CSE 250

Data Structures

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Lec 28: AVL Trees

Announcements

- WA4 due Sunday
- Midterm next Friday (See Piazza for details)
  - Extra review session from SAs tomorrow (see Piazza @384)
  - Review lecture on Wednesday
  - Practice exams will be up later today

BST Operations

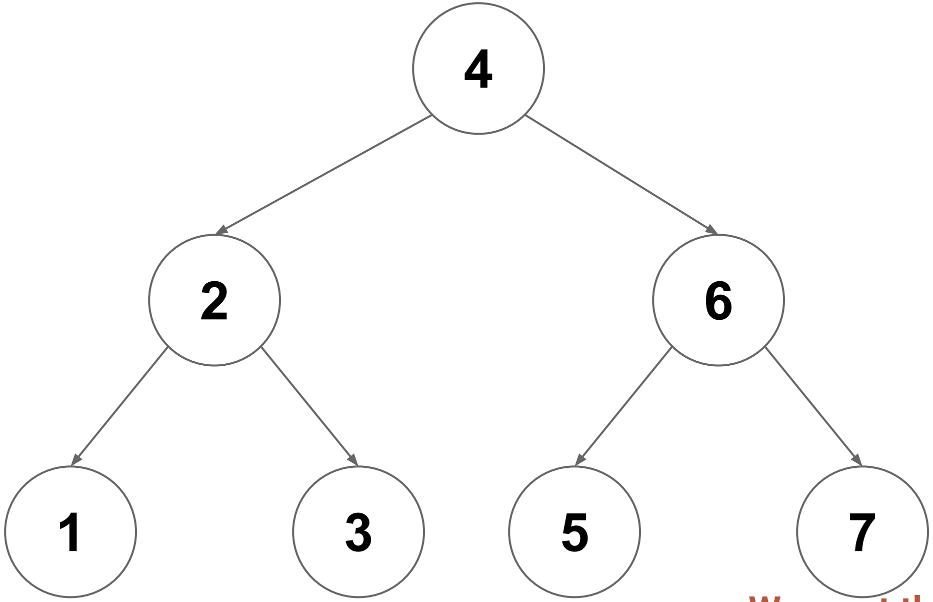
Operation	Runtime
`find`	$O(d)$
`insert`	$O(d)$
`remove`	$O(d)$

What is the runtime in terms of n*?* $O(n)$

$\log(n) \leq d \leq n$

Tree Depth vs Size

If height(left) ≈ height(right)



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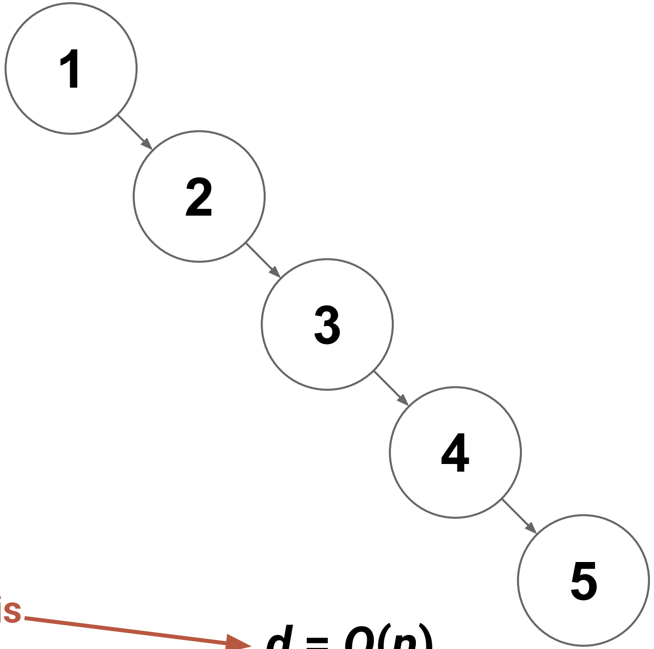


$d = O(\log(n))$

We want this, not this

$d = O(n)$

Keeping Depth Small - Two Approaches

Option 1

Keep tree **balanced**: subtrees **+/-1** of each other in height

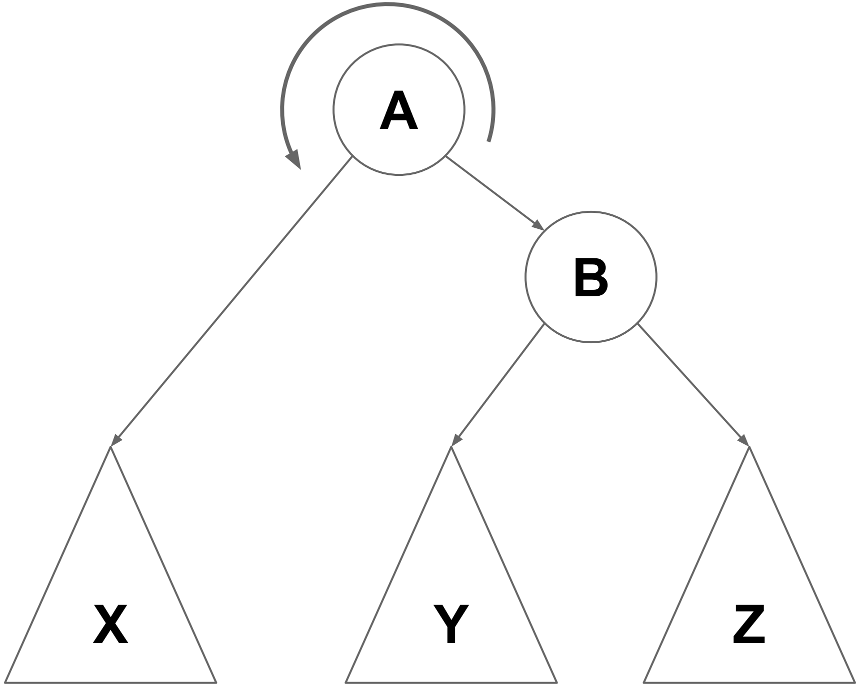
(add a field to track amount of "imbalance")

Option 2

Keep leaves at some minimum depth (***d*/2**)

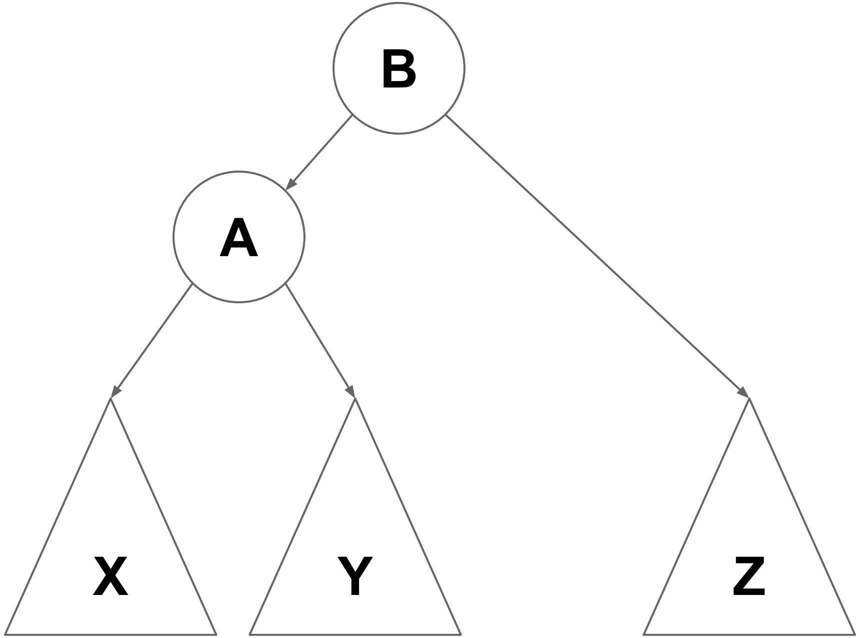
(Add a color to each node marking it as "red" or "black")

Rebalancing Trees (rotations)



Rotate(A, B)

Rebalancing Trees (rotations)



Rotate(A, B)

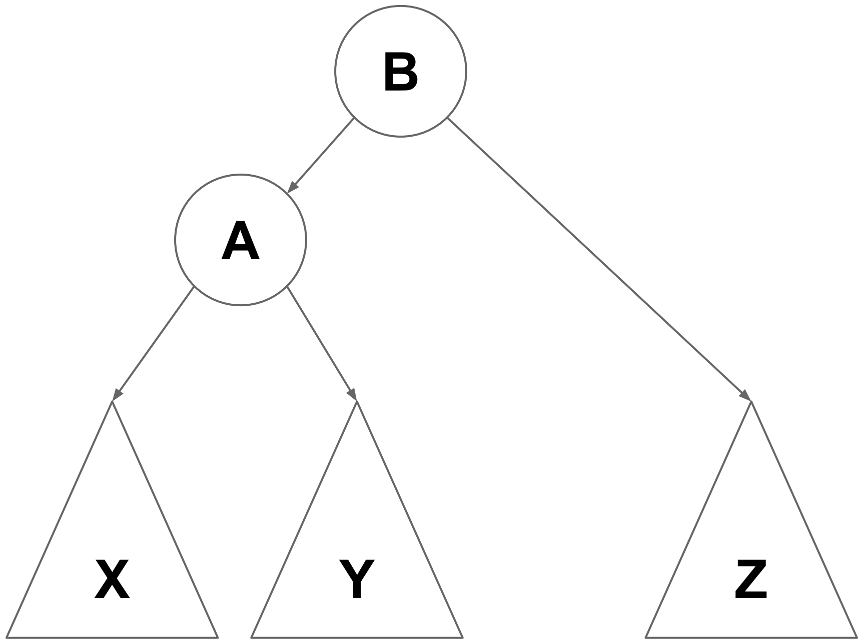
Rebalancing Trees (rotations)

A became **B**'s left child

B's left child became **A**'s right child

Is ordering maintained? **Yes!**

Complexity? ***O*(1)**



Rotate(A, B)

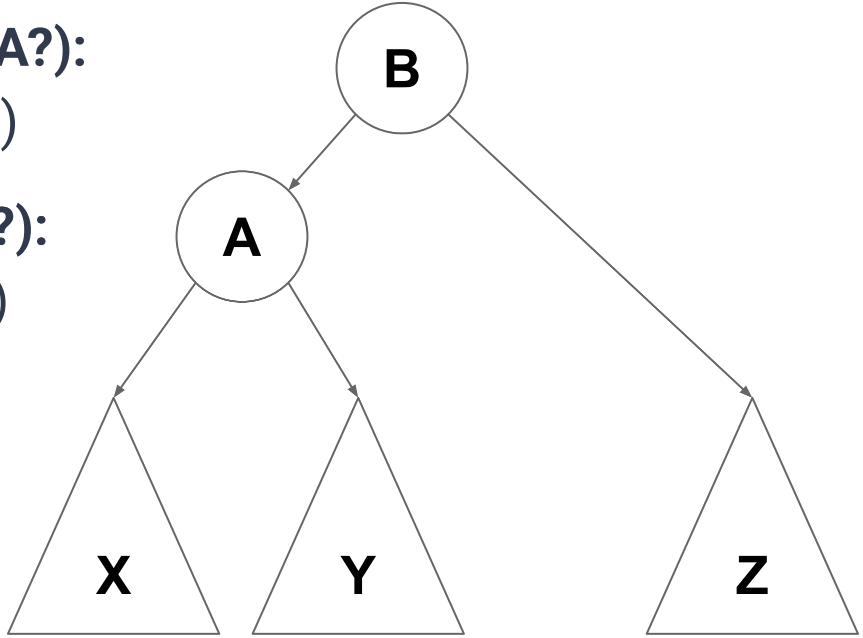
Rebalancing Trees (rotations)

Before Rotation (what is the height of A?):

h(A) = 1 + max(h(X), 1 + max(h(Y), h(Z))

After Rotation (what is the height of B?):

h(B) = 1+ max(1 + max(h(X),h(Y)), h(Z))



Rotate(A, B)

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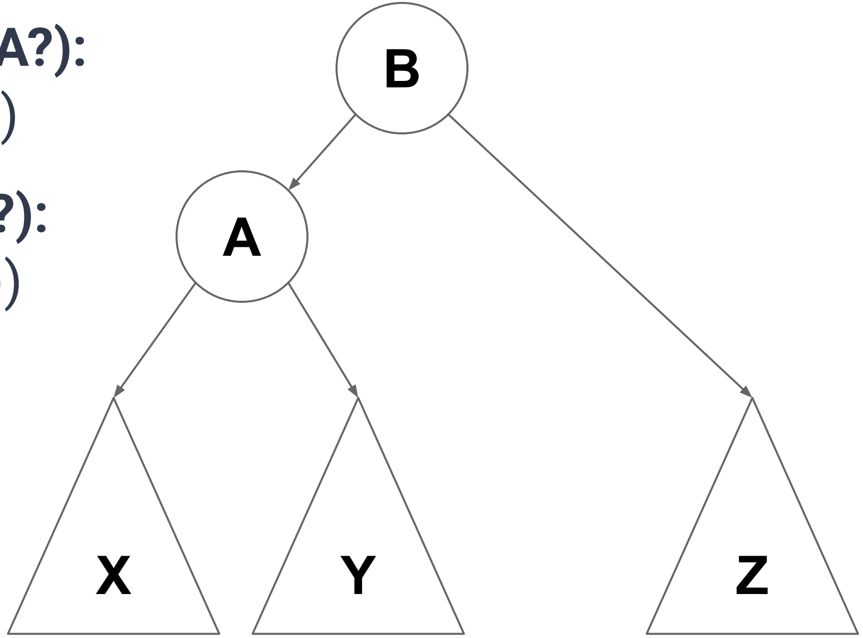
- If **X** was the tallest of **X,Y,Z** our total height increased by 1.
- If **Z** was the tallest our total height decreased by 1.
- If **X,Z** same height, or **Y** is the tallest then total is unchanged



Rotate(A, B)

Rebalancing Trees (rotations)

Before Rotation (what is the height of A?):

h(A) = 1 + ma

Therefore, a single left (or right) rotation can change the height of the tree by +1/0/-1

After Rotatio

h(B) = 1 + max(1 + max(h(X),h(Y)), h(Z))

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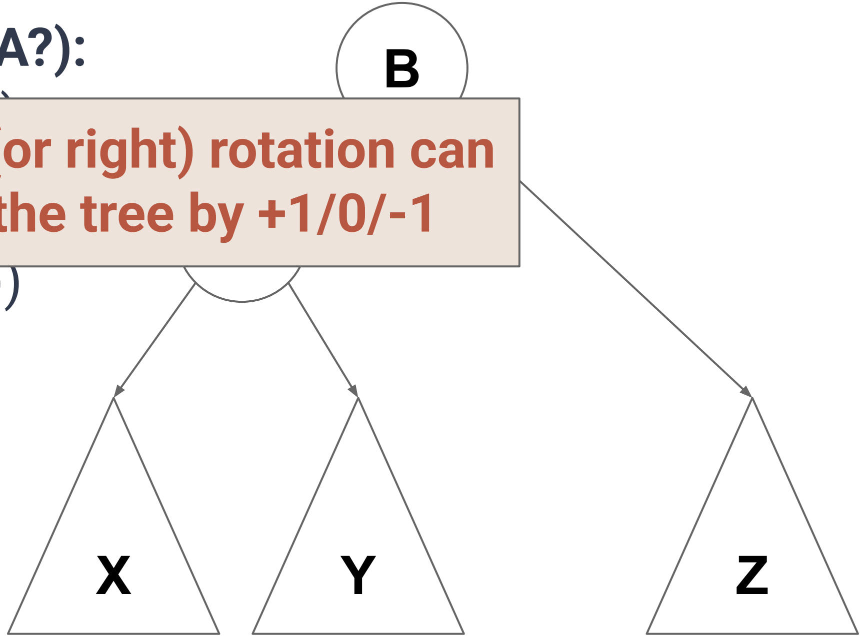

Rotate(A, B)

AVL Trees

AVL Trees

An **<u>AVL tree</u>** (**<u>A</u>**delson-**<u>V</u>**elsky and **<u>L</u>**andis) is a ***BST*** where **every** subtree is depth-balanced

Remember: Tree depth = height(root)

Balanced: |height(root.right) - height(root.left)| ≤ 1

AVL Trees

Define **`balance(`*`v`*`) = height(`*`v.right`*`) - height(`*`v.left`*`)`**

Goal: Maintaining **`balance(`*`v`*`)`** **∈ { -1, 0, 1 }**

- **`balance(`*`v`*`) = 0`** → "***v*** is balanced"
- **`balance(`*`v`*`) = -1`** → "***v*** is left-heavy"
- **`balance(`*`v`*`) = 1`** → "***v*** is right-heavy"

AVL Trees

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What does enforcing this gain us?

AVL Trees - Depth Bounds

Question: Does the AVL property result in any guarantees about depth?

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YES! Depth balance forces a maximum possible depth of **log(*n*)**

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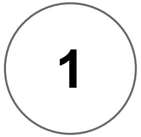
YES! Depth balance forces a maximum possible depth of **log(*n*)**

Proof Idea: An AVL tree with depth ***d*** has "enough" nodes

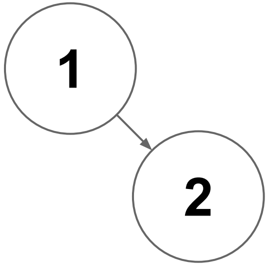
AVL Trees - Depth Bounds

Let **`minNodes(`*`d`*`)`** be the min number of nodes an in AVL tree of depth ***d***

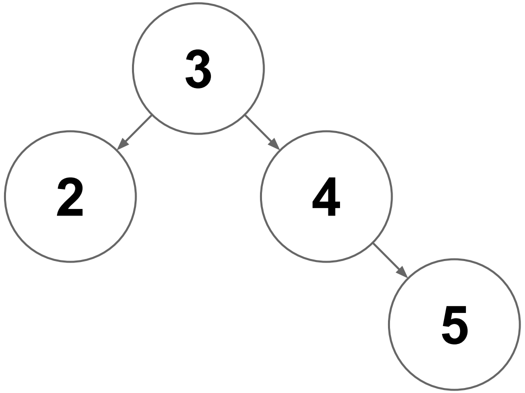
minNodes(0) = 1



minNodes(1) = 2



minNodes(2) = 4



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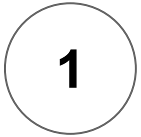

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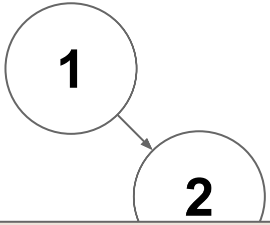

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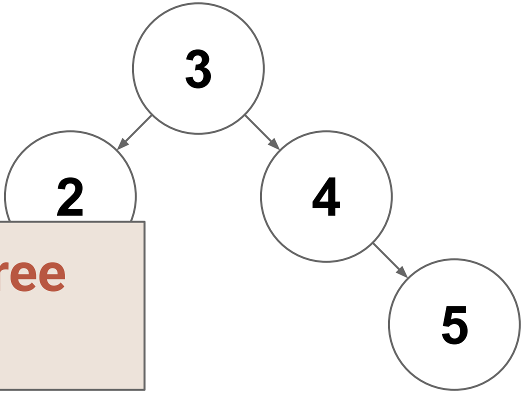
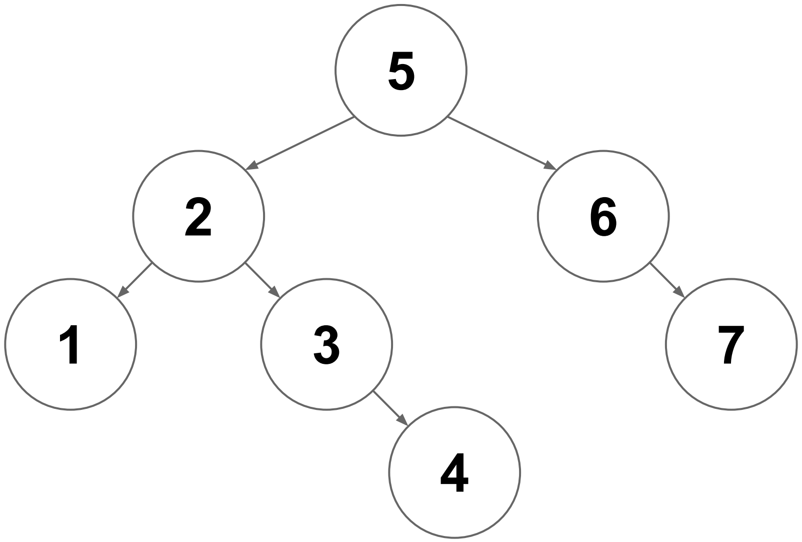


Work with you neighbors to come up with an AVL tree of depth 3 with the fewest nodes possible

AVL Trees - Depth Bounds

The minimum number of nodes for an AVL tree of depth 3…is 7!



AVL Trees - Depth Bounds

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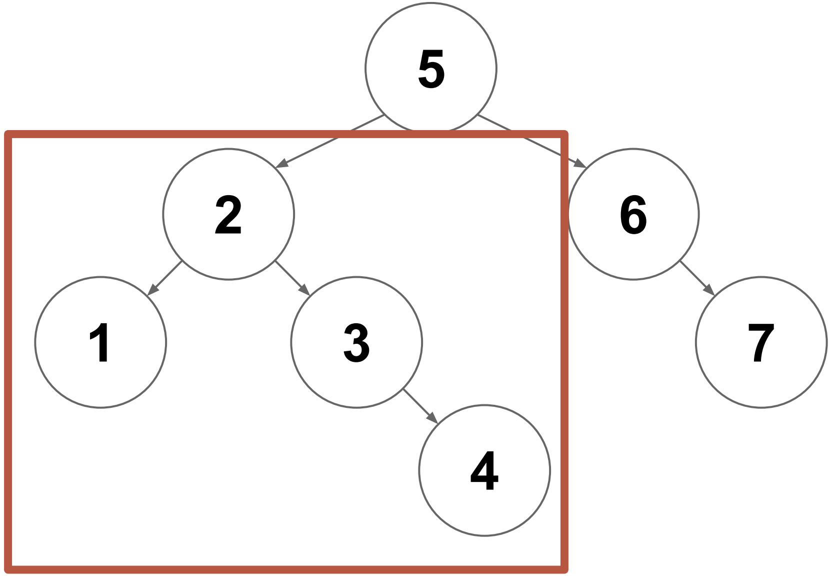
This is just the minimum AVL tree of depth 2



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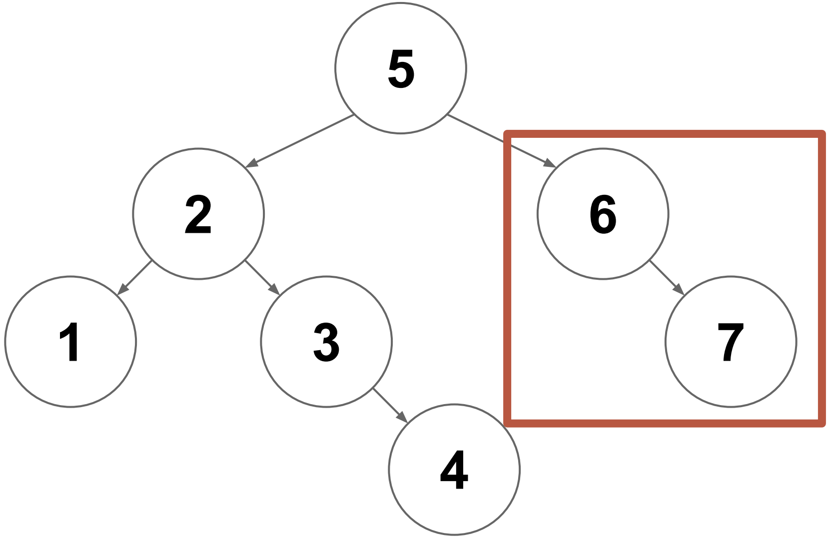
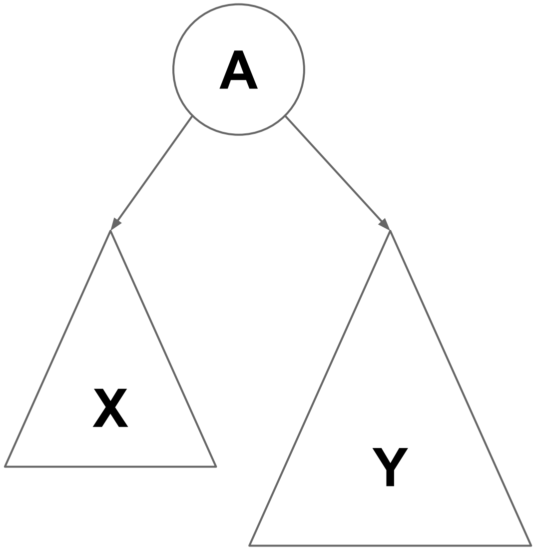


This is just the minimum AVL tree of depth 1

AVL Trees - Depth Bounds

For any tree of depth ***d***:



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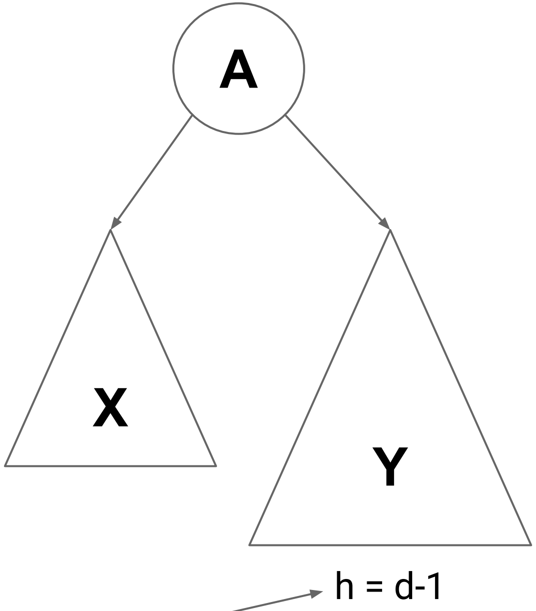

At least one subtree must have depth of ***d* - 1**
(because total depth is ***d***)

AVL Trees - Depth Bounds

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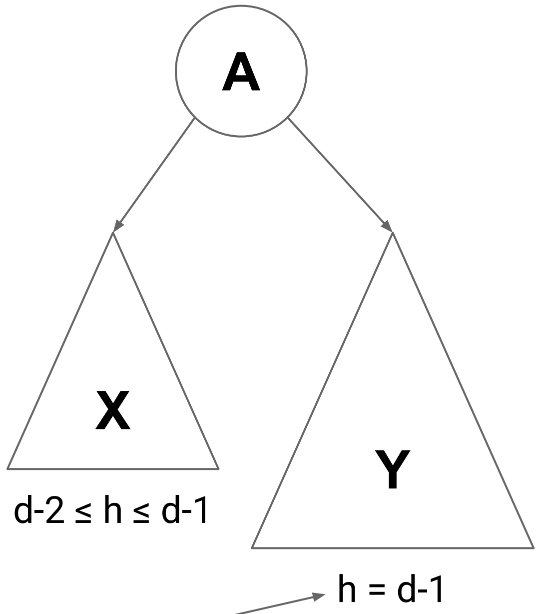

The other subtree must have a depth of at least ***d*** **- 2** because the AVL constraint does not allow it to differ by more than 1

At least one subtree must have depth of ***d*** **- 1** (because total depth is ***d***)

AVL Tree - Depth Bounds

For $\boldsymbol{d} < 1$:

minNodes($\boldsymbol{d}$) = 1 + minNodes($\boldsymbol{d}$ **- 1**) + minNodes($\boldsymbol{d}$ **- 2**)

AVL Tree - Depth Bounds

For $d < 1$:

$$\text{minNodes}(d) = 1 + \boxed{\text{minNodes}(d - 1) + \text{minNodes}(d - 2)}$$

This is the Fibonacci Sequence!

AVL Tree - Depth Bounds

For $d < 1$:

minNodes(d) = 1 + $\boxed{\text{minNodes}(\boldsymbol{d - 1}) + \text{minNodes}(\boldsymbol{d - 2})}$

This is the Fibonacci Sequence!

What is the d^{th} term of the Fibonacci sequence?

Coarse approximation: minNodes(d) = $\Omega(1.5^d)$

https://en.wikipedia.org/wiki/Fibonacci_sequence

AVL Tree - Depth Bounds

minNodes($\boldsymbol{d}$) = $\Omega(1.5^{\boldsymbol{d}})$

AVL Tree - Depth Bounds

minNodes(d) = $\Omega(1.5^d)$

$$n \geq c1.5^d$$

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All constants

AVL Tree - Depth Bounds

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$$d \in O(\log_2(n))$$

AVL Tree - Depth Bounds

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$$\log_2\left(\frac{n}{c}\right) \geq d$$

$$\log_2\left(\frac{n}{c}\right) \geq \log_2 \ldots$$

$$\frac{\ldots}{\log_2(1.5)} - \frac{\log_2(c)}{\log_2(1.5)} \geq d$$

Therefore if we enforce the AVL constraint, then a tree with *n* nodes will have logarithmic depth

$$\log_2\left(\frac{n}{c}\right) \geq d\log_2(1.5)$$

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AVL Tree - Depth Bounds

minNodes(***d***) = **Ω**(1.5^*d*^)

$$n \geq c1.5^d$$

$$\log_2\left(\frac{n}{c}\right) \geq \log$$

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So how do we enforce the constraint?

Enforcing the AVL Constraint

- Computing `balance()` on the fly is expensive
  - `balance()` calls `height()` twice
  - Computing `height()` requires visiting every node

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Idea: Store height of each node at the node

Better Idea: Just store the balance factor (only needs 2 bits)

Enforcing the AVL Constraint

```java
public class AVLTreeNode<T> {
  T value;
  Optional<AVLTreeNode<T>> parent; // We need a ref to parent to rotate
  Optional<AVLTreeNode<T>> leftChild;
  Optional<AVLTreeNode<T>> rightChild;
  Boolean isLeftHeavy;   // true if height(right) - height(left) == -1
  Boolean isRightHeavy;  // true if height(right) - height(left) == 1
}
```

Need to add 3 fields to our TreeNode class to make it an AVLTreeNode

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Assume we have a valid AVL tree and we modify it. How might this break our AVL constraint?

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Enforcing the AVL Constraint

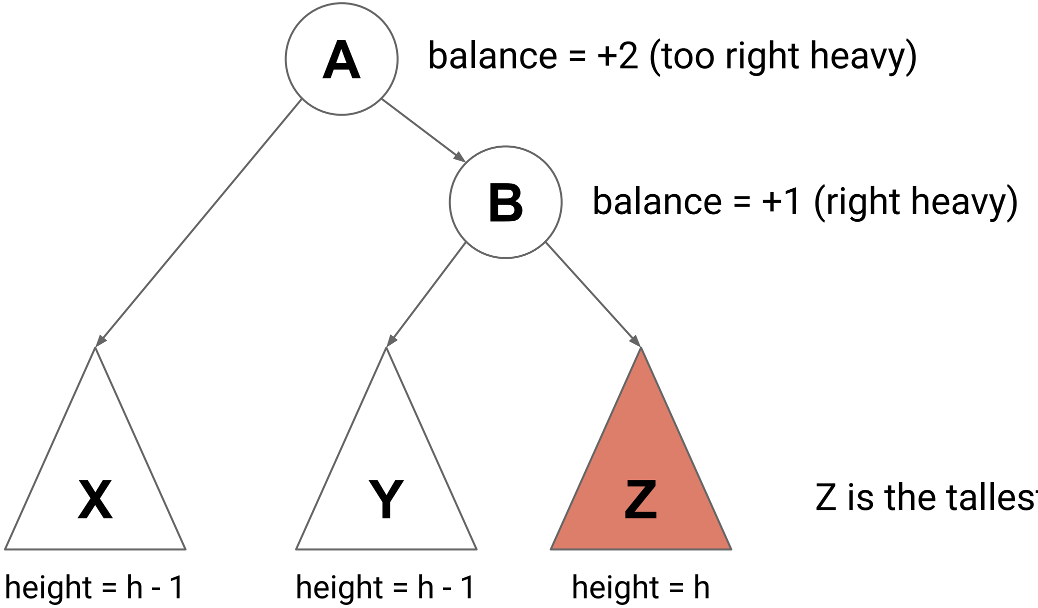
Assume we have a valid AVL tree and we modify it. How might this break our AVL constraint?

- What is the effect on the height of `insert`? Increases by **at most** 1
- What is the effect on the height of `remove`? Decreases by **at most** 1

Therefore after an operation that modifies an AVL tree, the difference in heights can be **at most** 2.

What are the exact ways this broken constraint might show up?

Enforcing the AVL Constraint: Case 1



How can we fix this?

Z is the tallest of X, Y, Z

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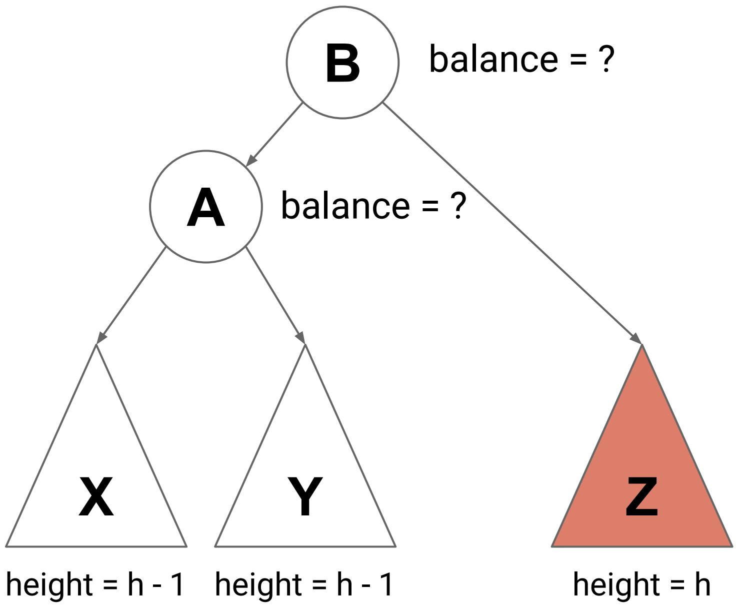

How can we fix this? **`rotate(A,B)`**

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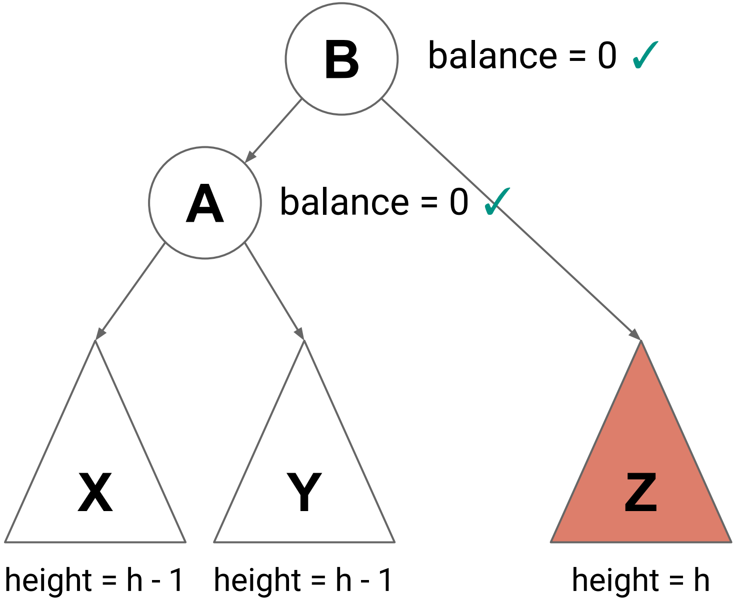

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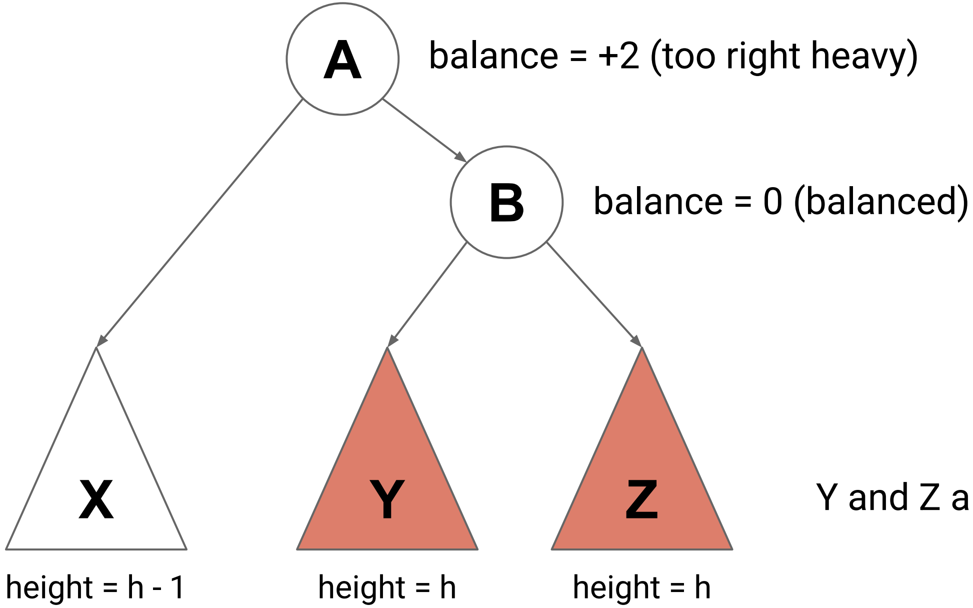
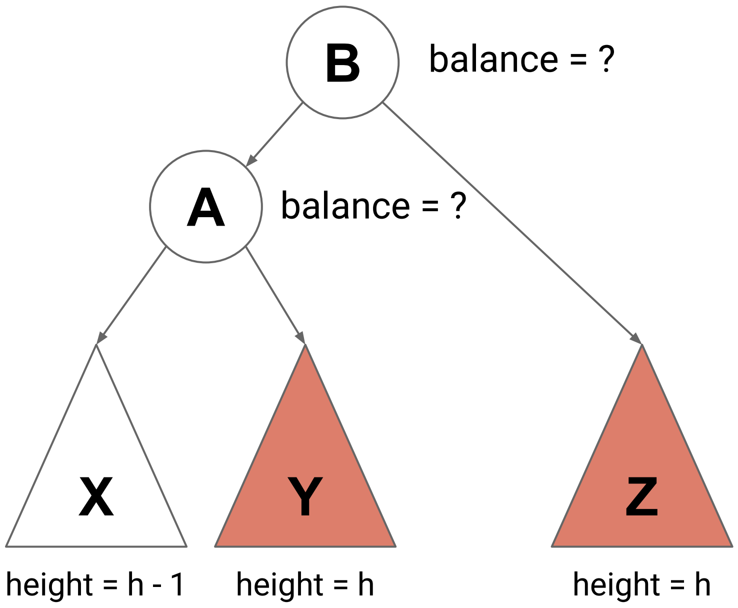
Enforcing the AVL Constraint: Case 2



How can we fix this?

Y and Z are taller than X

Enforcing the AVL Constraint: Case 2



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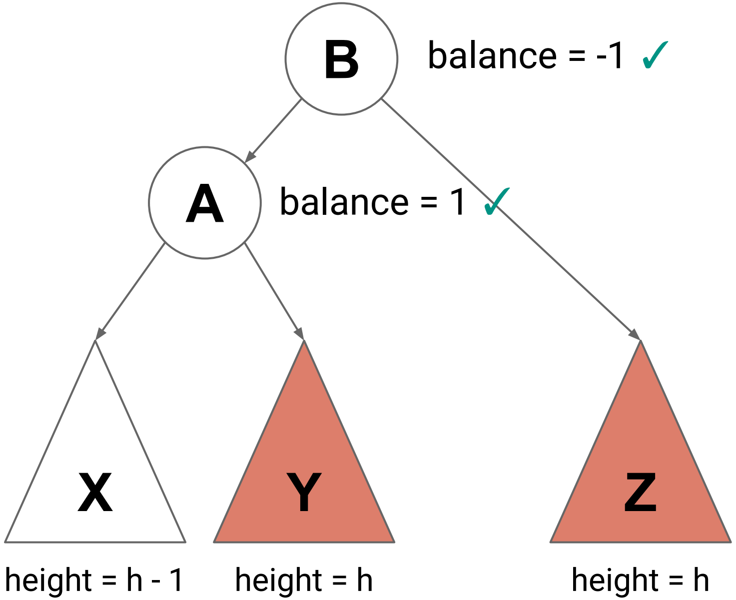

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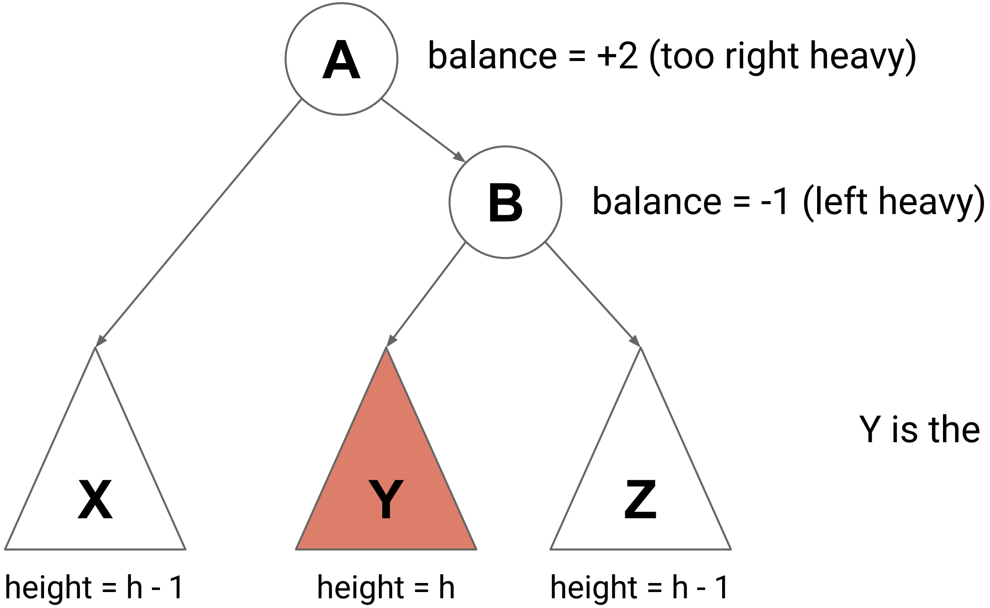
Enforcing the AVL Constraint: Case 3



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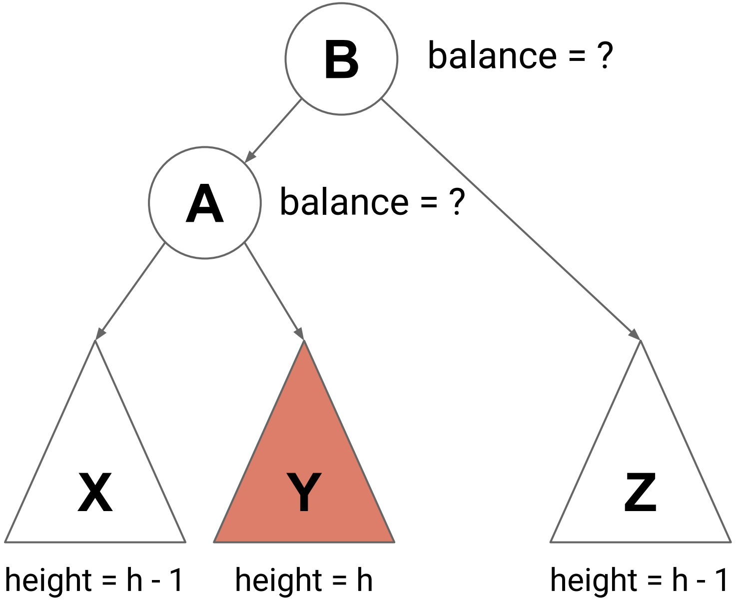
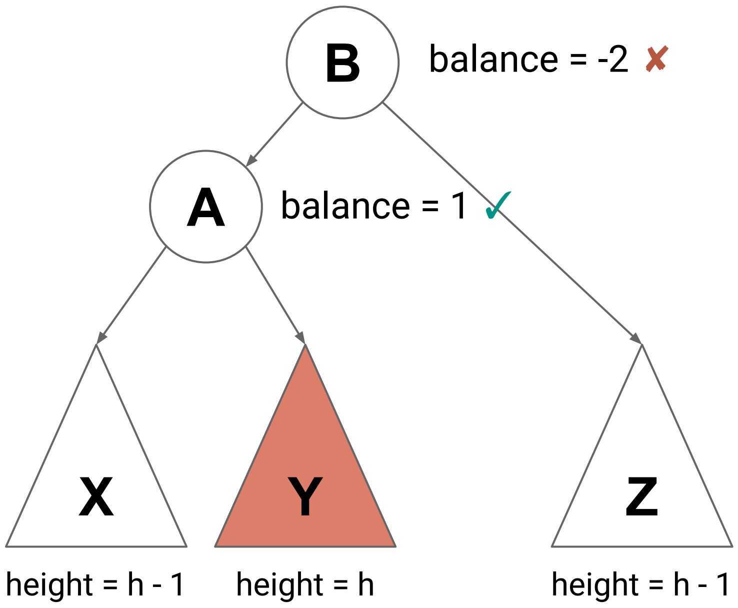
Enforcing the AVL Constraint: Case 3



How can we fix this?
Will just a single left rotation work?

Enforcing the AVL Constraint: Case 3



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Will just a single left rotation work? **No**

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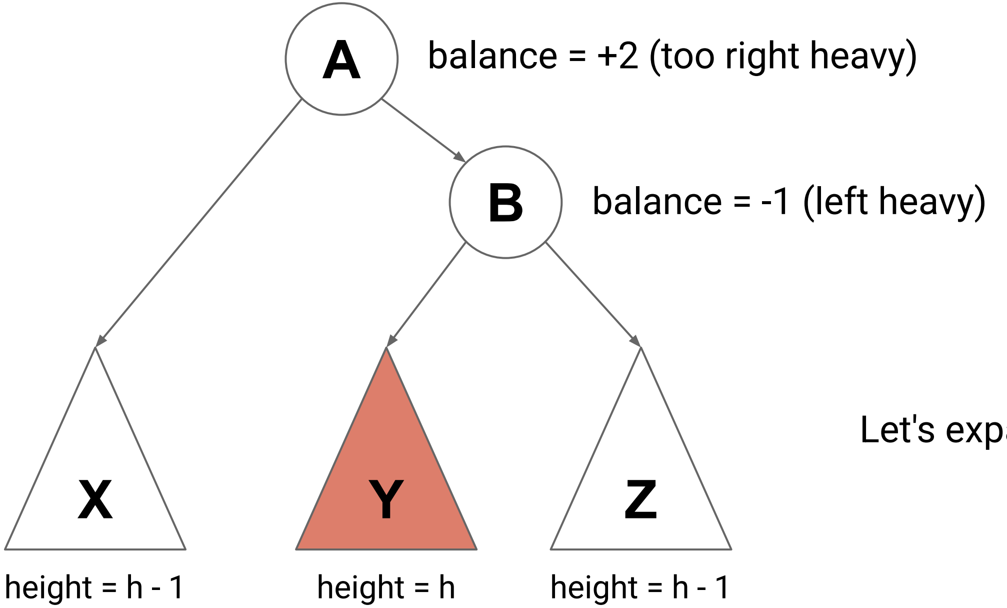

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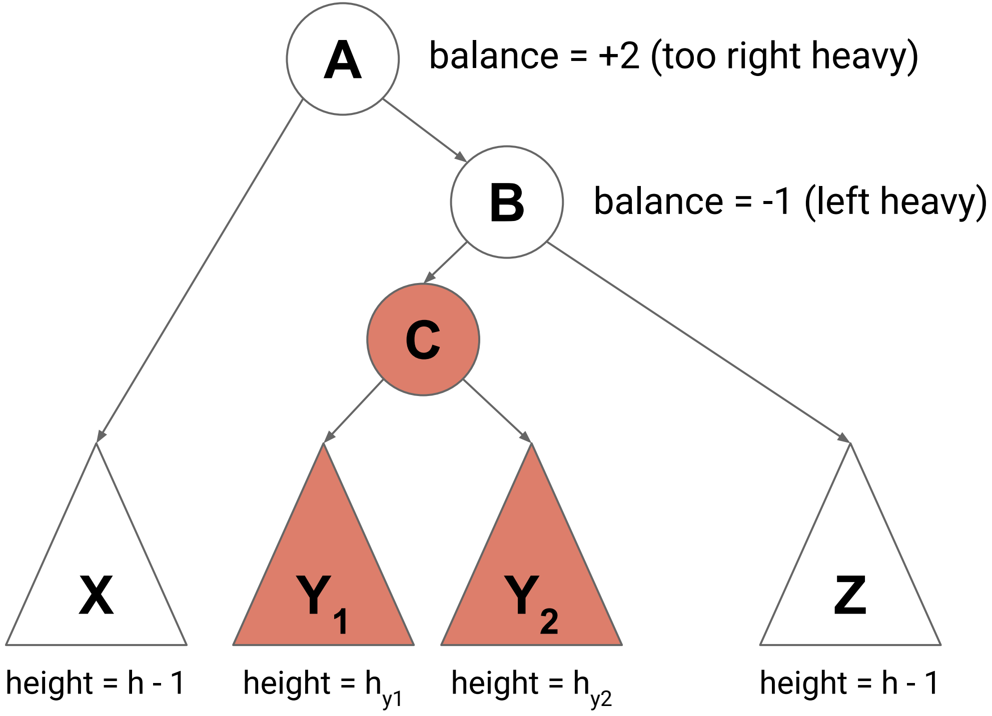
Let's expand Y to figure out what to do

Enforcing the AVL Constraint: Case 3



How can we fix this?

Height of **C** we know must be ***h***

Therefore At least one of h_{y1} or h_{y2} must be $h - 1$

The other can be $h - 2$, or $h - 1$

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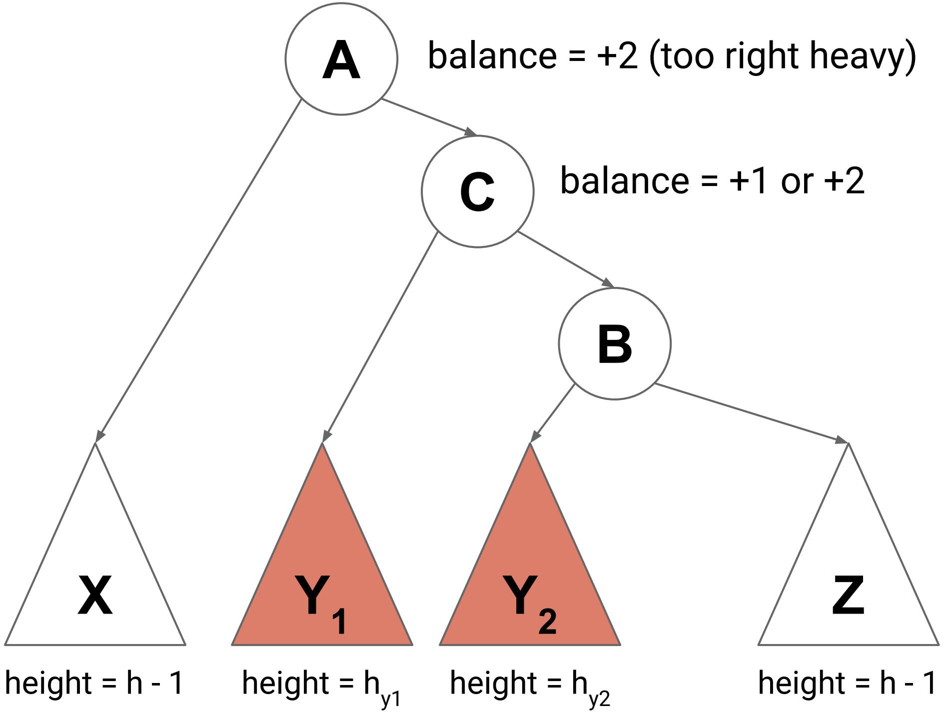

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Rotate right first: `rotate(B,C)`

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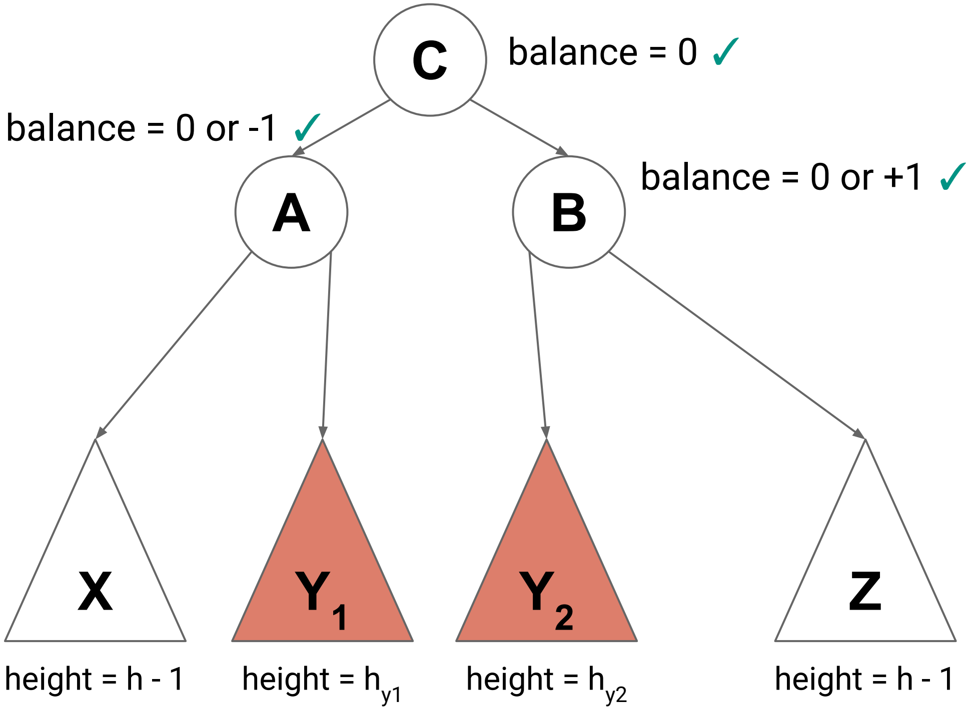

How can we fix this?
Rotate right first: **`rotate(B,C)`**
Then right left: **`rotate(A,C)`**

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Therefore At least one of $\boldsymbol{h_{y1}}$ or $\boldsymbol{h_{y2}}$ must be ***h* - 1**

The other can be ***h* - 2**, or ***h* - 1**

Enforcing the AVL Constraint

- If too right heavy (balance == +2)
  - If right child is right heavy (balance == +1) or balanced (balance == 0)
    - rotate left around the root
  - If right child is left heavy (balance == -1)
    - rotate right around root of right child, then rotate left around root
- If too left heavy (balance == -2)
  - Same as above but flipped

Therefore if we have a balance factor that is off, but all children are AVL trees, we can fix the balance factor in at most 2 rotations

Inserting Records

To insert a record into an AVL Tree:

1. Find the insertion point (remember it is a BST)
2. Insert the new leaf and set balance factor to 0
3. Trace path back up to root and update balance factors
   a. If a balance factor becomes +/-2 then rotate to fix

Inserting Records

To insert a record into an AVL Tree:

1. Find the insertion point (remember it is a BST) **$O(d) = O(\log n)$**
2. Insert the new leaf and set balance factor to 0 **$O(1)$**
3. Trace path back up to root and update balance factors **$O(d) = O(\log n)$**
    a. If a balance factor becomes +/-2 then rotate to fix **$O(1)$**

Inserting New Nodes

```java
public void insert(T value, AVLTreeNode<T> root) {
  // Use normal logic for inserting into a BST, then set heavy flags
  AVLTreeNode<T> newNode = insertIntoBST(value, root);
  newNode.isLeftHeavy = newNode.isRightHeavy = false;
  while (newNode.parent.isPresent()) {
    if (newNode.parent.get().leftChild.orElse(null) == newNode) {
      // Fix issues that occur from inserting into parents left subtree
    } else {
      // Fix issues that occur from inserting into parents right subtree
    }
    newNode = newNode.parent.get();
  }
}
```

Inserting New Nodes

```java
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      // Fix issues that occur from inserting into parents right subtree
    }
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  }
}
```

Find insertion point and create the new leaf ***O*(*d*) = *O*(log *n*)**

Inserting New Nodes

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      // Fix issues that occur from inserting into parents right subtree
    }
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  }
}
```

$O(d) = O(\log n)$ iterations

Inserting New Nodes

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  while (newNode.parent.isPresent()) {
    if (newNode.parent.get().leftChild.orElse(null) == newNode) {
      // Fix issues that occur from inserting into parents left subtree
    } else {
      // Fix issues that occur from inserting into parents right subtree
    }
    newNode = newNode.parent.get();
  }
}
```

What is the cost of each iteration?
How exactly do we fix the issues? (next slide)

Inserting New Nodes

```
if (newNode.parent.get().leftChild.orElse(null) == newNode) {
  // Fix issues that occur from inserting into parents left subtree
  if (newNode.parent.get().isRightHeavy) {
    newNode.parent.get().isRightHeavy = false;
    return
  } else if (newNode.parent.get().isLeftHeavy) {
    if (newNode.isLeftHeavy) newNode.parent.get().rotateRight();
    else newNode.parent.get().rotateLeftRight();
    return
  } else {
    newNode.parent.get().isLeftHeavy = true;
  }
}
```

Inserting New Nodes

```
if (newNode.parent.get().leftChild.orElse(null) == newNode) {
  // Fix issues that occur from inserting into parents left subtree
  if (newNode.parent.get().isRightHeavy) {
    newNode.parent.get().isRightHeavy = false;
    return
  } else if (newNode.parent.get().isLeftHeavy) {
    if (newNode.isLeftHeavy) newNode.parent.get().rotateRight();
    else newNode.parent.get().rotateLeftRight();
    return
  } else {
    newNode.parent.get().isLeftHeavy = true;
  }
}
```

If we inserted into the left of a right heavy subtree, then the subtree is no longer right heavy and we can stop here

Inserting New Nodes

```
if (newNode.parent.get().leftChild.orElse(null)
  // Fix issues that occur from inserting into
  if (newNode.parent.get().isRightHeavy) {
    newNode.parent.get().isRightHeavy = false;
    return
  } else if (newNode.parent.get().isLeftHeavy) {
    if (newNode.isLeftHeavy) newNode.parent.get().rotateRight();
    else newNode.parent.get().rotateLeftRight();
    return
  } else {
    newNode.parent.get().isLeftHeavy = true;
  }
}
```

If we inserted into the left of a left heavy subtree, then we just created imbalance, and need to rotate. But then we can stop.

Inserting New Nodes

```
if (newNode.parent.get().leftChild.orElse(null) == newNode) {
  // Fix issues that occur from inserting into parents left subtree
  if (newNode.parent.get().isRightHeavy) {
    newNode.parent.get().isRightHeavy = false;
    return
  } else if (newNode.parent.get().isLeftHeavy)
    if (newNode.isLeftHeavy) newNode.parent.get
    else newNode.parent.get().rotateLeftRight()
    return
  } else {
    newNode.parent.get().isLeftHeavy = true;
  }
}
```

If we inserted into the left of a balanced subtree, then we mark it as now being left heavy, and continue up the tree

Inserting New Nodes

```
public void insert(T value, AVLTreeNode<T> root) {
  // Use normal logic for inserting into a BST, then set heavy flags
  AVLTreeNode<T> newNode = insertIntoBST(value, root);
  newNode.isLeftHeavy = newNode.isRightHeavy = false;
  while (newNode.parent.isPresent()) {
    if (newNode.parent.get().leftChild.orElse(null) == newNode) {
      // Fix issues that occur from inserting into parents left subtree
    } else {
      // Fix issues that occur from inserting into parents right subtree
    }
    newNode = newNode.parent.get();
  }
}
```

What is the cost of each iteration? **O(1)**

Inserting New Nodes

```
public void insert(T value, AVLTreeNode<T> root) {
  // Use normal logic for inserting into a BST, then set heavy flags
  AVLTreeNode<T> newNode = insertIntoBST(value, root);
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      // Fix issues that occur from inserting into parents left subtree
    } else {
      // Fix issues that occur from inserting into parents right subtree
    }
    newNode = newNode.parent.get();
  }
}
```

Therefore, our total insertion cost is **O(d) = O(log(*n*))**

Removing Records

- Removal follows essentially the same process as insertion
    - Do a normal BST removal
    - Go back up the tree adjusting balance factors
    - If you discover a balance factor that goes to +2/-2, rotate to fix

Summary

- We want shallow BSTs (it makes `find`, `insert`, `remove` faster)

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- Enforcing AVL constraints makes our BSTs shallow
    - The constraints are |height(right) - height(left)| ≤ 1
    - It will guarantee $d = O(\log(n))$

Summary

- We want shallow BSTs (it makes **`find`**, **`insert`**, **`remove`** faster)
- Enforcing AVL constraints makes our BSTs shallow
  - The constraints are |height(right) - height(left)| ≤ 1
  - It will guarantee $\boldsymbol{d = O(\log(n))}$
- Adding/removing from a BST changes height by at most 1
- A rotation can also change a BST height by at most 1

Summary

- We want shallow BSTs (it makes **`find`**, **`insert`**, **`remove`** faster)
- Enforcing AVL constraints makes our BSTs shallow
  - The constraints are |height(right) - height(left)| ≤ 1
  - It will guarantee $d = O(\log(n))$
- Adding/removing from a BST changes height by at most 1
- A rotation can also change a BST height by at most 1
- Therefore after **`insert`**/**`remove`** into an AVL tree, we can reinforce AVL constraints with one (or two) rotations
  - We only need to make one trip back up the tree to do so
  - Therefore **`insert`**/**`remove`** is still $O(d) = O(\log(n))$