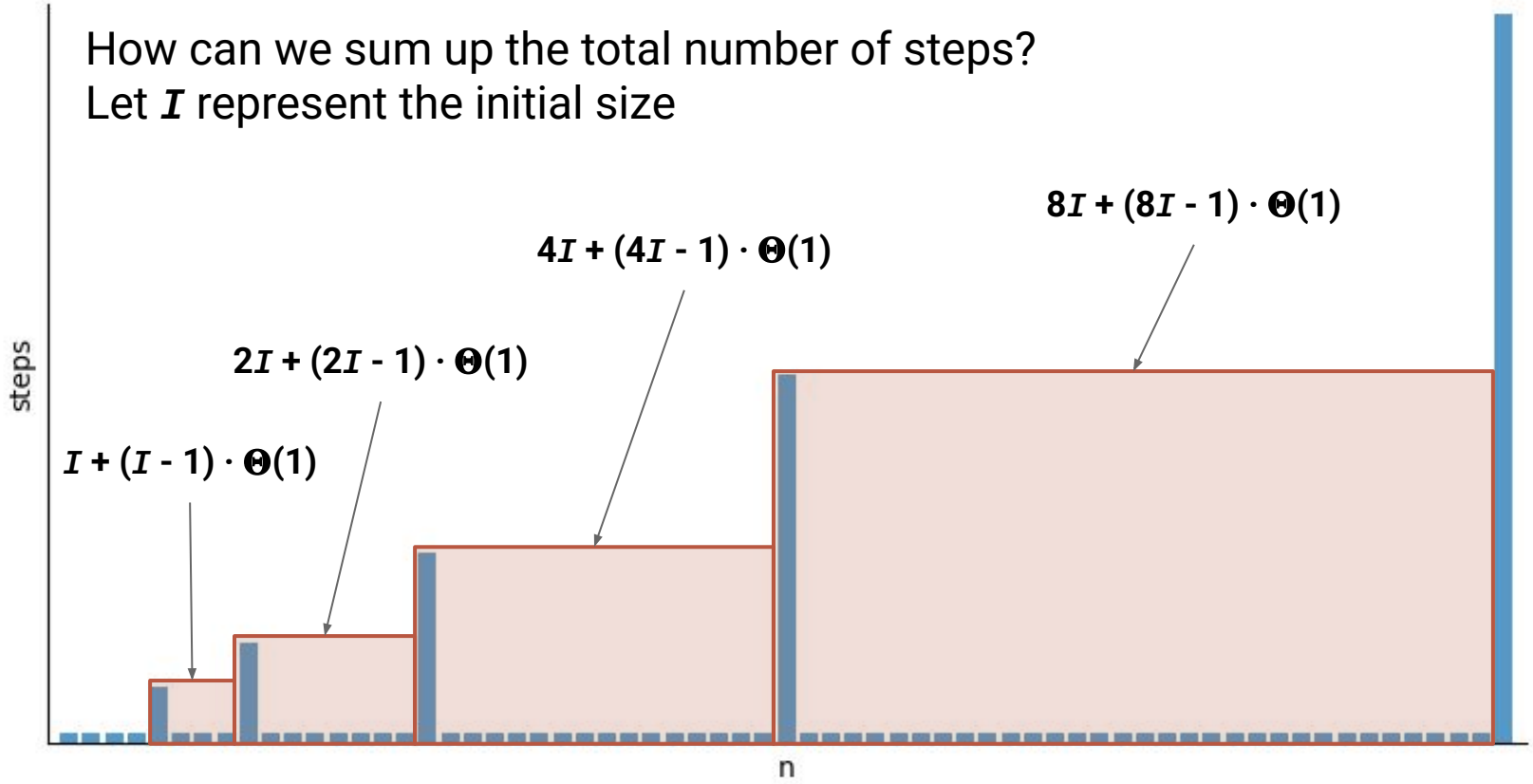

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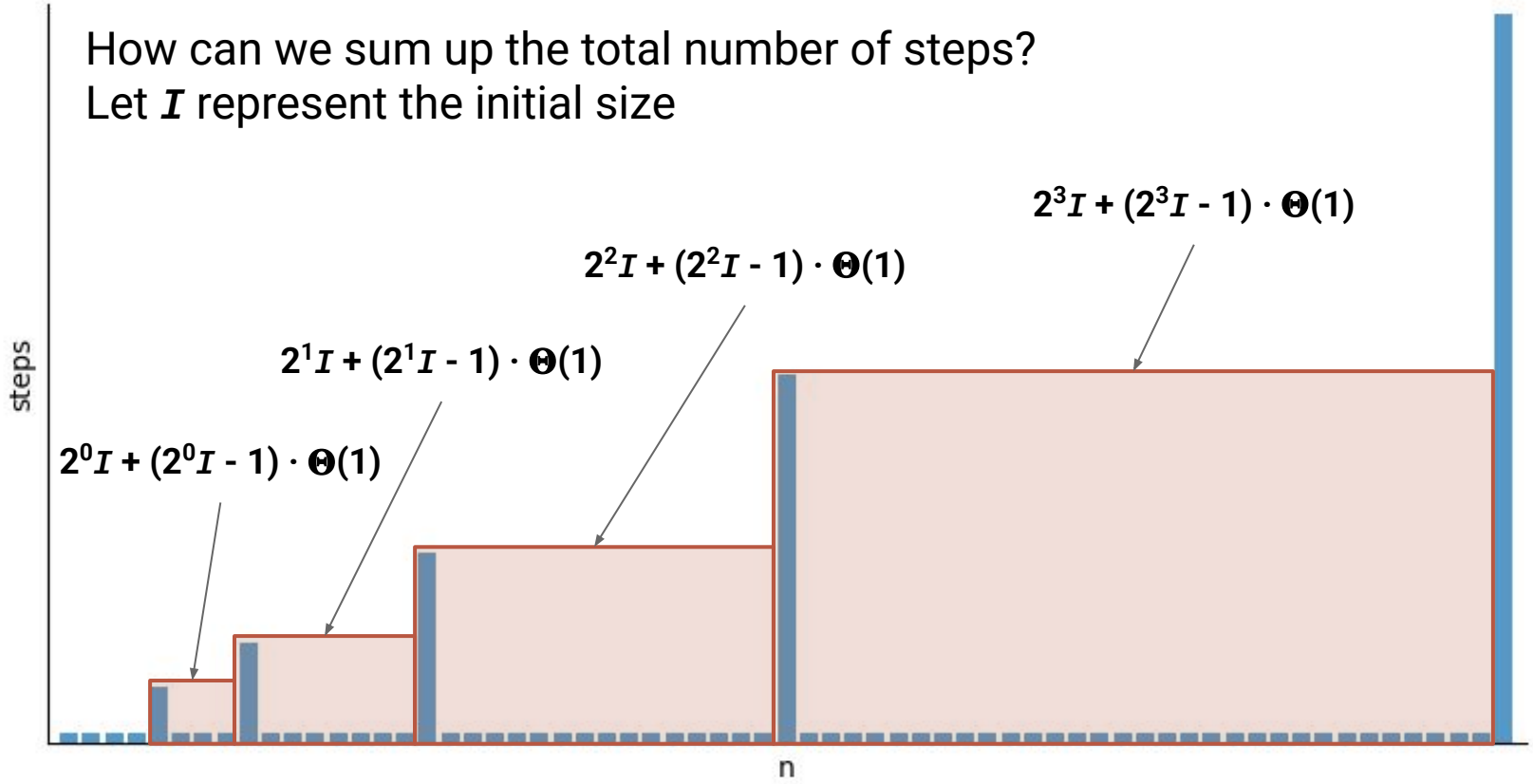

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Doubling the Array Size

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How many steps in total?

Doubling the Array Size

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How many red boxes are there for ***n*** inserts? $\Theta(\log(n))$

How many steps in total? $\sum_{i=0}^{\log(n)} 2^i$

Doubling the Array Size

$$\sum_{i=0}^{\log(n)} 2^i = 2^{\log(n)+1} - 1$$

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$$= 2n - 1$$

$$= O(n)$$

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Let's assume we have a FULL array with 16 elements…What do the next 16 adds look like?

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Let's assume we have a FULL array with 16 elements…What do the next 16 adds look like?

- Create a new array of size 32
- Copy over the 16 elements
- Assign the new element to the next free spot 16 times

Total: Copy 16 elements, assign 16 elements = 2 * 16 operations

Another Perspective

Let's assume we have a FULL array with 32 elements…What do the next 32 adds look like?

- Create a new array of size 64
- Copy over the 32 elements
- Assign the new element to the next free spot 32 times

Total: Copy 32 elements, assign 32 elements = 2 * 32 operations

Another Perspective

Let's assume we have a FULL array with 64 elements…What do the next 64 adds look like?

- Create a new array of size 128
- Copy over the 64 elements
- Assign the new element to the next free spot 64 times

Total: Copy 64 elements, assign 64 elements = 2 * 64 operations

Another Perspective

Let's assume we have a FULL array with ***n*** elements…What do the next ***n*** adds look like?

- Create a new array of size 2****n***
- Copy over the ***n*** elements
- Assign the new element to the next free spot ***n*** times

Total: Copy ***n*** elements, assign ***n*** elements = 2 * ***n*** operations = $\Theta(n)$

Another Perspective

Let's assume we have a FULL array with ***n*** elements…What do the next ***n*** adds look like?

- Create a n
- Copy over
- Assign the new element to the next free spot ***n*** times

Each chunk starts with a big copy, but that big copy also "pays" for the rest of the insertions in that chunk!

Total: Copy ***n*** elements, assign ***n*** elements = 2 * ***n*** operations = $\Theta(n)$

Doubling the Array Size

Wait…so one call to add is ***O*(*n*)**...

But ***n*** calls to add are ***O*(*n*)** as well?

MOST calls only require constant time…

The total cost of ***n*** calls is **guaranteed** ***O*(*n*)** steps

Amortized Runtime

If n calls to a function take $O(f(n))$...

We say the **Amortized Runtime** is $O(f(n) / n)$

The **amortized runtime** of `add` on an `ArrayList` is: $O(n/n) = O(1)$
The **unqualified runtime** of `add` on an `ArrayList` is: $O(n)$

List Runtimes (so far…)

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`size()`	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
`add(...)`	$O(n)$, Amortized $\Theta(1)$	$\Theta(idx)$ or $O(n)$	$\Theta(1)$
`remove(...)`	$O(n)$	$\Theta(idx)$ or $O(n)$	$\Theta(1)$

Follow-Up Questions

What is the amortized runtime of **`add`** for a **`LinkedList`**?

What is the runtime of **`add(int idx, E elem)`** for an **`ArrayList`**?

Follow-Up Questions

What is the amortized runtime of **`add`** for a **`LinkedList`**?

Each **`add`** is $O(1)$. Total for n calls is $O(n)$. Amortized is $O(n/n) = O(1)$

What is the runtime of **`add(int idx, E elem)`** for an **`ArrayList`**?

To **`add`** between two elements requires the rest of the elements to be shifted to the right (opposite of **`remove`**), so runtime is always $O(n)$.

What Data Structure is Best?

Scenario #1: You need to read in the lines of a CSV file, store them in a List, and later be able to access individual records based on index.

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`ArrayList`

Since the amortized runtime of add for **`ArrayList`** and **`LinkedList`**, adding the ***n*** lines of the CSV file will take ***O*(*n*)** time for both…

But **`ArrayLists`** will then have an advantage because looking up records by index will be ***O*(1)** whereas **`LinkedLists`** will be ***O*(*n*)**

What Data Structure is Best?

Scenario #2: Users logging onto an online game need to be efficiently added to a List in the order they log on. From time to time you must be able to iterate through the list from beginning to end.

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`LinkedList`

The enumeration will cost a total of $O(n)$ for both types of List

But some users will experience longer waits being added to the List if implemented as an `ArrayList` due to the need for it to occasionally resize