CSE 250

Data Structures

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Lec 13: Divide and Conquer

Announcements

- WA2 due Sunday 2/23 @ 11:59PM
- Midterm Review Session this Saturday! 2/22/25 @ 12PM (room TBD)
- PA1 Complete
  - Amnesty Reminder
  - Submission counts (50+ submissions is too many)
  - Imposter Syndrome

Recap

- **Recursion:** A big problem made up of one or more instances of a smaller problem
    - Factorial: `f(n) = n * f(n-1)`
    - Fibonacci: `f(n) = f(n-1) + f(n-2)`
    - Towers of Hanoi: `move(n) = move(n-1)`, `move(1)`, `move(n-1)` again
- **Inductive Proofs:**
    - Come up with a hypothesis
    - Prove it on the base case
    - Assume it works for ***n' < n***; Prove for ***n*** based on that assumption

Inductive Proof for Towers of Hanoi

- Base case is one ring. I can move one ring.
- Assume I can move ***n*** **- 1** rings; Can I prove that I can move ***n***? Yes
  - Move ***n*** **- 1** (which we can do based on our assumption)
  - Move 1 ring
  - Move ***n*** **- 1** (which we can do based on our assumption.
  - Therefore, if we can move ***n*** **- 1**, we can move ***n***.

** Note this is just a proof that we* ***can*** *solve it for any value of n. The actual number of steps required can also be shown by induction…*

Fibonacci

What is the complexity of **`fib(n)`**?

```java
public int fib(int n) {
    if(n < 2) { return 1; }
    else { return fib(n - 1) + fib(n - 2); }
}
```

Fibonacci

$$T(n) = \begin{cases} \Theta(1) & \textbf{if } n < 2 \\ T(n-1) + T(n-2) + \Theta(1) & \textbf{otherwise} \end{cases}$$

Solve for *T*(*n*)...How?

Divide and Conquer

Remember the Towers of Hanoi…

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1. You can move *n* blocks if you know how to move *n*-1 blocks

Divide and Conquer

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Divide and Conquer

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2. You can move *n*-1 blocks if you know how to move *n*-2 blocks
3. You can move *n*-2 blocks if you know how to move *n*-3 blocks

Divide and Conquer

Remember the Towers of Hanoi…

1. You can move *n* blocks if you know how to move *n*-1 blocks
2. You can move *n*-1 blocks if you know how to move *n*-2 blocks
3. You can move *n*-2 blocks if you know how to move *n*-3 blocks
4. You can move *n*-3 blocks if you know how to move *n*-4 blocks

Divide and Conquer

Remember the Towers of Hanoi…

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2. You can move *n*-1 blocks if you know how to move *n*-2 blocks
3. You can move *n*-2 blocks if you know how to move *n*-3 blocks
4. You can move *n*-3 blocks if you know how to move *n*-4 blocks

…

You can always move 1 block

Divide and Conquer

To solve the problem at *n*:

Divide and Conquer

To solve the problem at *n*:

Divide the problem into smaller problems (size *n*-1 and 1 in this case)

Divide and Conquer

To solve the problem at *n*:

Divide the problem into smaller problems (size *n*-1 and 1 in this case)

Conquer the smaller problems

Divide and Conquer

To solve the problem at *n*:

Divide the problem into smaller problems (size *n*-1 and 1 in this case)

Conquer the smaller problems

Combine the smaller solutions to get the bigger solution

Merge Sort

Input: An array with elements in an unknown order.

Output: An array with elements in sorted order.

Merge Sort - Questions

Divide (break the array into smaller arrays)

What's the smallest list I could try to sort?

Merge Sort - Questions

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Merge Sort - Questions

Divide (break the array into smaller arrays)
What's the smallest list I could try to sort? size n = 1

Conquer (sort the smaller arrays)
How do I sort it?

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Conquer (sort the smaller arrays)

How do I sort it? It's already sorted!!!

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Combine (combine the sorted arrays into a bigger sorted array)
How can I do this, and how long does it take?

Merge Sort - Questions

Divide (break the array into smaller arrays)
What's the smallest list I could try to sort? size n = 1

Conquer (sort the smaller arrays)
How do I sort it? It's already sorted!!!

Combine (combine the sorted arrays into a bigger sorted array)
How can I do this, and how long does it take? Merge…

How do we Merge Two Sorted Arrays?

24	37	62	73	95
15	31	55	61	88

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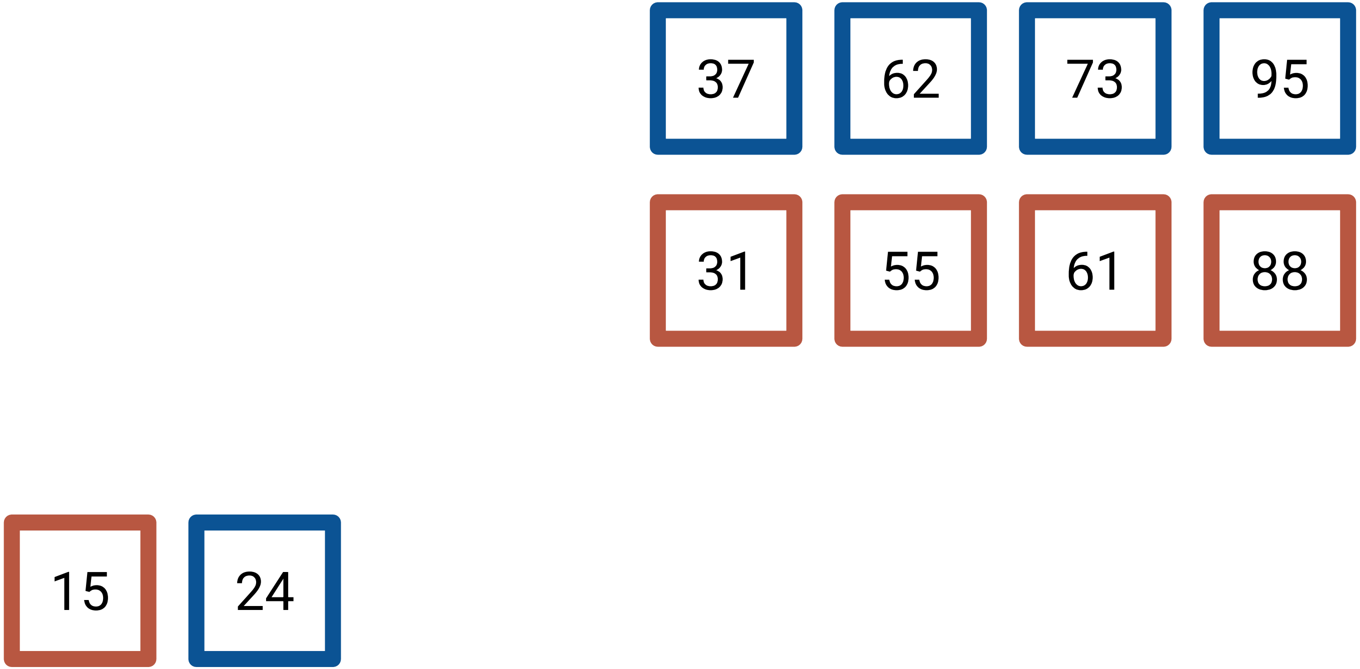
15

How do we Merge Two Sorted Arrays?



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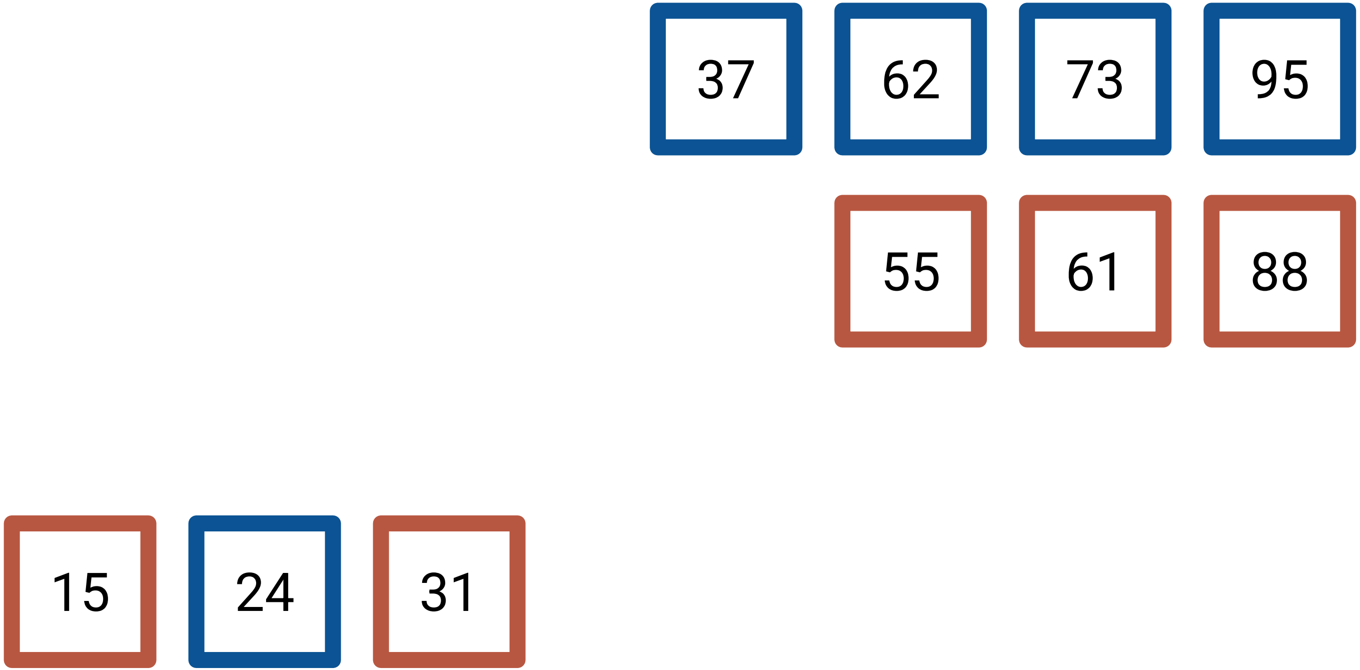

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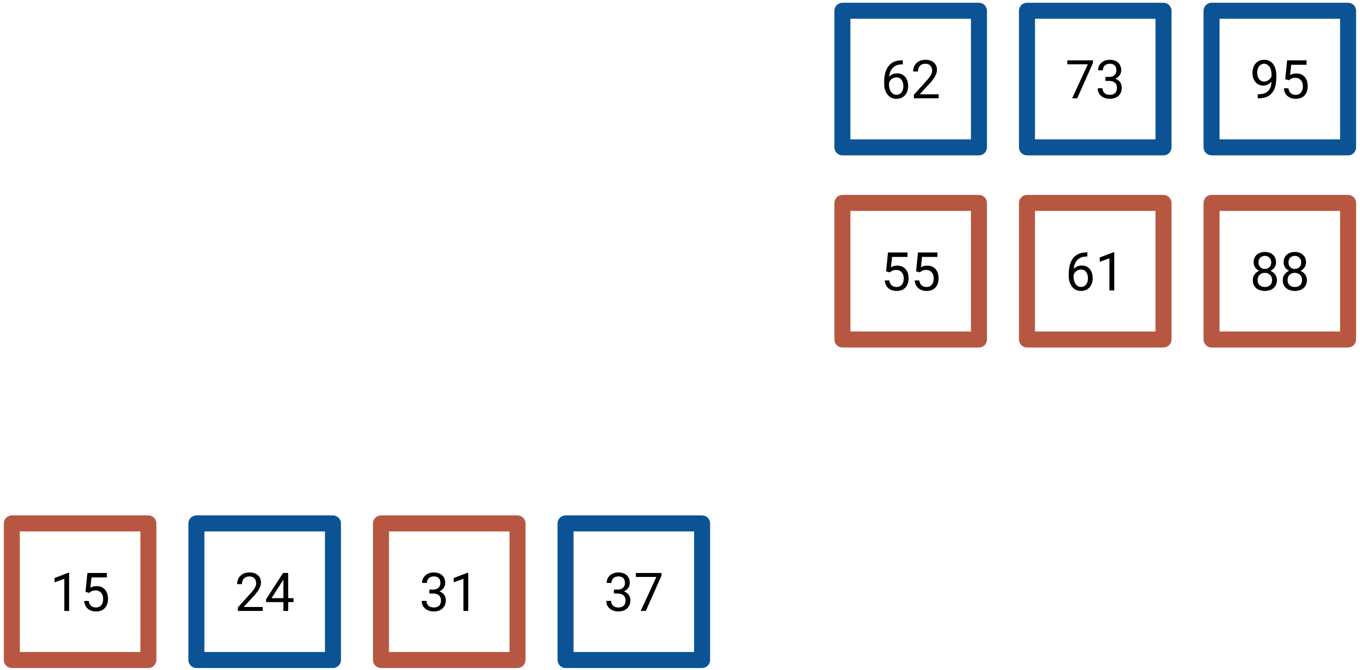

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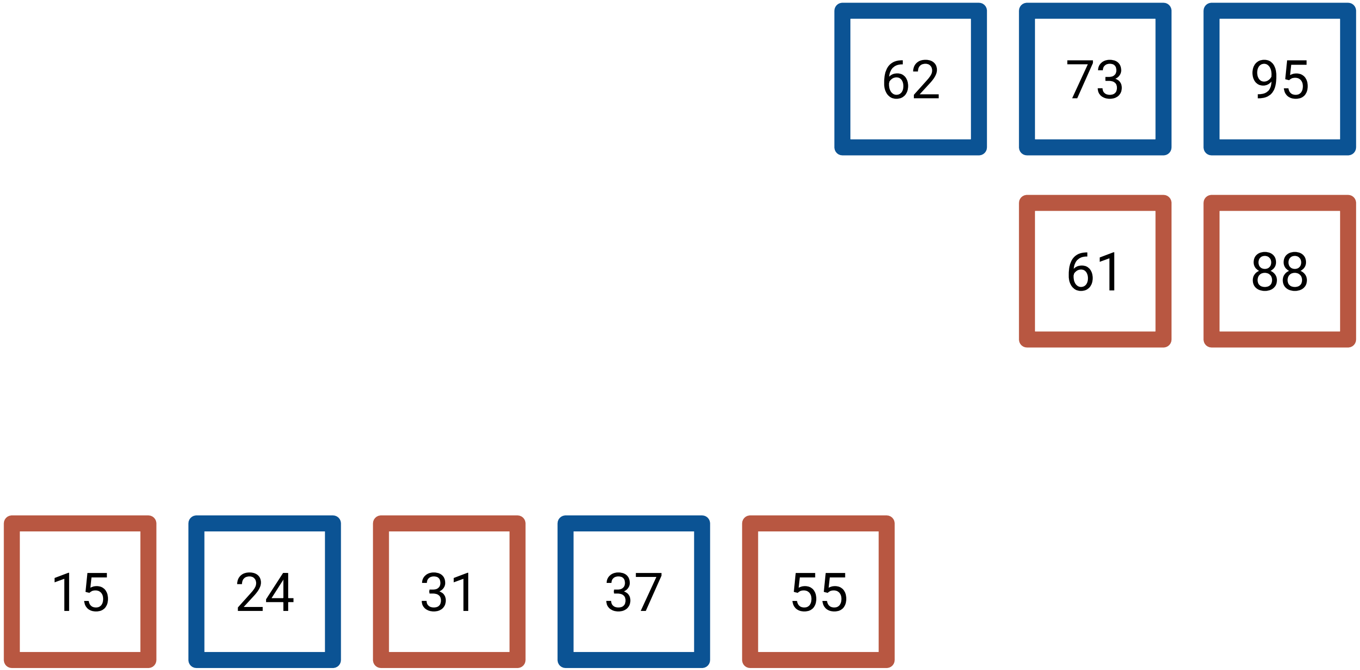

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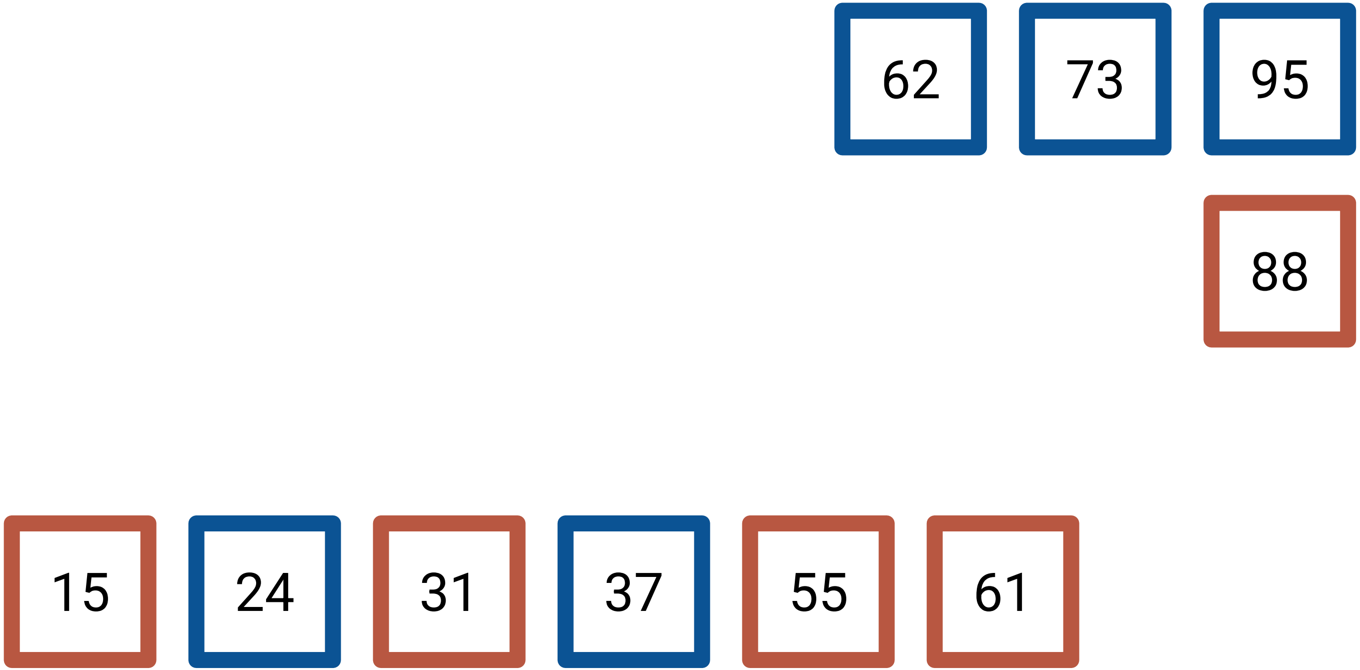

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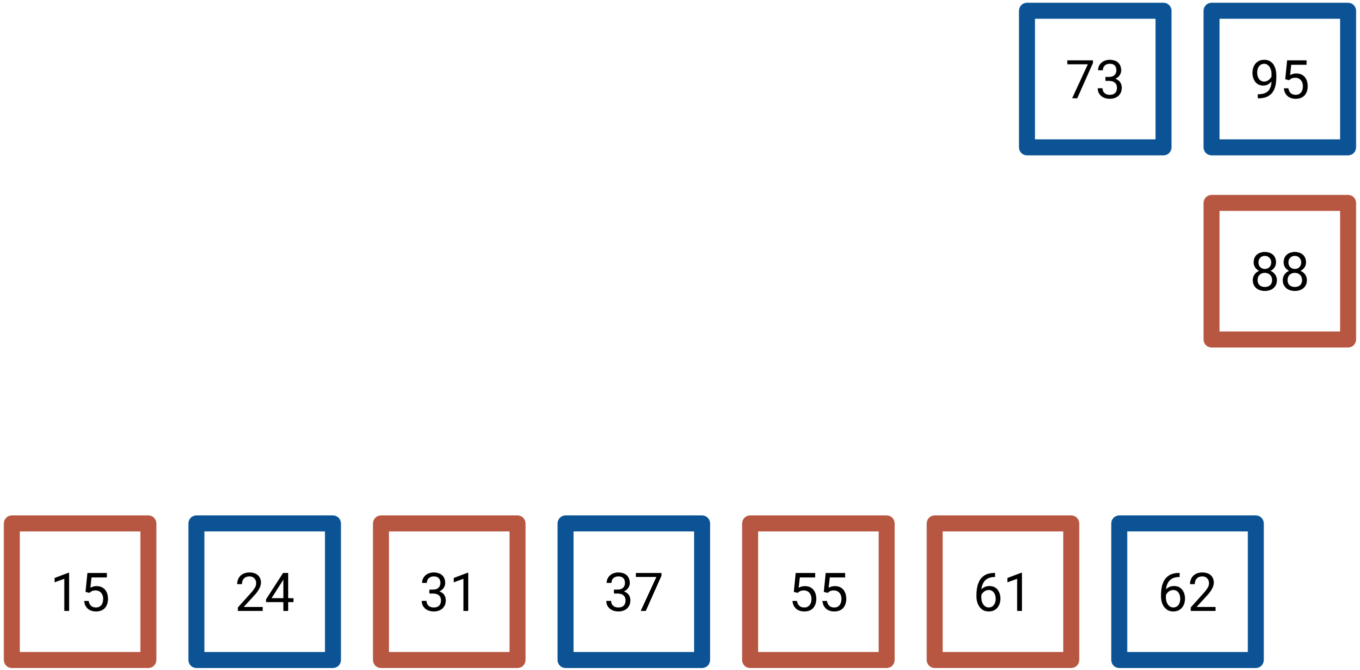

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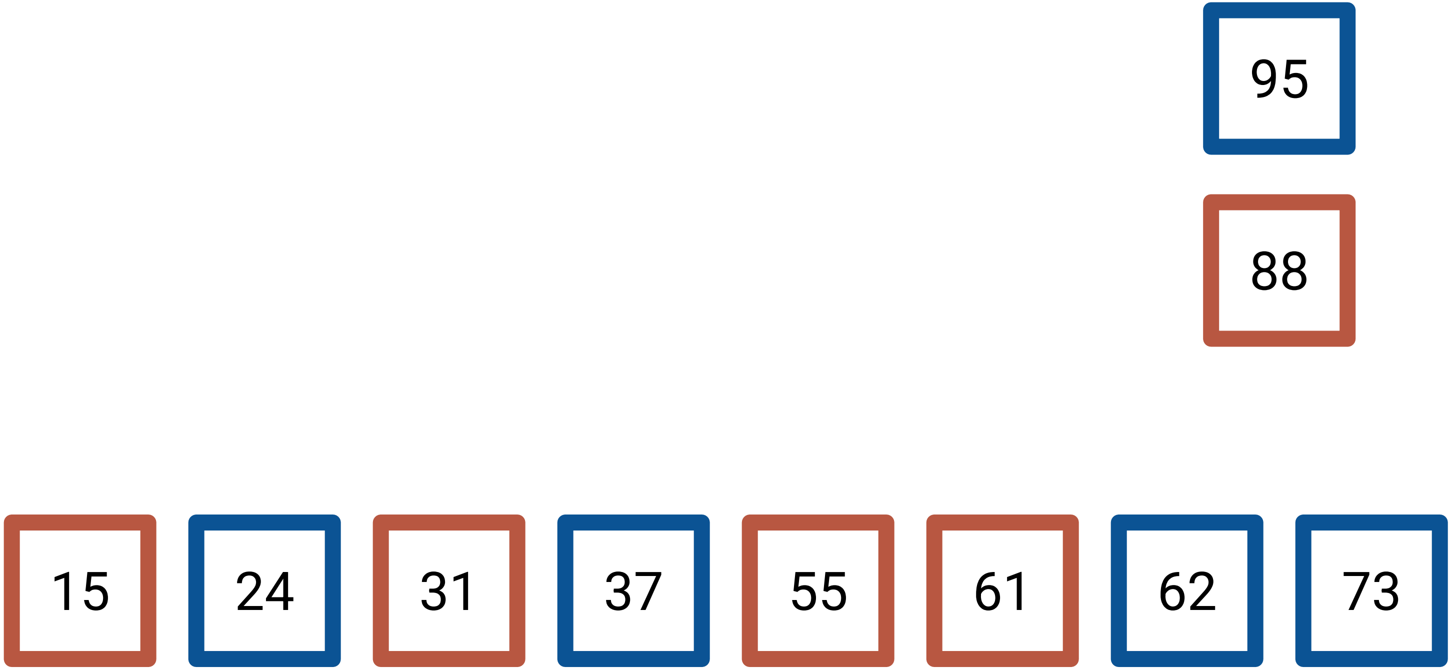

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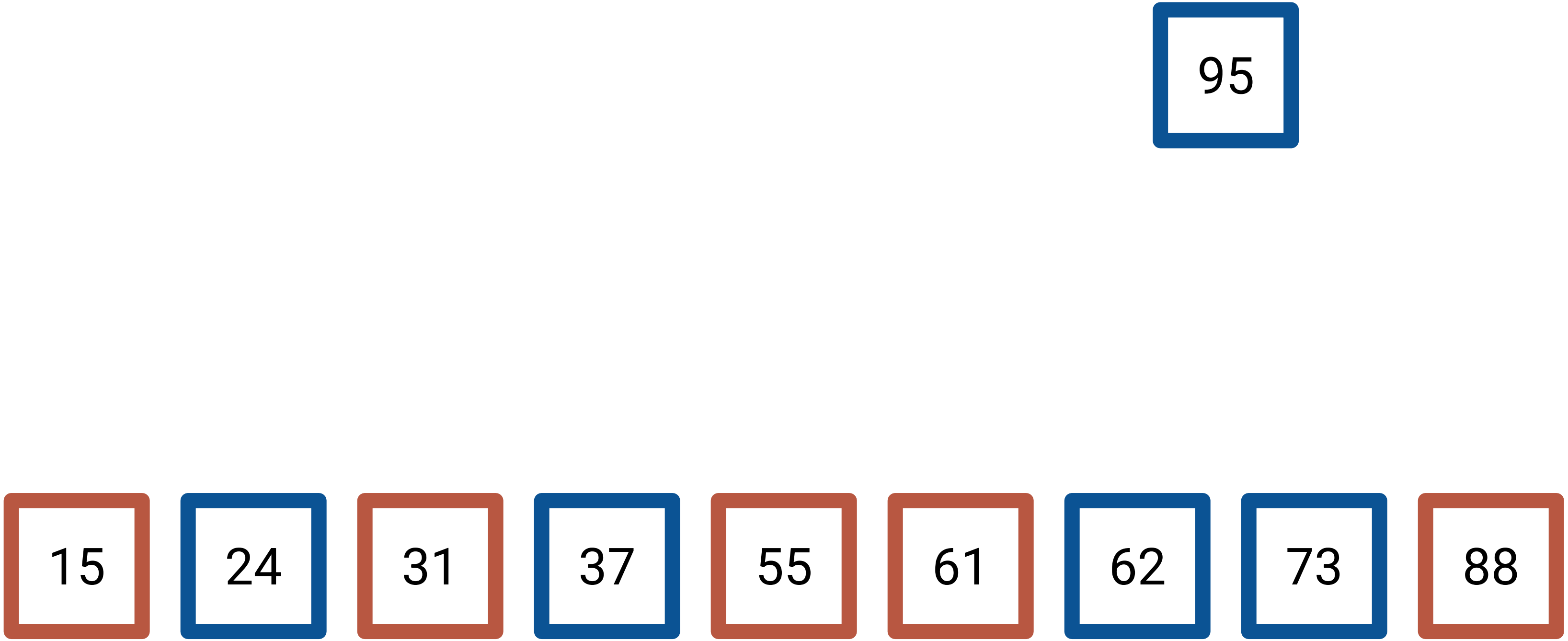


How do we Merge Two Sorted Arrays?

15	24	31	37	55	61	62	73	88	95

How do we Merge Two Sorted Arrays?

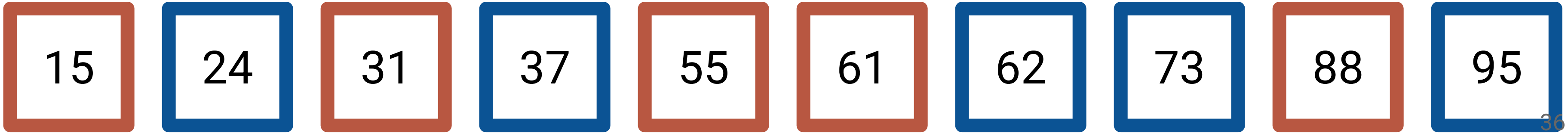
What was the complexity?

15	24	31	37	55	61	62	73	88	95

How do we Merge Two Sorted Arrays?

What was the complexity?

Each comparison was $\Theta(1)$...



How do we Merge Two Sorted Arrays?

What was the complexity?

Each comparison was $\Theta(1)$...

How many comparisons? $\Theta(|red| + |blue|)$

15	24	31	37	55	61	62	73	88	95

Divide

- We know how to combine sorted arrays
- We know that in a base case of $n = 1$ how to sort
- How do we divide our problem to get there?

Divide

- We know how to combine sorted arrays
- We know that in a base case of $n = 1$ how to sort
- How do we divide our problem to get there?

Let's divide our array in half (recursively)!

Visualization - Divide

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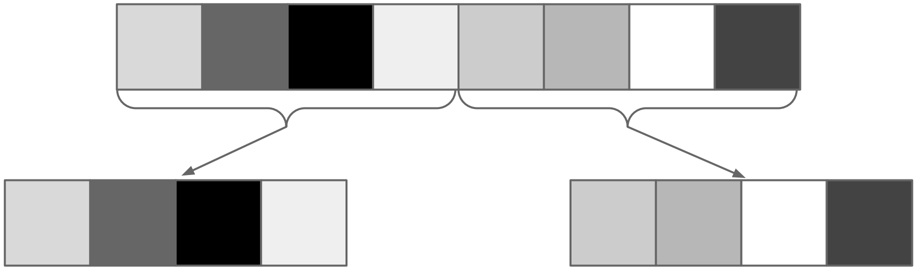
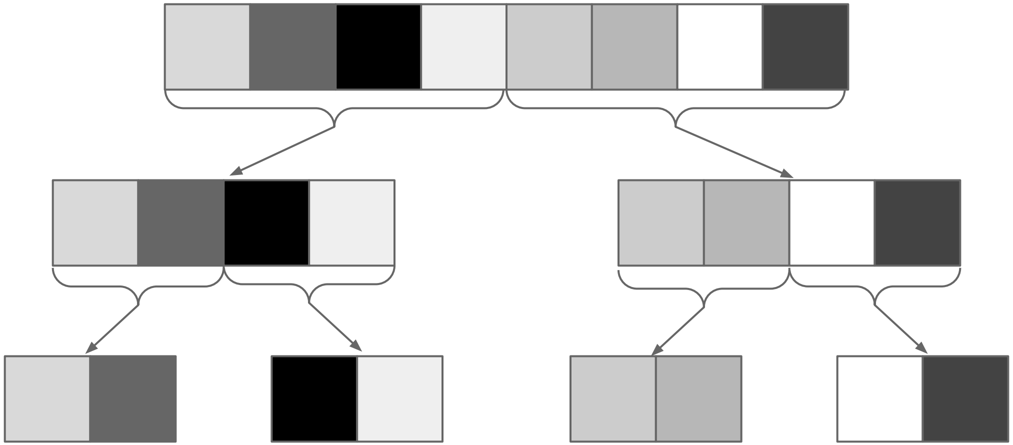


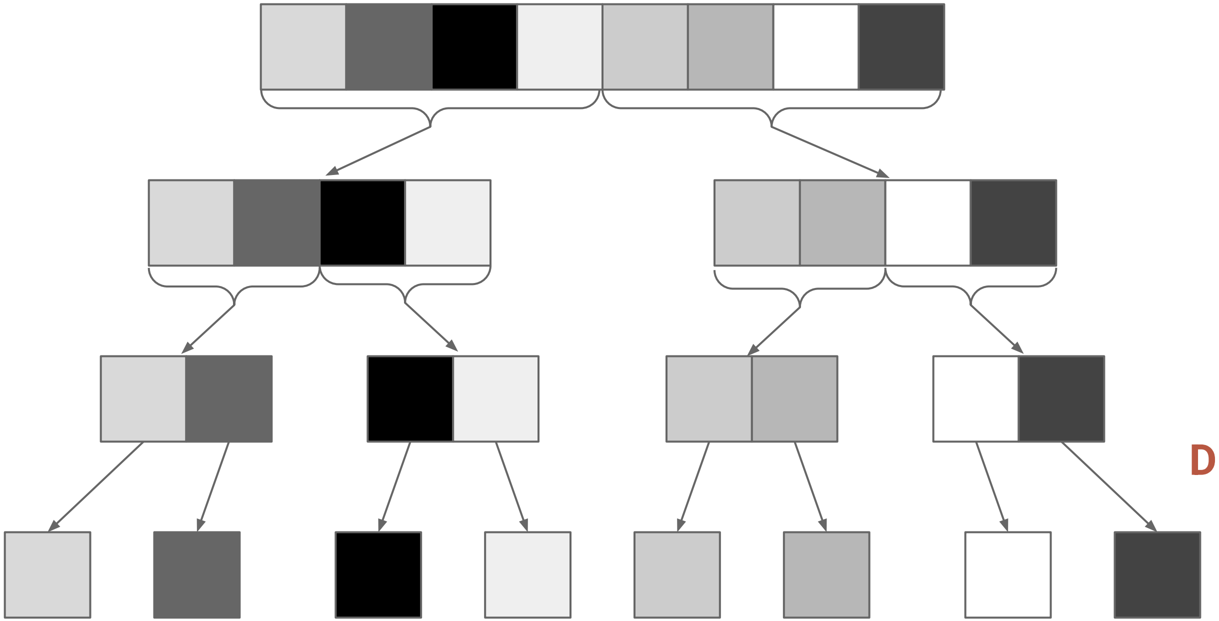
Divide the input in half

Visualization - Divide



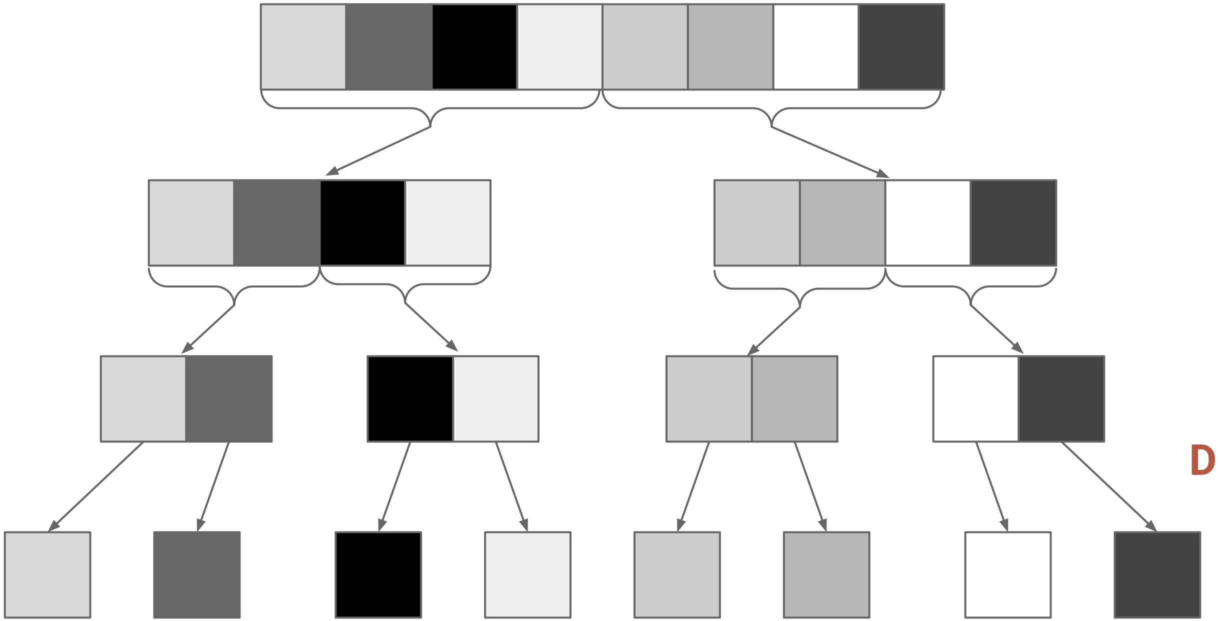
Divide each half in half

Visualization - Divide



Divide each half in half again...

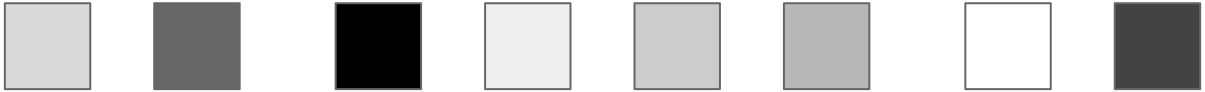
Visualization - Conquer



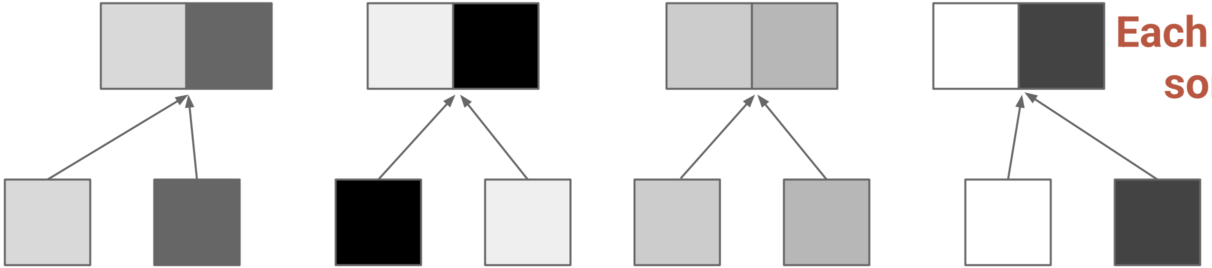
Divide each half in half again...

We can't divide in half anymore (base case)

Visualization - Combine

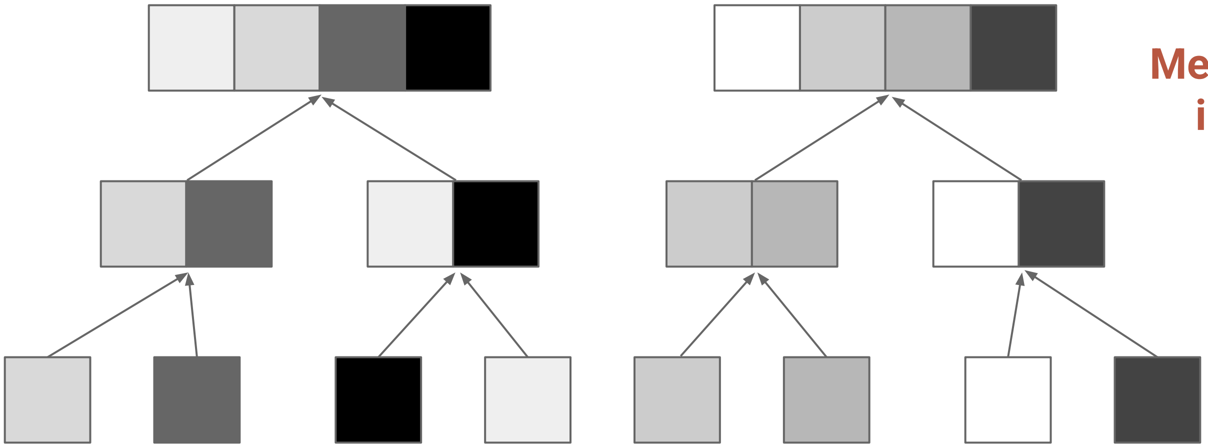


Visualization - Combine



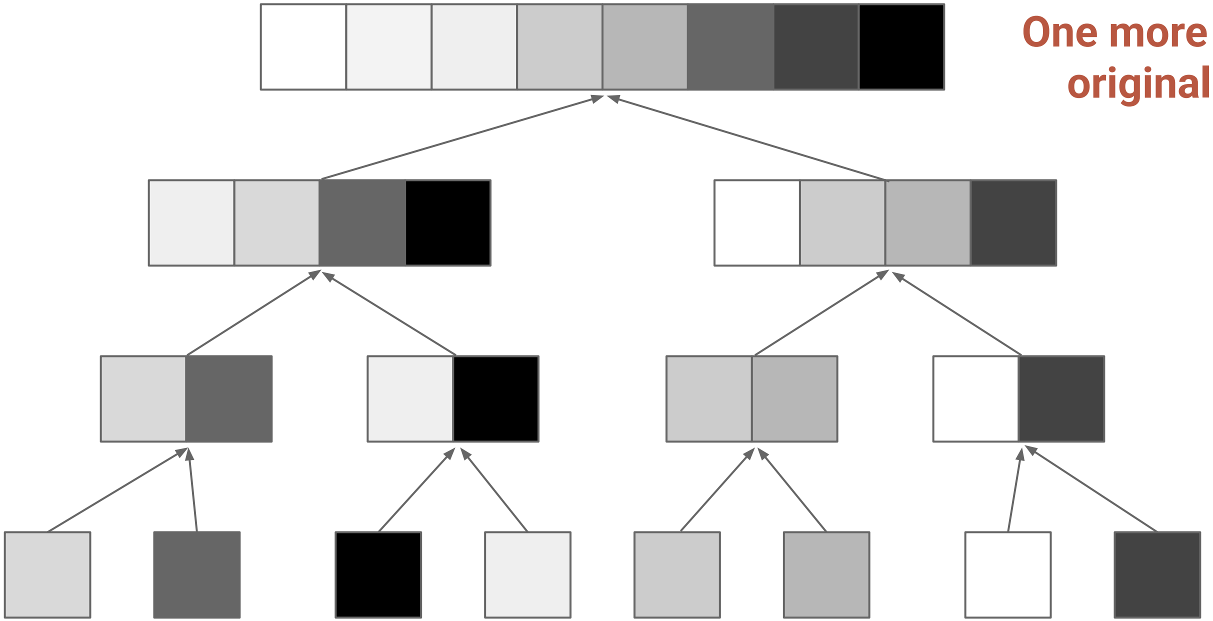
Each single item list is sorted…merge each pair into a bigger sorted list

Visualization - Combine



Merge each pair of 2 into sorted lists of size 4

Visualization - Combine



One more merge gets our original list fully sorted

Complexity

If we solve a problem of size n by:

- Dividing it into a sub-problems
  - Where each problem is of size n/b (usually $b = a$)
  - …and stop recurring at $n \leq c$
  - …and the cost of dividing is $D(n)$
  - …and the cost of combining is $C(n)$

Then our total cost will be…

Complexity

$$T(n) = \begin{cases} \Theta(1) & \textbf{if } n \leq c \\ a \cdot T(\frac{n}{b}) + D(n) + C(n) & \textbf{otherwise} \end{cases}$$

a subproblems of size ***n/b**,* base case of ***n ≤ c***

divide cost of ***D*(*n*)**

and combine cost of ***C*(*n*)**

Merge Sort

Divide: Split the sequence in half

$D(n) = \Theta(n)$ (can we do it faster?)

Conquer: Sort left and right halves

$a = 2$, $b = 2$, $c = 1$

Combine: Merge halves together

$C(n) = \Theta(n)$

Merge Sort

Divide: Split the sequence in half

$D(n) = \Theta(n)$ (can we do it faster? $\Theta(1)$ for ArrayList)

Conquer: Sort left and right halves

$a = 2$, $b = 2$, $c = 1$

Combine: Merge halves together

$C(n) = \Theta(n)$

Merge Sort

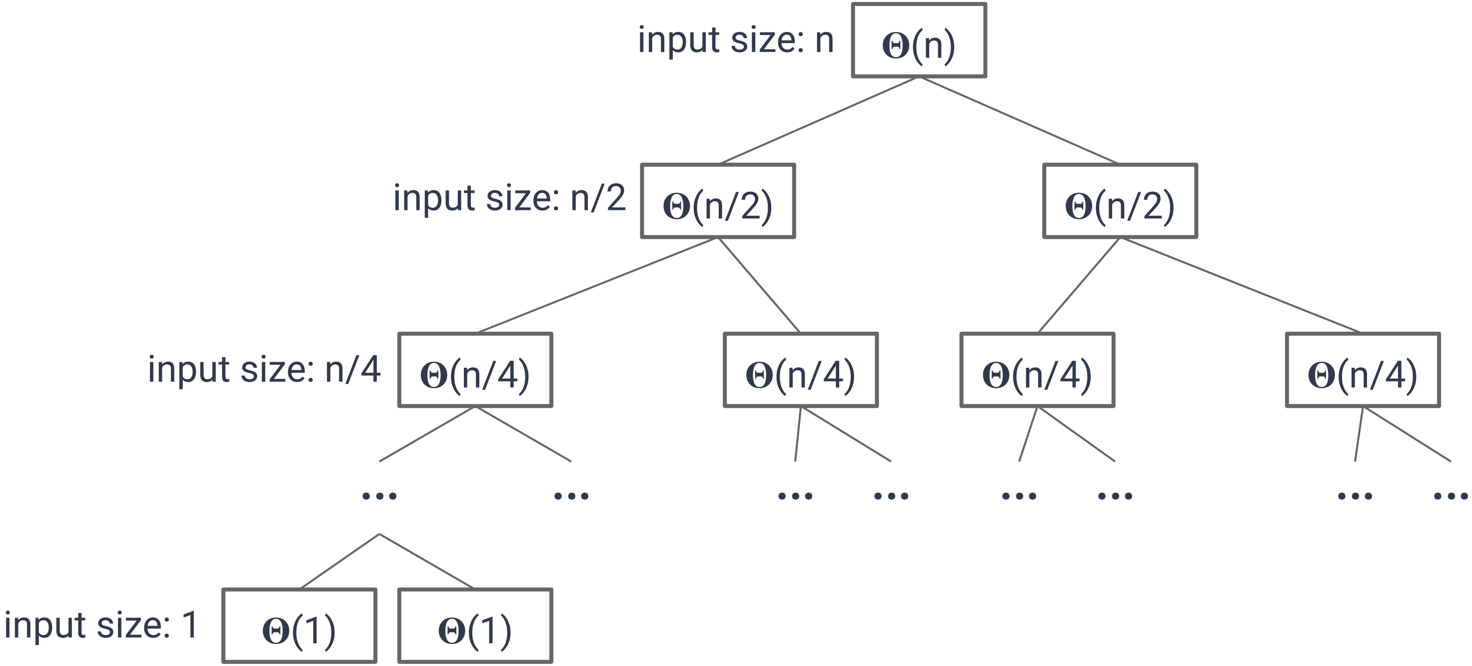
$$T(n) = \begin{cases} \Theta(1) & \textbf{if } n \leq 1 \\ 2 \cdot T(\frac{n}{2}) + \Theta(1) + \Theta(n) & \textbf{otherwise} \end{cases}$$

Merge Sort

$$T(n) = \begin{cases} \Theta(1) & \textbf{if } n \leq 1 \\ 2 \cdot T(\frac{n}{2}) + \Theta(1) + \Theta(n) & \textbf{otherwise} \end{cases}$$

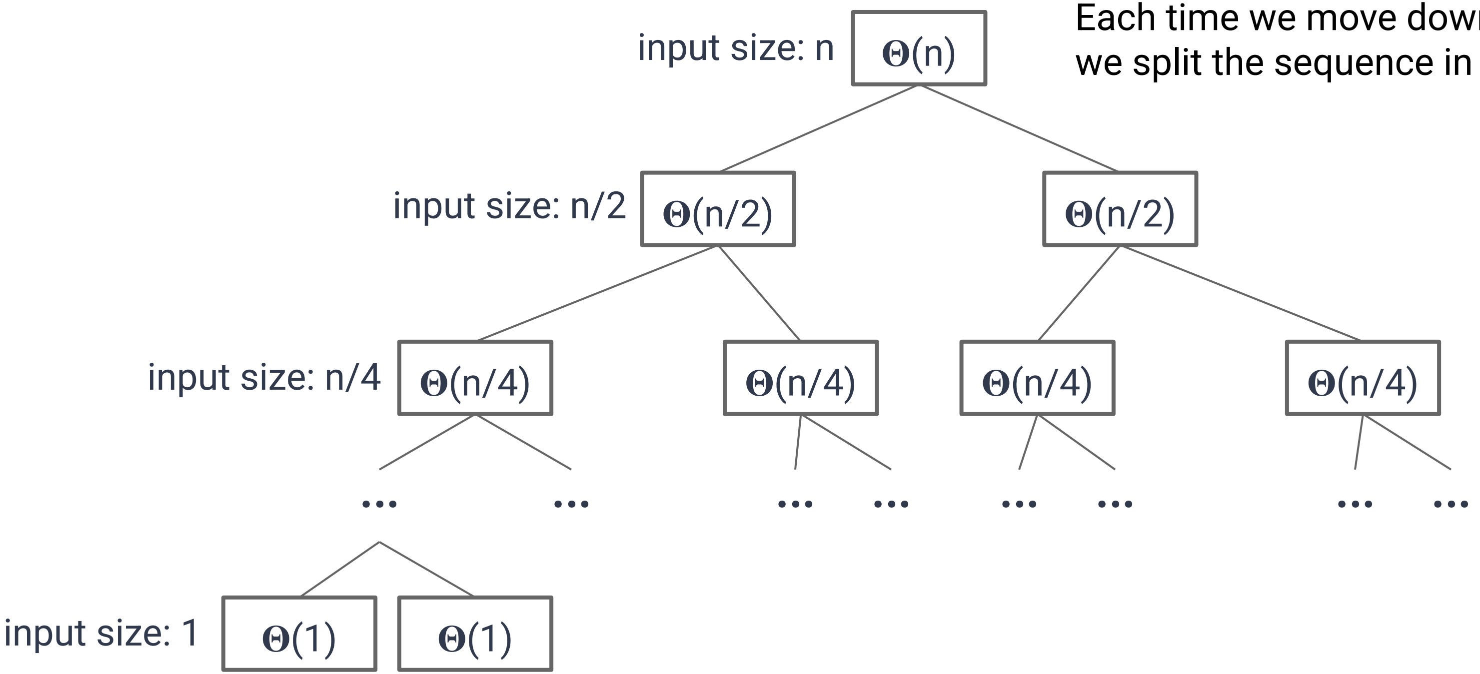
How do we find a closed-form hypothesis?

Merge Sort: Recursion Tree



Merge Sort: Recursion Tree

Each time we move down a level, we split the sequence in half



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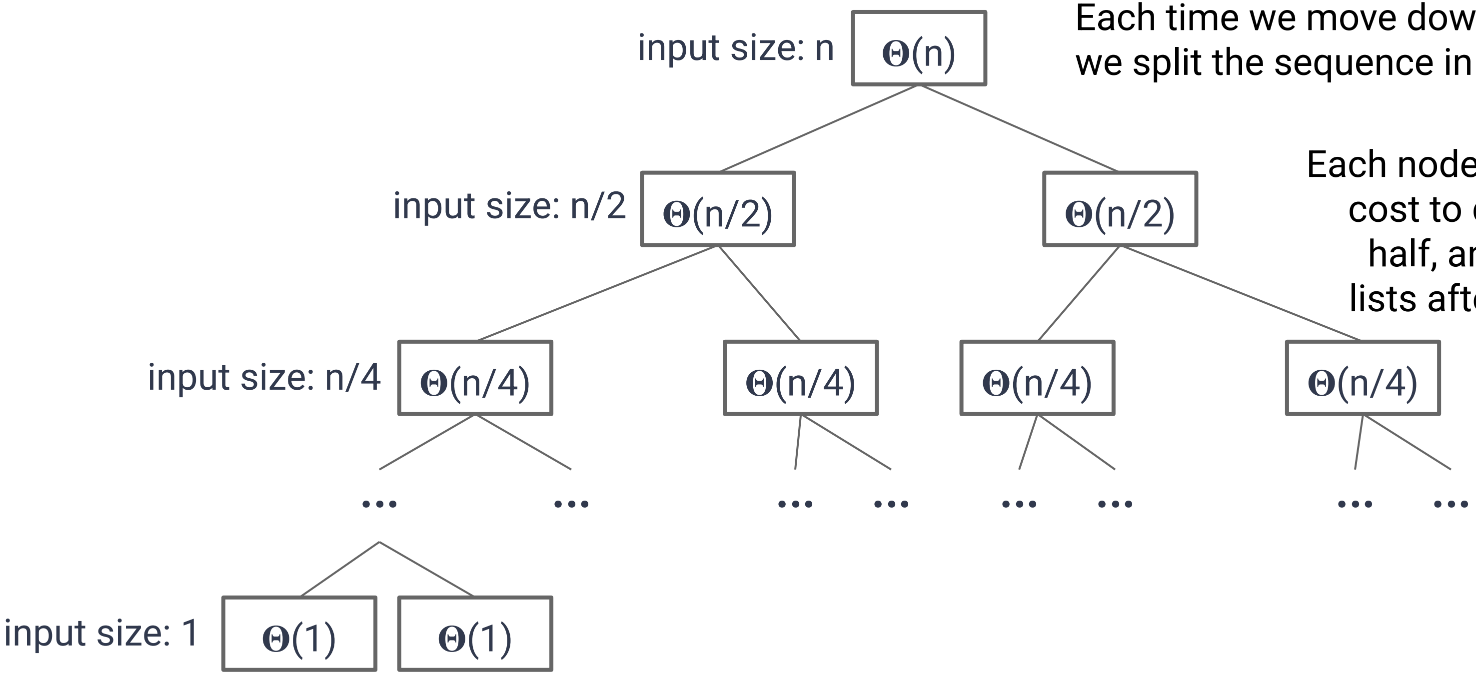

Each time we move down a level, we split the sequence in half

Each node is labeled with the total cost to dividing the sequence in half, and combining the sorted lists after they are sorted by the lower levels

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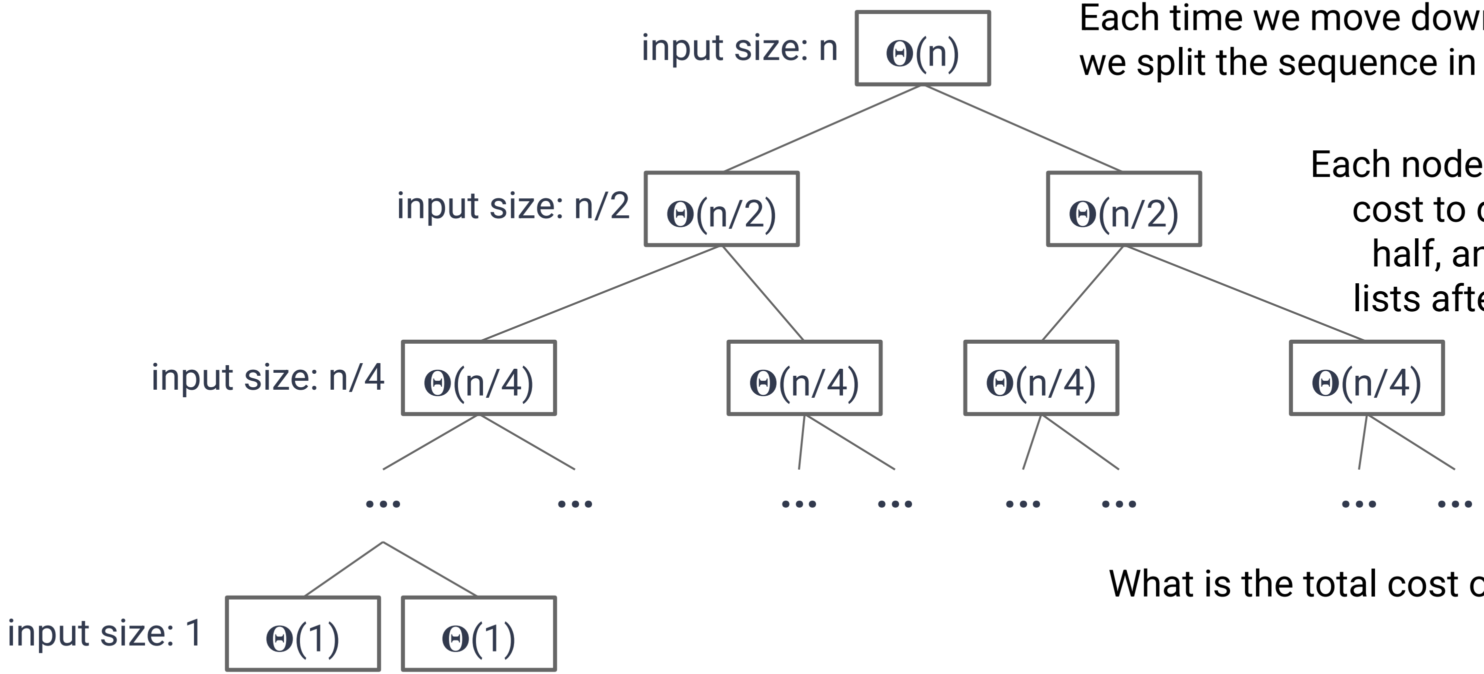

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What is the total cost of each level?

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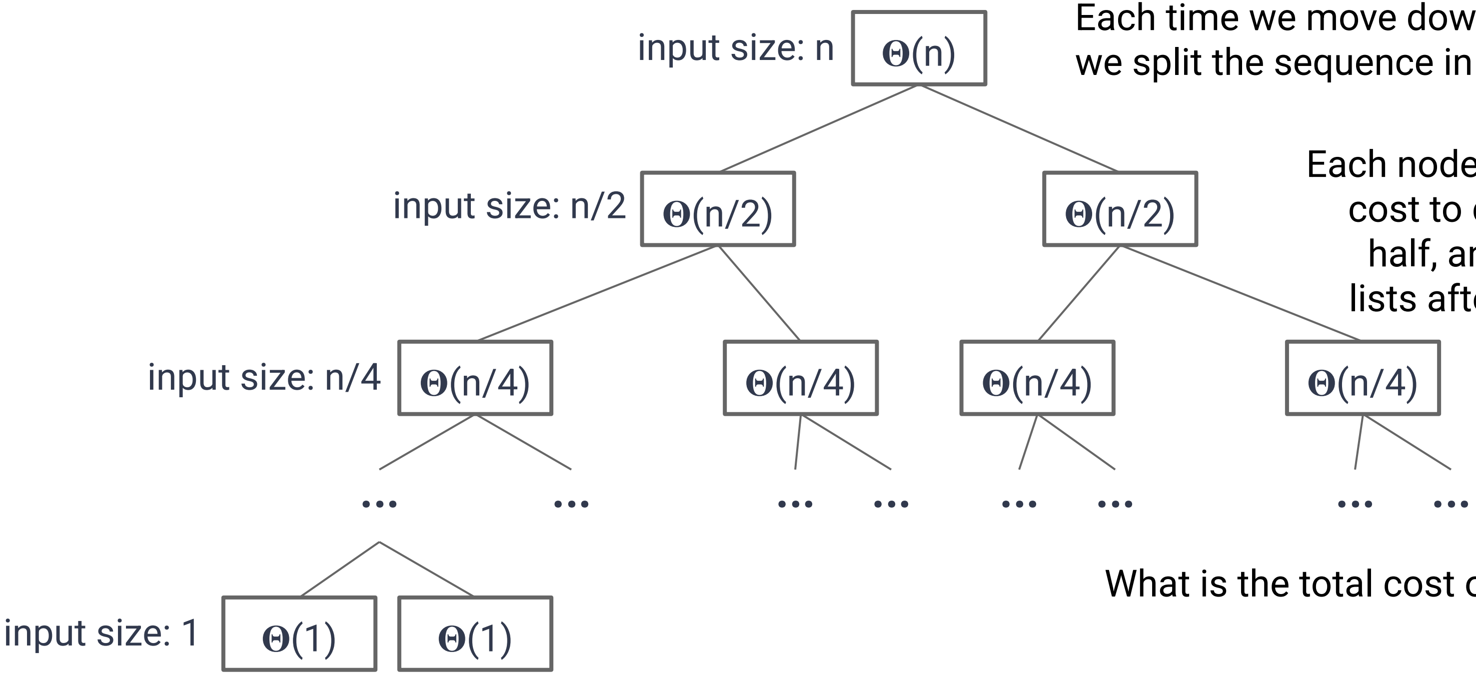


Each time we move down a level, we split the sequence in half

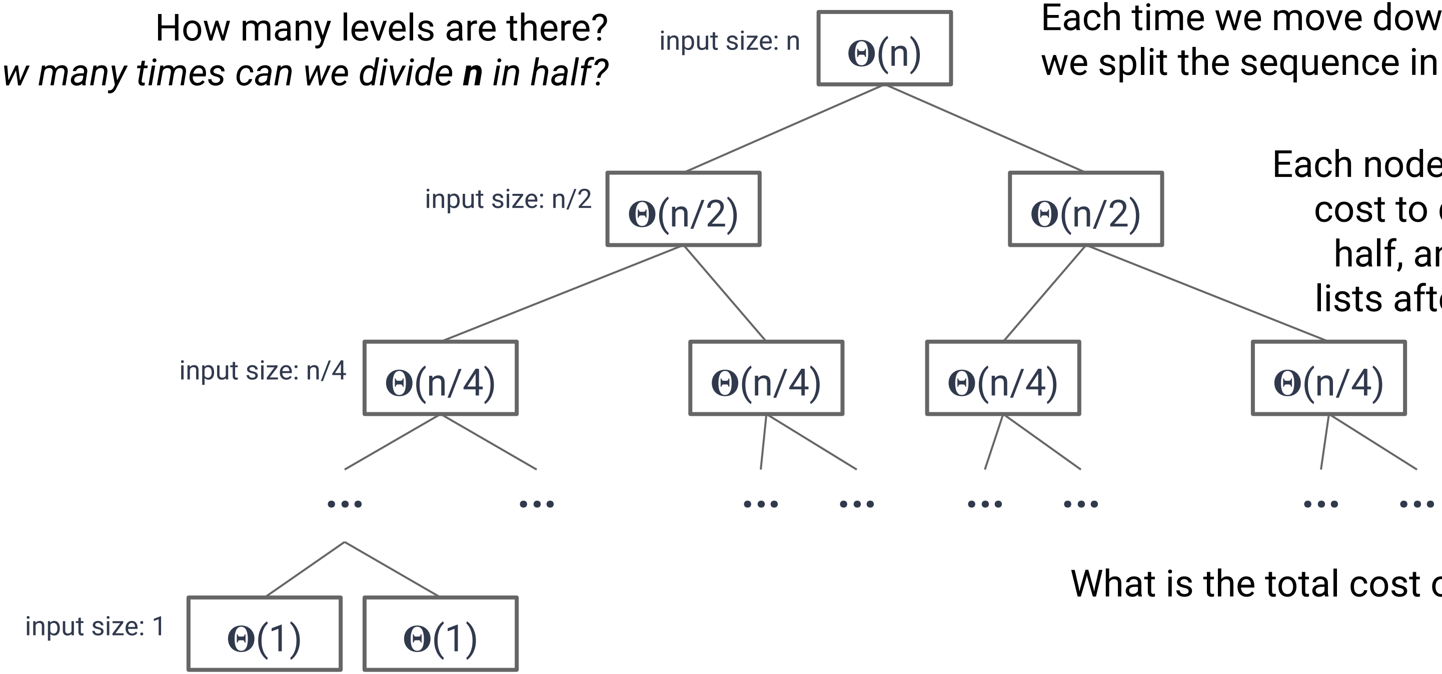
Each node is labeled with the total cost to dividing the sequence in half, and combining the sorted lists after they are sorted by the lower levels

What is the total cost of each level? $\Theta(n)$

Merge Sort: Recursion Tree

How many levels are there?
*How many times can we divide **n** in half?*

Each time we move down a level, we split the sequence in half



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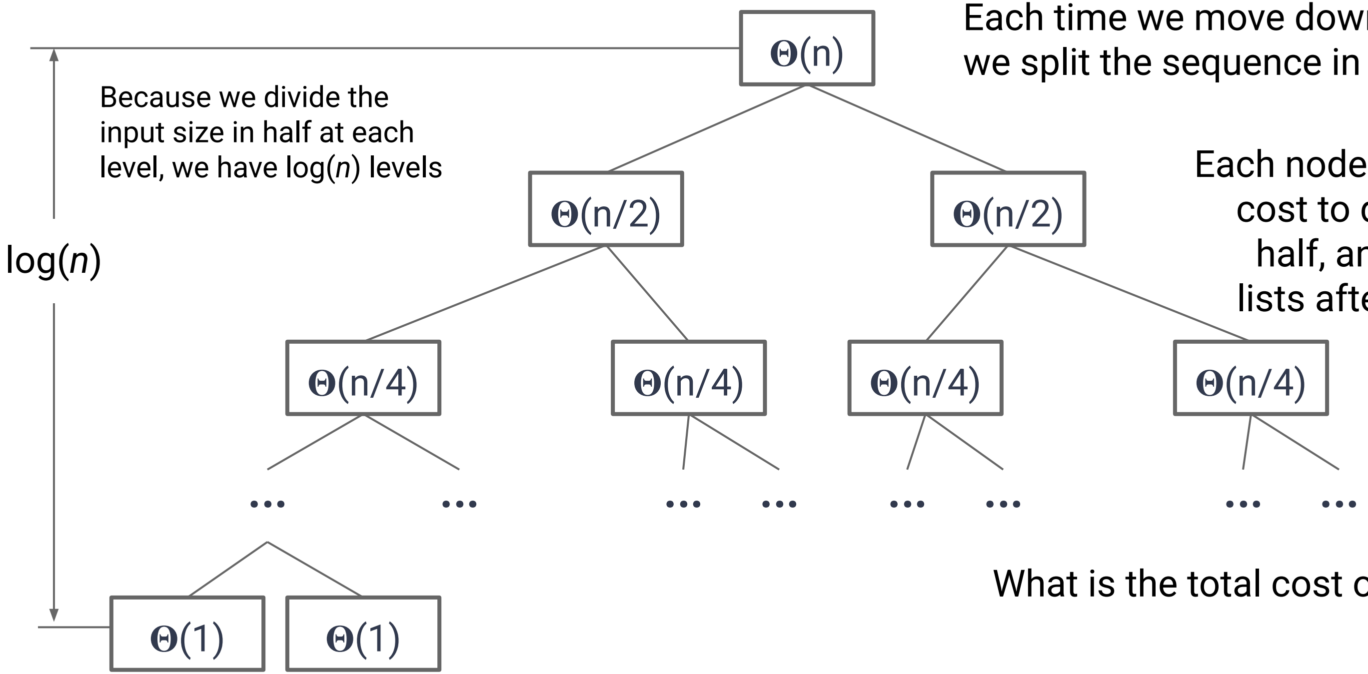

Because we divide the input size in half at each level, we have log(n) levels

Each node is labeled with the total cost to dividing the sequence in half, and combining the sorted lists after they are sorted by the lower levels

What is the total cost of each level? $\Theta(n)$

Hypothesis: The cost of merge sort is n log(n)

Merge Sort: Recursion Tree Details

At level i there are 2^i tasks, each with runtime $\Theta(n/2^i)$,
and there are $\log(n)$ levels.

$$\sum_{i=0}^{\log(n)} \sum_{j=1}^{2^i} \Theta(\frac{n}{2^i})$$

For Merge Sort: Recursion Trees

At level *i* there are 2^i tasks, each with runtime $\Theta(n/2^i)$,
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Merge Sort Runtime

$$\sum_{i=0}^{\log(n)} \sum_{j=1}^{2^i} \Theta(\frac{n}{2^i})$$

Merge Sort Runtime

$$\sum_{i=0}^{\log(n)} \sum_{j=1}^{2^i} \Theta(\frac{n}{2^i})$$

$$\sum_{i=0}^{\log(n)} (2^i + 1 - 1)\Theta(\frac{n}{2^i})$$

Merge Sort Runtime

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$$\sum_{i=0}^{\log(n)} 2^i\Theta(\frac{n}{2^i})$$

Merge Sort Runtime

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$$\sum_{i=0}^{\log(n)} \Theta(n)$$

Merge Sort Runtime

$$\sum_{i=0}^{\log(n)} 2^i \Theta(\frac{n}{2^i})$$

$$\sum_{i=0}^{\log(n)} \Theta(n)$$

$$(\log(n) - 0 + 1)\Theta(n)$$

Merge Sort Runtime

$$\sum_{i=0}^{\log(n)} 2^i \Theta(\frac{n}{2^i})$$

$$\sum_{i=0}^{\log(n)} \Theta(n)$$

$$(\log(n) - 0 + 1)\Theta(n)$$

$$\Theta(n \log(n)) + \Theta(n)$$

Merge Sort Runtime

$$\sum_{i=0}^{\log(n)} 2^i \Theta(\frac{n}{2^i})$$

$$\sum_{i=0}^{\log(n)} \Theta(n)$$

$$(\log(n) - 0 + 1)\Theta(n)$$

$$\Theta(n\log(n)) + \Theta(n)$$

$$\Theta(n\log(n))$$

Merge Sort Runtime: Inductive Proof

Now we can use induction to prove that there is a *c*, n_0 s.t. ***T*(*n*) ≤ c *n*log(*n*)** for any $n > n_0$

$$T(n) = \begin{cases} c_0 & \textbf{if } n \le 1 \\ 2 \cdot T(\frac{n}{2}) + c_1 + c_2 \cdot n & \textbf{otherwise} \end{cases}$$

Merge Sort Runtime: Inductive Proof

Base Case: $T(1) \leq c\ 1 \log(1)$

~~$c_0 \leq 0$~~

$T(2) \leq c\ 2 \log(2)$

$2c_0 + c_1 + 2c_2 \leq 2c$

True when $c = c_0 + c_1 + c_2$

Merge Sort Runtime: Inductive Proof

Assume: $T(n/2) \leq c\,(n/2)\log(n/2)$

Show: $T(n) \leq cn\log(n)$

Merge Sort Runtime: Inductive Proof

Assume: $T(n/2) \leq c\,(n/2)\log(n/2)$

Show: $T(n) \leq cn\log(n)$

How did we choose our smaller problem size?

Merge Sort Runtime: Inductive Proof

Assume: $T(n/2) \leq c\,(n/2)\log(n/2)$
Show: $T(n) \leq cn\log(n)$

How did we choose our smaller problem size?

Our runtime for ***n*** relies on the runtime for ***n/2***

$$T(n) = \begin{cases} c_0 & \textbf{if } n \leq 1 \\ 2 \cdot \boxed{T(\frac{n}{2})} + c_1 + c_2 \cdot n & \textbf{otherwise} \end{cases}$$

Merge Sort Runtime: Inductive Proof

Assume: $T(n/2) \leq c\,(n/2)\log(n/2)$

Show: $T(n) \leq cn\log(n)$

$$2 \cdot T(\frac{n}{2}) + c_1 + c_2 n \leq cn\log(n)$$

Merge Sort Runtime: Inductive Proof

Assume: $T(n/2) \leq c\,(n/2)\log(n/2)$

Show: $T(n) \leq cn\log(n)$

$$2 \cdot \boxed{T(\frac{n}{2})} + c_1 + c_2 n \leq cn\log(n)$$

This matches the left hand side of our assumption!
We can substitute the right hand side, and use transitivity

Merge Sort Runtime: Inductive Proof

Assume: $T(n/2) \leq c\,(n/2)\log(n/2)$

Show: $T(n) \leq cn\log(n)$

$$2 \cdot T(\frac{n}{2}) + c_1 + c_2 n \leq cn\log(n)$$

By the assumption, and transitivity, we just need to show:

$$2c\frac{n}{2}\log\left(\frac{n}{2}\right) + c_1 + c_2 n \leq cn\log(n)$$

Merge Sort Runtime: Inductive Proof

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$$c_1 + c_2 n \leq cn\log(2)$$

Merge Sort Runtime: Inductive Proof

$$c_1 + c_2 n \leq cn\log(2)$$

Merge Sort Runtime: Inductive Proof

$$c_1 + c_2 n \leq cn\log(2)$$

$$\frac{c_1}{n\log(2)} + \frac{c_2}{\log(2)} \leq c$$

Merge Sort Runtime: Inductive Proof

$$c_1 + c_2 n \leq cn\log(2)$$

$$\frac{c_1}{n\log(2)} + \frac{c_2}{\log(2)} \leq c$$

Which is true for any

$$n_0 \geq \frac{c_1}{\log(2)} \quad \text{and} \quad c > \frac{c_2}{\log(2)} + 1$$