CSE 250
Data Structures

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Lec 06: Asymptotic Analysis

Announcements and Feedback

- PA1 out now
  - Testing due Sunday 2/8 (no late submissions, no grace days)
  - Implementation due Sunday 2/15 (closes Tuesday, can use grace days)
- WA1 due yesterday
  - Late submission (for 50% penalty) closes today. No grace days.

Analysis Checklist

1. Don't think in terms of wall-time, think in terms of "number of steps"
2. To give a useful solution, we should take "scale" into account
   - How does the runtime change as we change the size of the input?
3. Focus on "large" inputs
   - Rank functions based on how they behave at large scales
4. Decouple algorithm from infrastructure/implementation

Attempt #1: Wall-clock time?

- What is fast?
  - 10s? 100ms? 10ns?
  - …it depends on the task
- Algorithm vs Implementation
  - Compare Grace Hopper's implementation to yours
- What machine are you running on?
  - Your old laptop? A lab machine? The newest, shiniest processor?
- What bottlenecks exist? CPU vs IO vs Memory vs Network…

Wall-clock time is not terribly useful… 4

Attempt #2: Growth Functions

Not a function in code...but a mathematical function:

$$T(n)$$

n: The "size" of the input

   ie: number of users,rows, pixels, etc

T(n): The number of "steps" taken for input of size n

   ie: 20 steps per user, where n = |Users|, is 20 x n

Some Basic Assumptions:

Problem sizes are non-negative integers

$$n \in \{0, 1, 2, 3, ...\} = \{0\} \cup \mathbb{Z}^+$$

We can't reverse time…(obviously)

$$T(n) > 0$$

Smaller problems aren't harder than bigger problems

$$n_1 < n_2 \Rightarrow T(n_1) \leq T(n_2)$$

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$$n_1 < n_2 \Rightarrow T(n_1) \leq T(n_2)$$

$T: \{0\} \cup \mathbb{Z}^+ \rightarrow \mathbb{R}^+$

T is non-decreasing

First Problem…

We are still implementation dependent…

$$T_1(n) = 19n$$

$$T_2(n) = 20n$$

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Does 1 extra step per element really matter...?

Is this just an implementation detail?

First Problem…

We are still implementation dependent…

$$T_1(n) = 19n$$

$$T_2(n) = 20n$$

$$T_3(n) = 2n^2$$

T_1 and T_2 are much more "similar" to each other than they are to T_3

First Problem...

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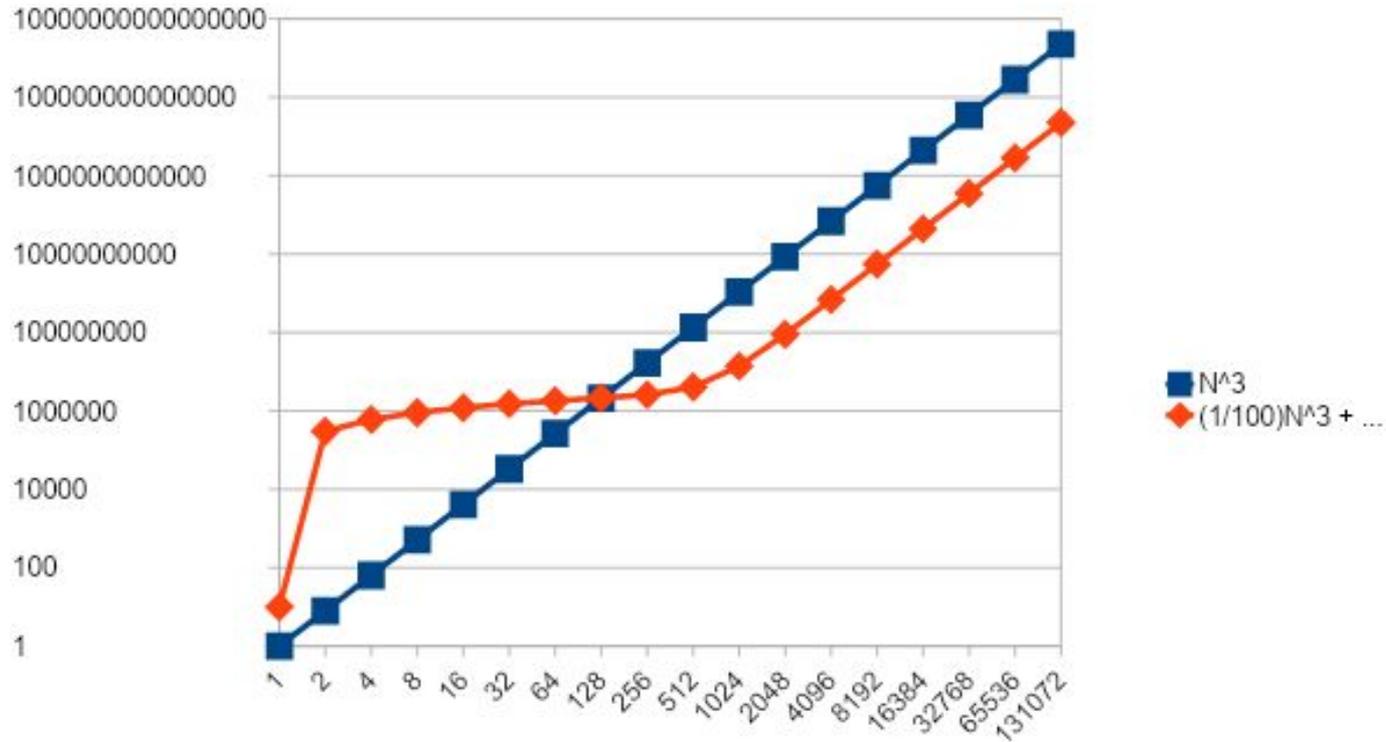
How do we capture this idea formally?

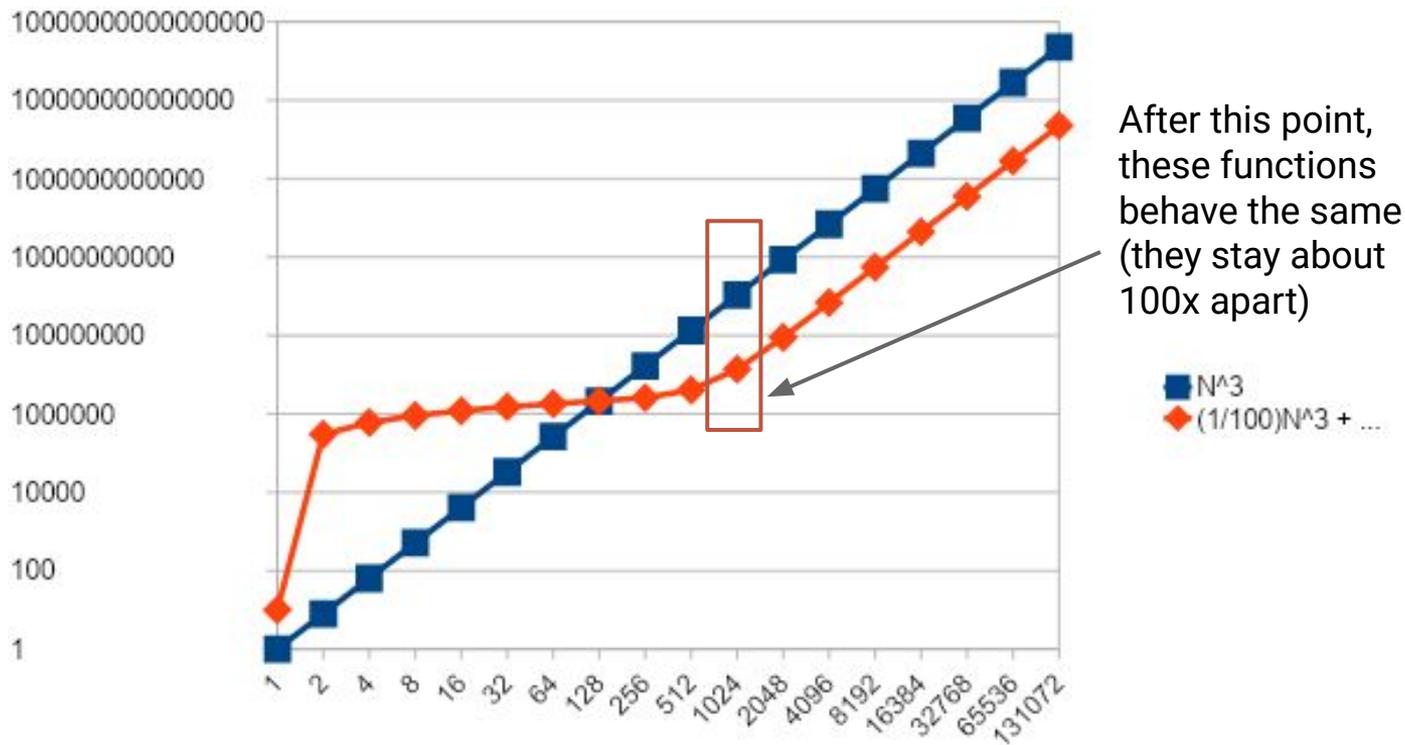# How Do We Capture Behavior at Scale?

Consider the following two functions:

$$\frac{1}{100}n^3 + 10n + 1000000 \log(n)$$

$$n^3$$

How Do We Capture Behavior at Scale?

How Do We Capture Behavior at Scale?

After this point, these functions behave the same (they stay about 100x apart)

N^3
(1/100)N^3 + ...

Attempt #3: Asymptotic Analysis

We want to organize runtimes (growth functions) into different *Complexity Classes*

Within the same complexity class, runtimes "behave the same"/"have the same shape" (at scale)
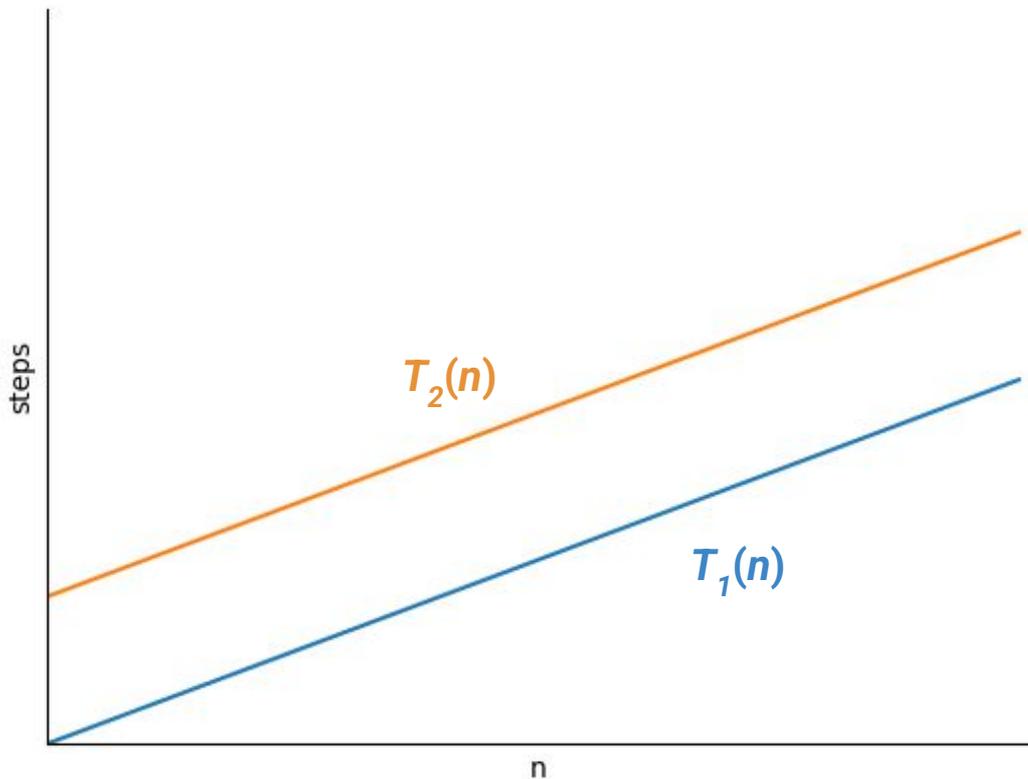
Getting More Formal

When do we consider two functions to have the same shape?

Additive Factors

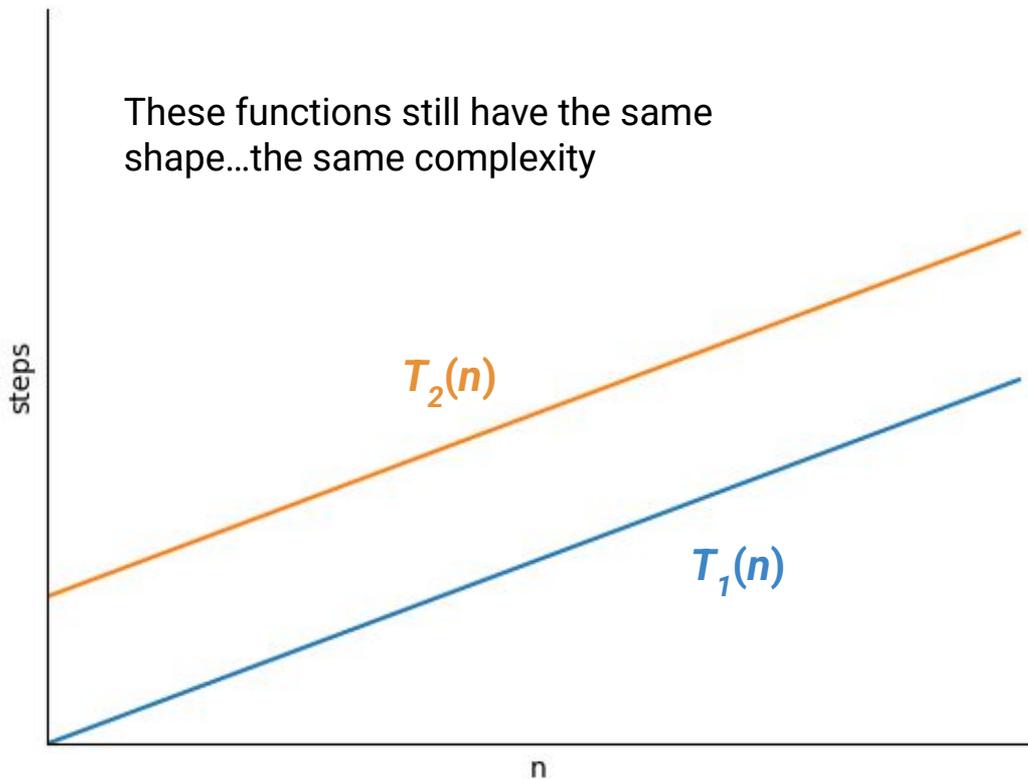Consider two growth functions:

$T_1(n) = 3n$

$T_2(n) = 3n + 3$

Additive Factors

Consider two growth functions:

$T_1(n) = 3n$

$T_2(n) = 3n + 3$

These functions still have the same shape...the same complexity
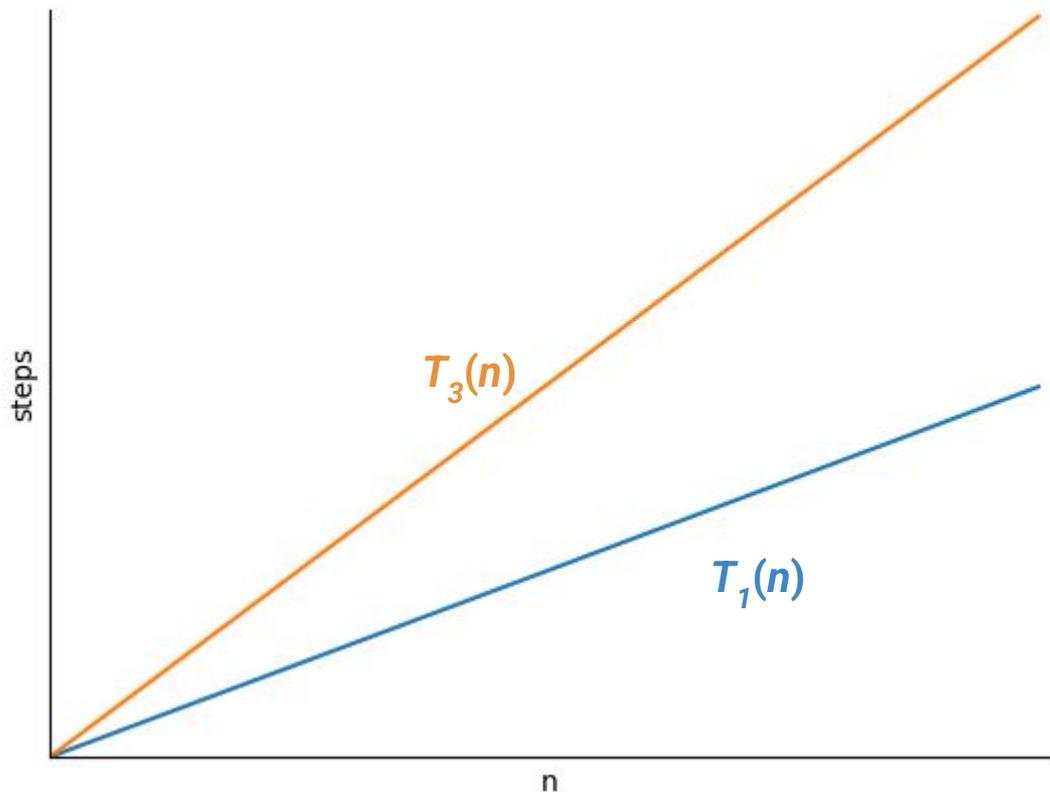
$T_2(n)$

$T_1(n)$

steps

n

Multiplicative Factors

Consider two growth functions:

$T_1(n) = 3n$

$T_3(n) = 6n$

Multiplicative Factors

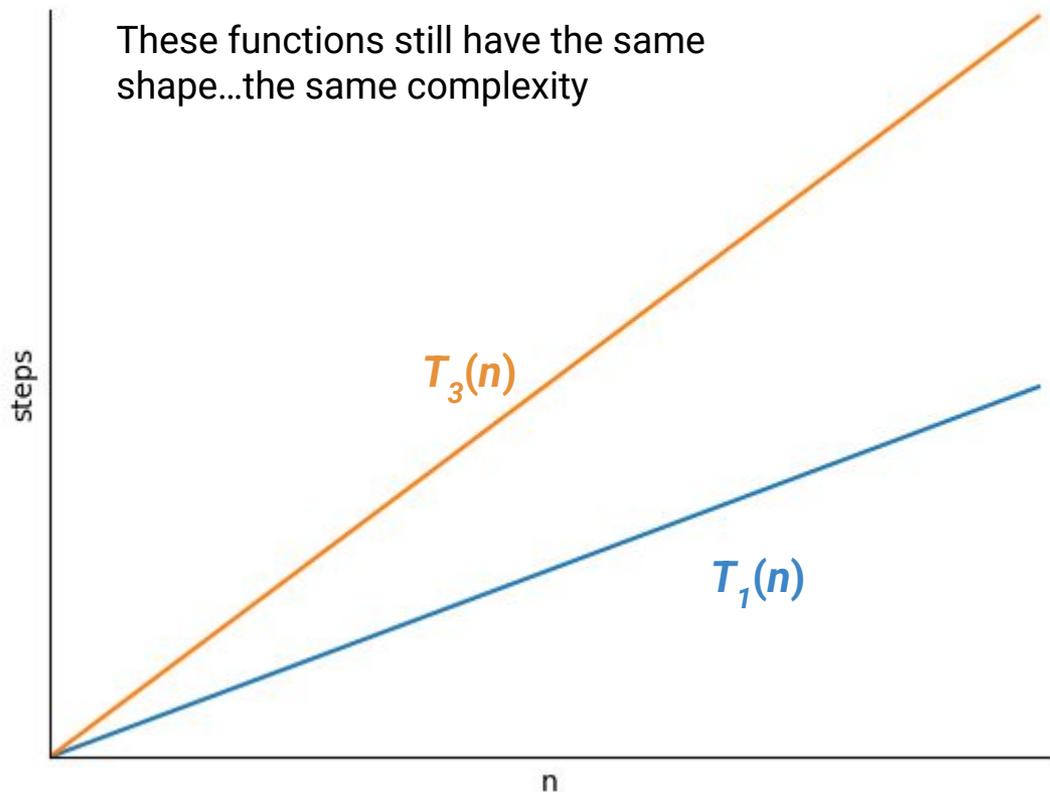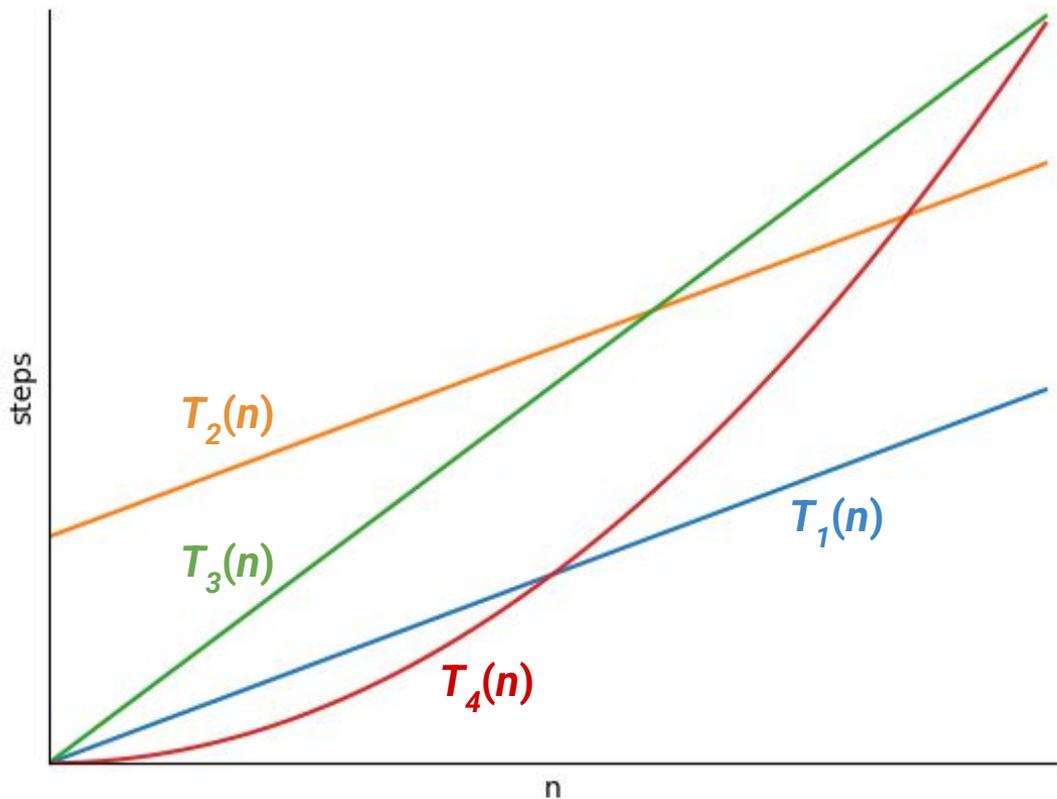Consider two growth functions:
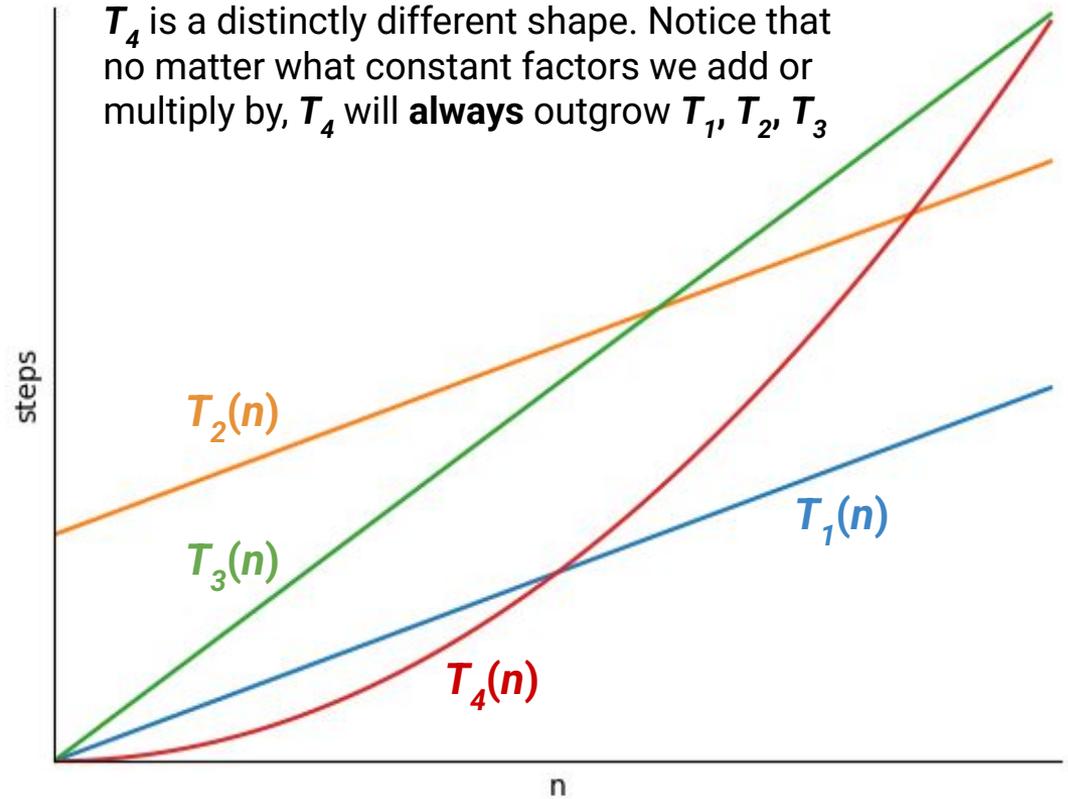
$T_1(n) = 3n$

$T_3(n) = 6n$

These functions still have the same shape...the same complexity

steps

$T_3(n)$

$T_1(n)$

n

A Counter Example

Now consider:

$T_4(n) = n^2$

steps

$T_2(n)$

$T_3(n)$

$T_1(n)$

$T_4(n)$

n

A Counter Example

Now consider:

$T_4(n) = n^2$

T_4 is a distinctly different shape. Notice that no matter what constant factors we add or multiply by, T_4 will **always** outgrow T_1, T_2, T_3

steps

$T_2(n)$

$T_3(n)$

$T_1(n)$

$T_4(n)$

n

A Counter Example

Now consider:

$T_4(n) = n^2$

T_1, T_2, T_3 are in the same complexity class

T_4 is not

steps

$T_2(n)$

$T_3(n)$

$T_1(n)$

$T_4(n)$

n
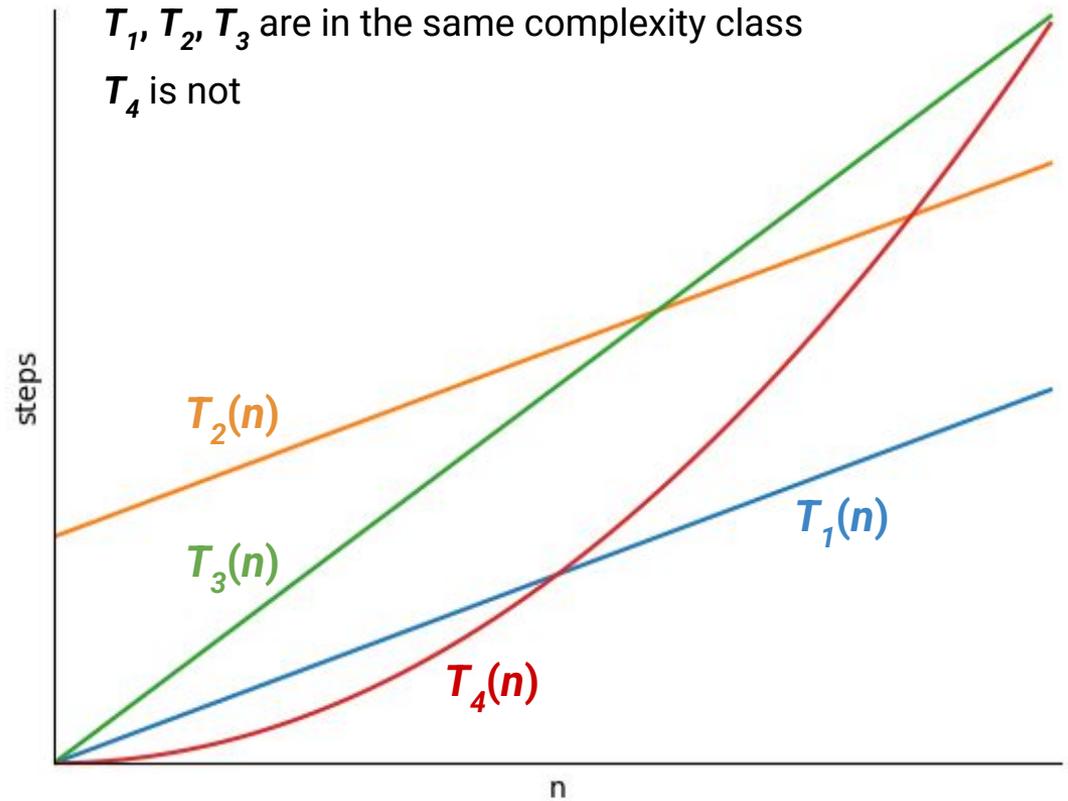
Complexity (so far...)

If there are constants c_1 and c_2 such that:

$$T_1(n) = c_1 + c_2 T_2(n)$$

then we say T_1 and T_2 are in the same complexity class*

not a complete definition...but we are getting there

Back To Growth Functions

So what exactly counts as a step?

Back To Growth Functions

So what exactly counts as a step?
- An arithmetic operation
- Accessing a variable
- Printing to the screen
- etc

but…

Counting Steps

How many steps in each of these snippets?

```
1  x = 10;
```

```
1  x = 10;
2  y = 20;
```

Counting Steps

How many steps in each of these snippets?

```
1  x = 10;
```

$T_1(n) = 1$

```
1  x = 10;
2  y = 20;
```

Counting Steps

How many steps in each of these snippets?

```
1 x = 10;
```

$T_1(n) = 1$

```
1 x = 10;
2 y = 20;
```

$T_2(n) = 2$

Counting Steps

How many steps in each of these snippets?

```
1 x = 10;
```

$T_1(n) = 1$

```
1 x = 10;
2 y = 20;
```

$T_2(n) = 2$

$$T_2(n) = T_1(n) + 1$$

They are in the same complexity class…in 250 we treat them as the same

Counting Steps

A *step* therefore is *any code* that always has the same runtime, regardless of input size (n)

Notation - Big Theta

$\Theta(f(n))$ is the **set** of all functions in the same complexity class as *f*

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$\Theta(\mathbf{\textit{f}(\textit{n})})$ is the **set** of all functions in the same complexity class as **_f_**

Example: $\Theta(3\textit{n} + 4) = \{$

  _n_,

  _n_ - 6,

  15_n_,

  …

}

Notation - Big Theta

$\Theta(\mathit{f(n)})$ is the **set** of all functions in the same complexity class as **f**

Example: $\Theta(3\mathit{n} + 4) = \{$
    n,
    n - 6,
    15n,
    …
$\}$

$\mathit{g(n)} \in \Theta(\mathit{f(n)})$ means **g** and **f** are in the same complexity class

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Common Shorthand

$g(n) = \Theta(f(n))$ is common shorthand for $g(n) \in \Theta(f(n))$

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Algorithm Foo is in $\Theta(f(n))$ is common shorthand for $T(n) \in \Theta(f(n))$ where $T(n)$ is the growth function describing the runtime of Foo

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Algorithm Foo is in $\Theta(f(n))$ is common shorthand for $T(n) \in \Theta(f(n))$ where $T(n)$ is the growth function describing the runtime of Foo

Moving forward: $f(n), g(n), f_1(n)$, etc will be used to name any mathematical function that's a growth function
$T(n), T_1(n)$, etc will be used for growth functions for specific algorithms

Complexity Class Names

$\Theta(1)$: Constant

$\Theta(\log(n))$: Logarithmic

$\Theta(n)$: Linear

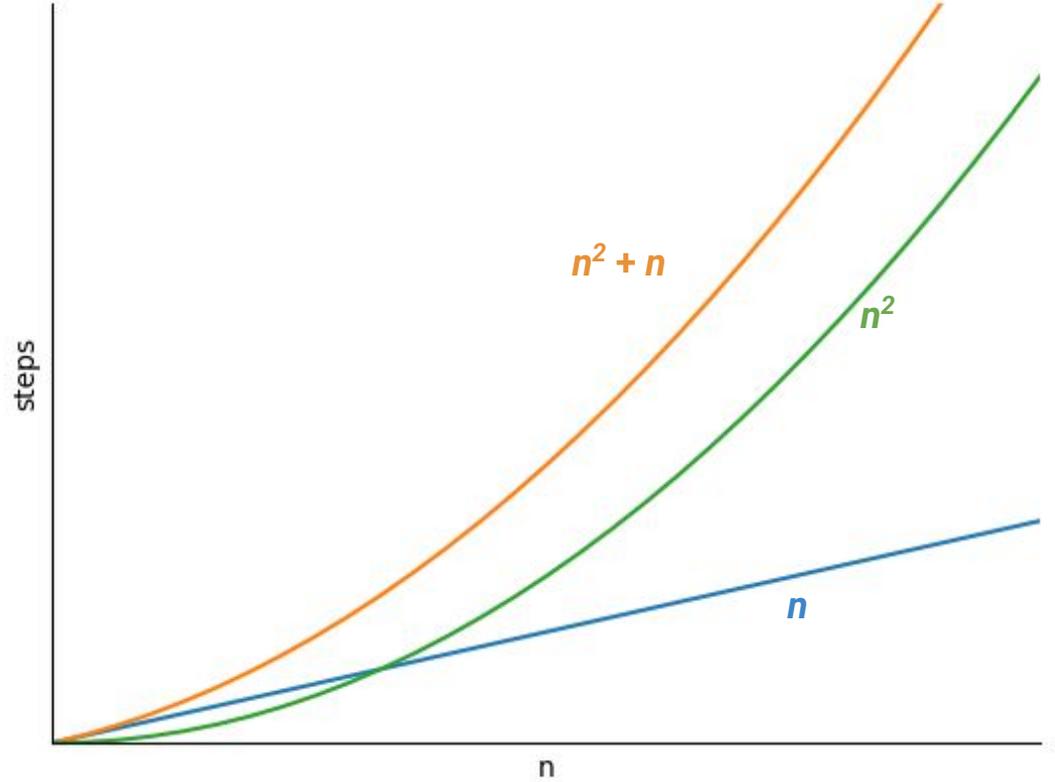$\Theta(n \log(n))$: Log-Linear

$\Theta(n^2)$: Quadratic

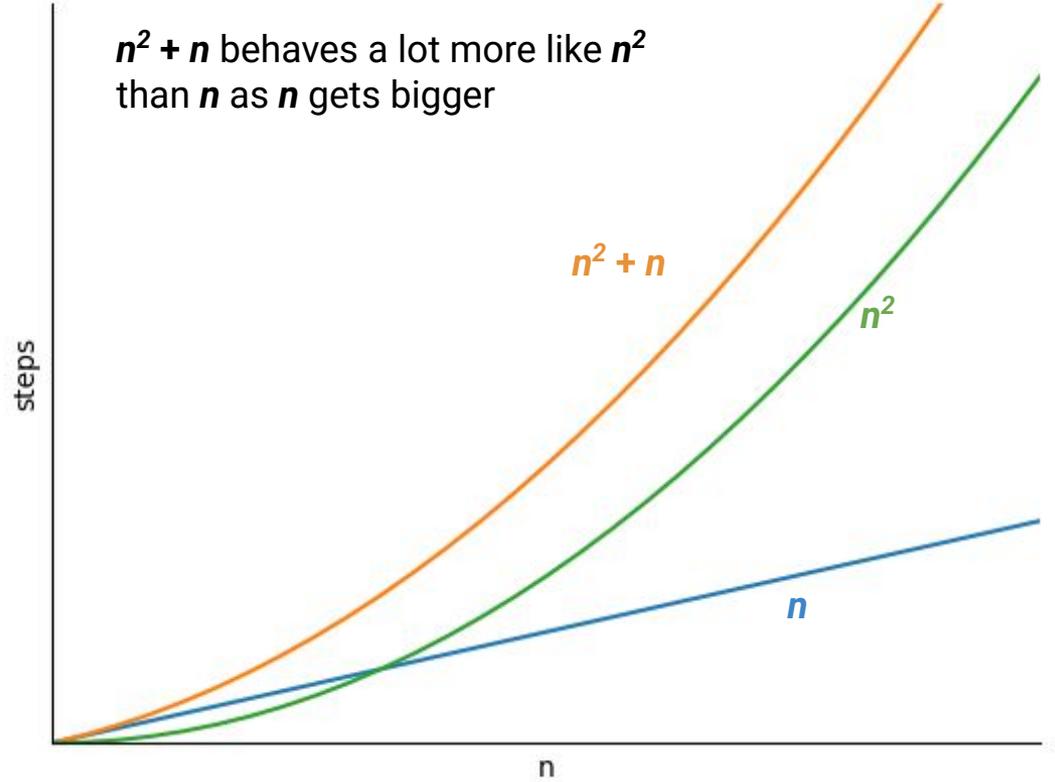$\Theta(n^k)$: Polynomial

$\Theta(2^n)$: Exponential

Combining Classes
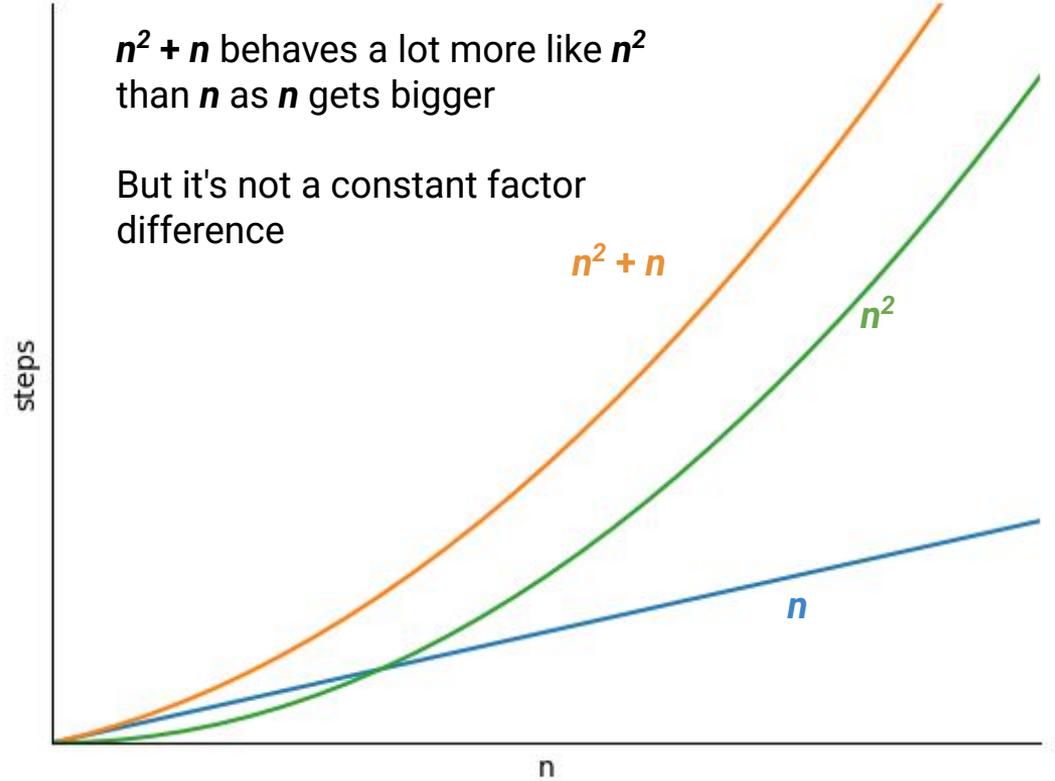
What complexity class is $g(n) = n + n^2$ in?

Combining Classes

Combining Classes

$n^2 + n$ behaves a lot more like n^2 than n as n gets bigger

$n^2 + n$

n^2

n

steps

n

Combining Classes

$n^2 + n$ behaves a lot more like n^2 than n as n gets bigger

But it's not a constant factor difference

$n^2 + n$

n^2

n

steps

n

Combining Classes

Consider the fact that n^2 and $2n^2$ are in the same complexity class…

How does $n^2 + n$ relate to these two functions?

Combining Classes

Consider the fact that n^2 and $2n^2$ are in the same complexity class…

$$1 \leq n$$

remember, we only care about problems with non-negative input sizes

Combining Classes

Consider the fact that n^2 and $2n^2$ are in the same complexity class…

$$1 \leq n$$

$$n \leq n^2 \qquad \text{multiply both sides by } n$$

Combining Classes

Consider the fact that n^2 and $2n^2$ are in the same complexity class…

$$1 \leq n$$

$$n \leq n^2$$

$$n + n^2 \leq 2n^2 \qquad \text{add } n^2 \text{ to both sides}$$

Combining Classes

Consider the fact that n^2 and $2n^2$ are in the same complexity class…

$$0 \leq n \qquad \text{obviously true}$$

Combining Classes

Consider the fact that n^2 and $2n^2$ are in the same complexity class…

$$0 \leq n$$

$$n^2 \leq n + n^2 \qquad \text{add } n^2 \text{ to both sides}$$

Combining Classes

Consider the fact that n^2 and $2n^2$ are in the same complexity class…

$$n^2 \leq n + n^2 \leq 2n^2$$

So $n^2 + n$ should probably be in $\Theta(n^2)$ too…
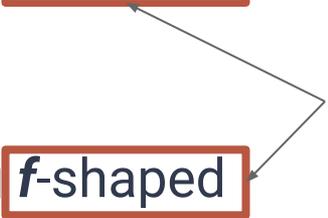
Complexity: A More Complete Definition

f and *g* are in the same complexity class iff:

g is bounded from above by something *f*-shaped

and

g is bounded from below by something *f*-shaped

Complexity: A More Complete Definition

f and g are in the same complexity class iff:

g is bounded from above by something **f**-shaped

and

g is bounded from below by something **f**-shaped

f shifted or stretched by a constant factor

Complexity: A More Complete Definition

f and *g* are in the same complexity class iff:

g is **bounded from above** by something *f*-shaped

and

g is **bounded from below** by something *f*-shaped

What do we mean by bounded from above/below?

Bounding from Above: Big O

$g(n)$ is bounded from above by $f(n)$ if:

There exists a constant $n_0 \geq 0$ and a constant $c > 0$ such that:

For all $n \geq n_0$, $g(n) \leq c \cdot f(n)$
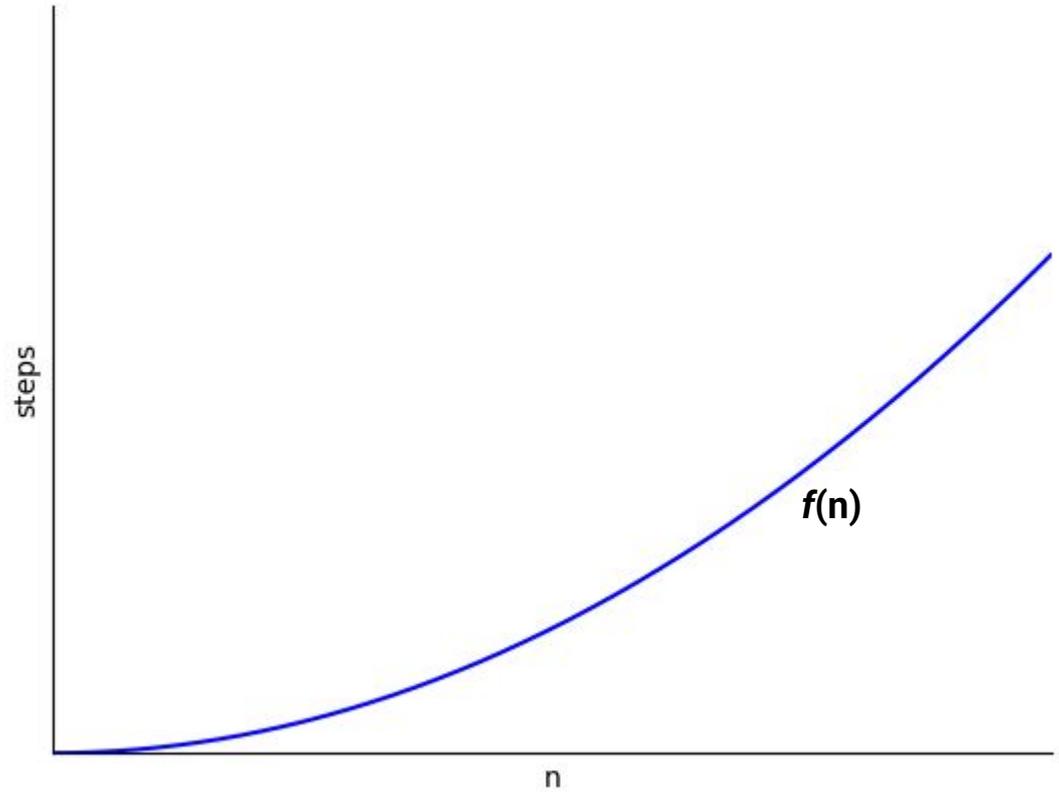
Bounding from Above: Big O

$g(n)$ is bounded from above by $f(n)$ if:

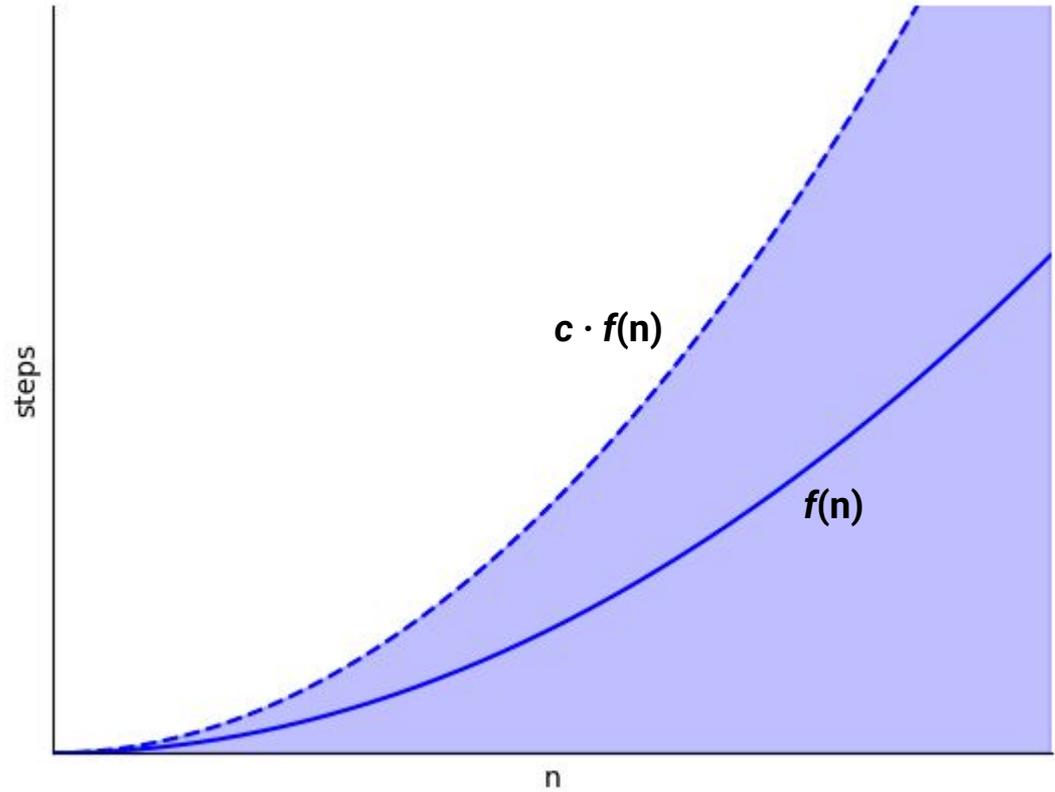There exists a constant $n_0 \geq 0$ and a constant $c > 0$ such that:

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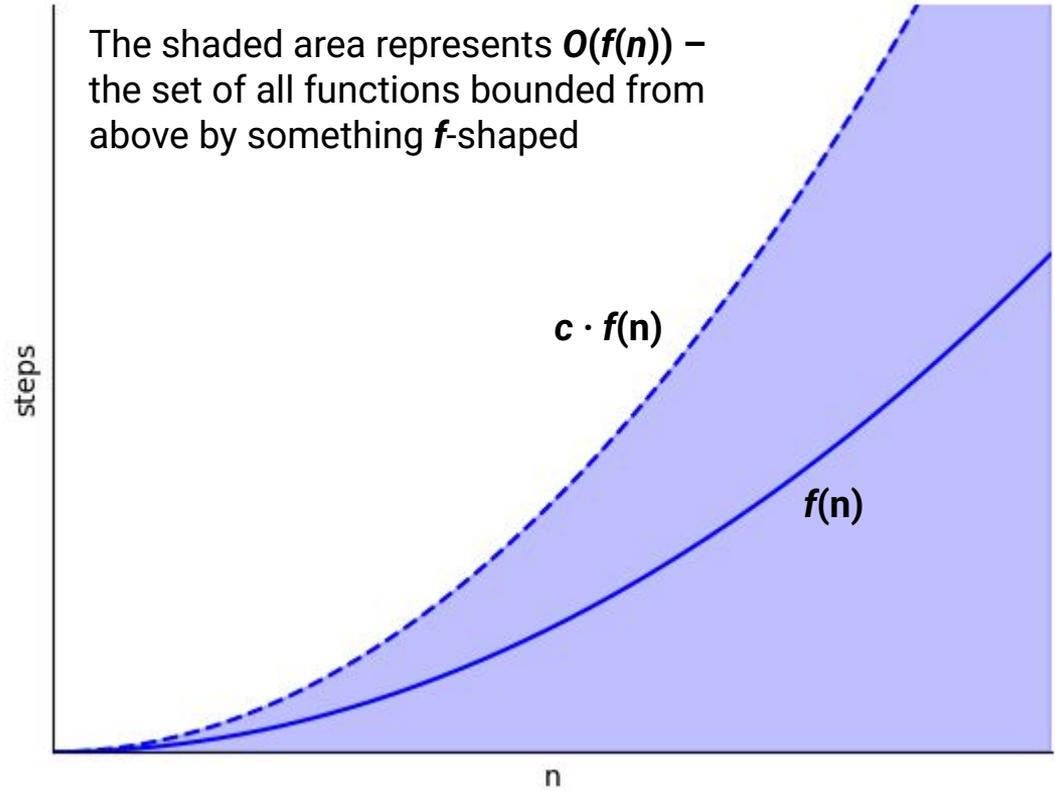
In this case, we say that $g(n) \in O(f(n))$

Bounded from Above: Big O

Bounded from Above: Big O

Bounded from Above: Big O

The shaded area represents $O(f(n))$ – the set of all functions bounded from above by something f-shaped

$c \cdot f(n)$

$f(n)$

steps

n

Bounding from Below: Big Omega

$g(n)$ is bounded from below by $f(n)$ if:

There exists a constant $n_0 \geq 0$ and a constant $c > 0$ such that:

For all $n \geq n_0$, $g(n) \geq c \cdot f(n)$

Bounding from Below: Big Omega
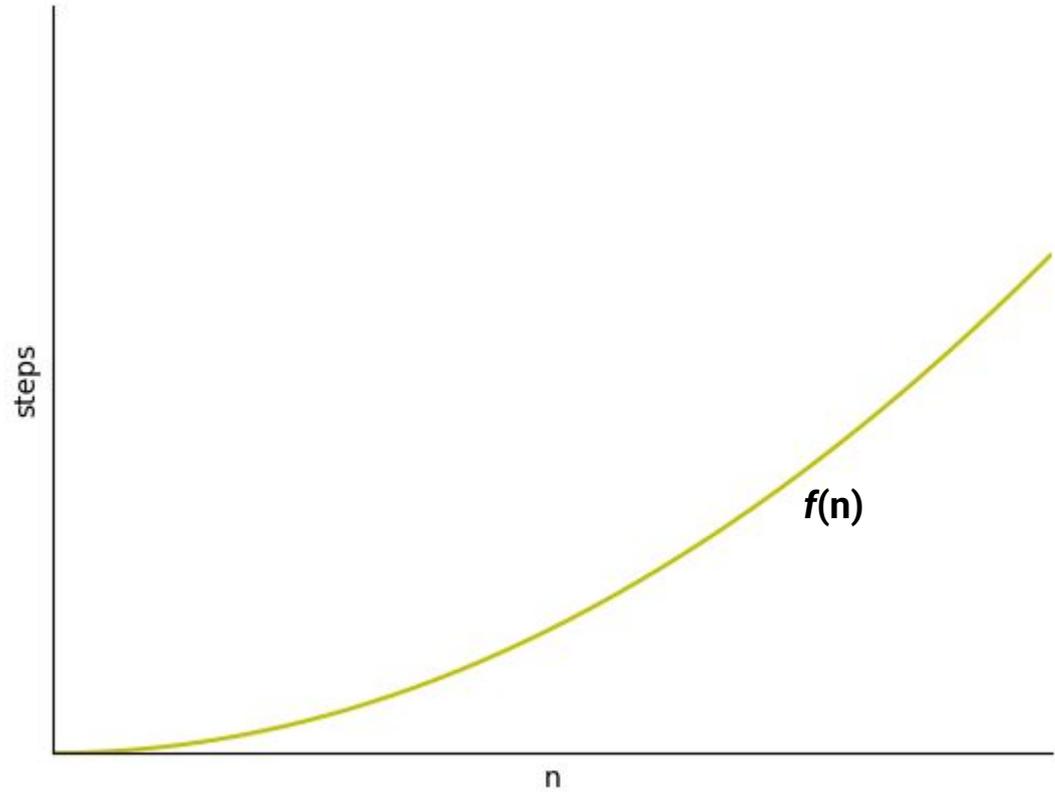
$g(n)$ is bounded from below by $f(n)$ if:

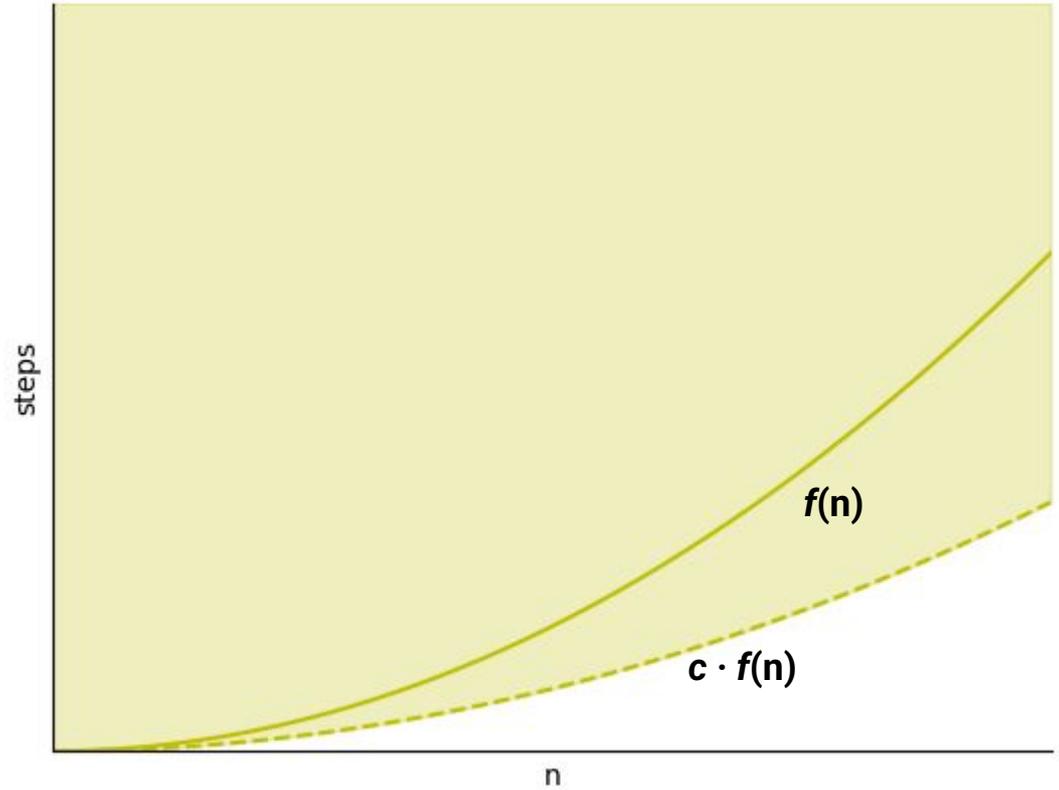There exists a constant $n_0 \geq 0$ and a constant $c > 0$ such that:

For all $n \geq n_0$, $g(n) \geq c \cdot f(n)$

In this case, we say that $g(n) \in \Omega(f(n))$

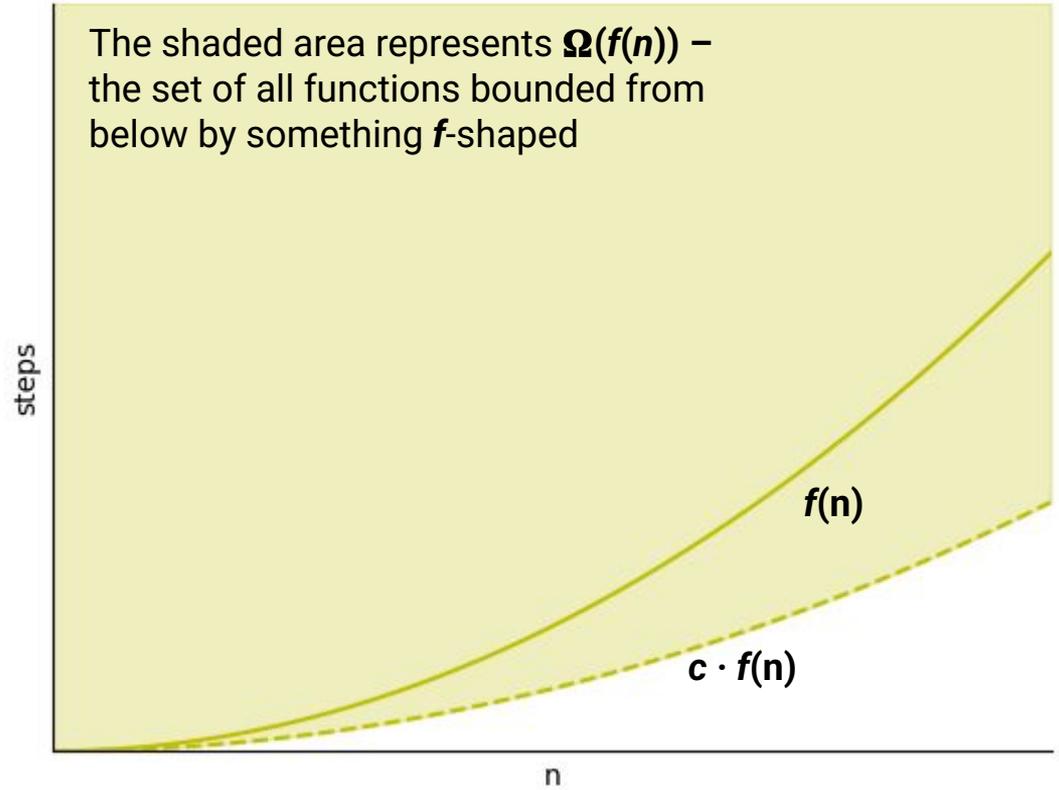# Bounded from Below: Big Ω

steps

f(n)

n

Bounded from Below: Big Ω

Bounded from Below: Big Ω

The shaded area represents $\Omega(f(n))$ – the set of all functions bounded from below by something f-shaped

steps

f(n)

$c \cdot f$(n)

n

Complexity: A More Complete Definition

f and *g* are in the same complexity class iff:

    g is bounded from above by something *f*-shaped

    and

    g is bounded from below by something *f*-shaped

Complexity: A More Complete Definition

$g(n) \in \Theta(f(n))$ iff:

g is bounded from above by something *f*-shaped

and

g is bounded from below by something *f*-shaped

Complexity: A More Complete Definition

$g(n) \in \Theta(f(n))$ iff:

$g(n) \in O(f(n))$

and

g is bounded from below by something **f**-shaped

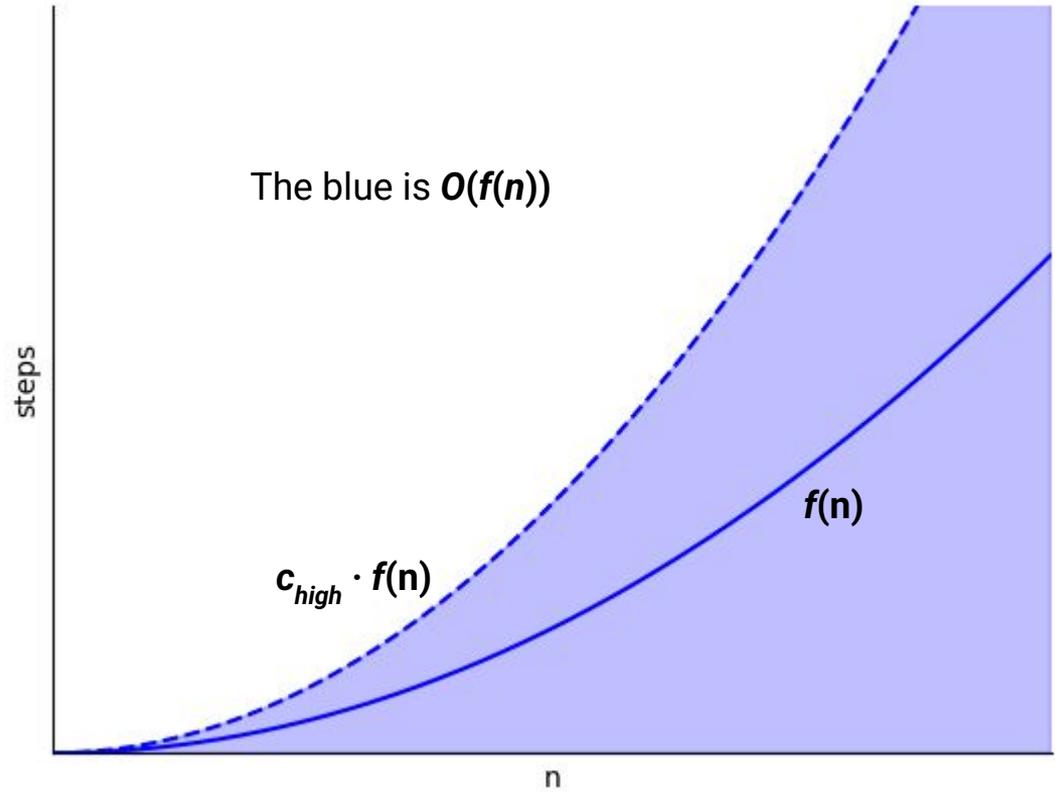# Complexity: A More Complete Definition

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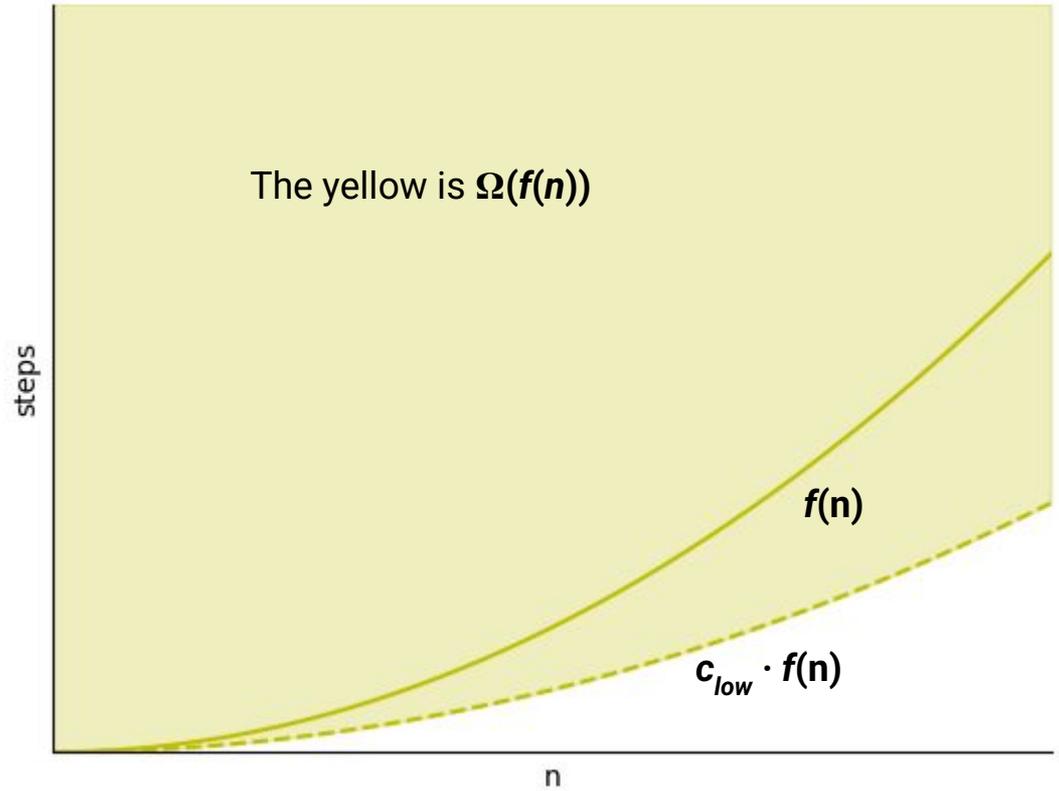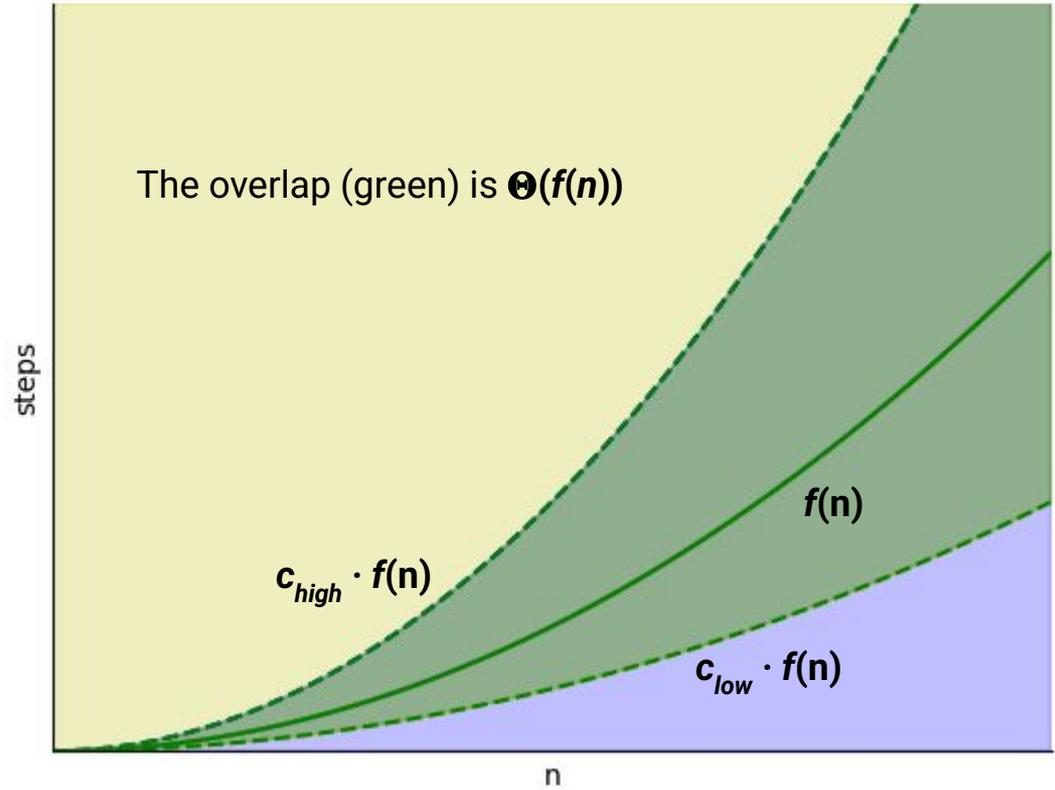
and

$g(n) \in \Omega(f(n))$

Complexity Class: Big Θ

The blue is $O(f(n))$

$c_{high} \cdot f(n)$

$f(n)$

steps

n

Complexity Class: Big Θ

The yellow is $\Omega(f(n))$

$f(n)$

$c_{low} \cdot f(n)$

steps

n

Complexity Class: Big Θ

The overlap (green) is $\Theta(f(n))$

$c_{high} \cdot f(n)$

$f(n)$

$c_{low} \cdot f(n)$

steps

n

Complexity Class: Big Θ

$\Theta(f(n))$ is the set of functions that will stay between $c_{high} \cdot f(n)$ and $c_{low} \cdot f(n)$ (after some constant n_0)

steps

$f(n)$

$c_{high} \cdot f(n)$

$c_{low} \cdot f(n)$

n

71
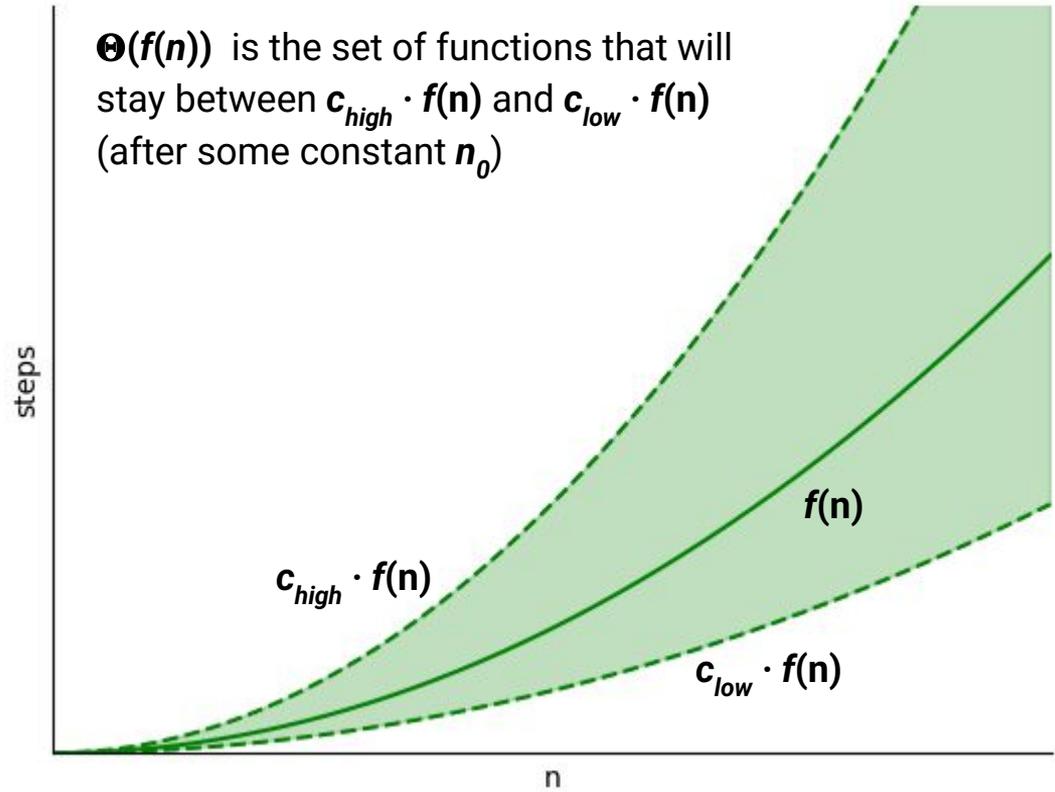
Complexity Class Ranking
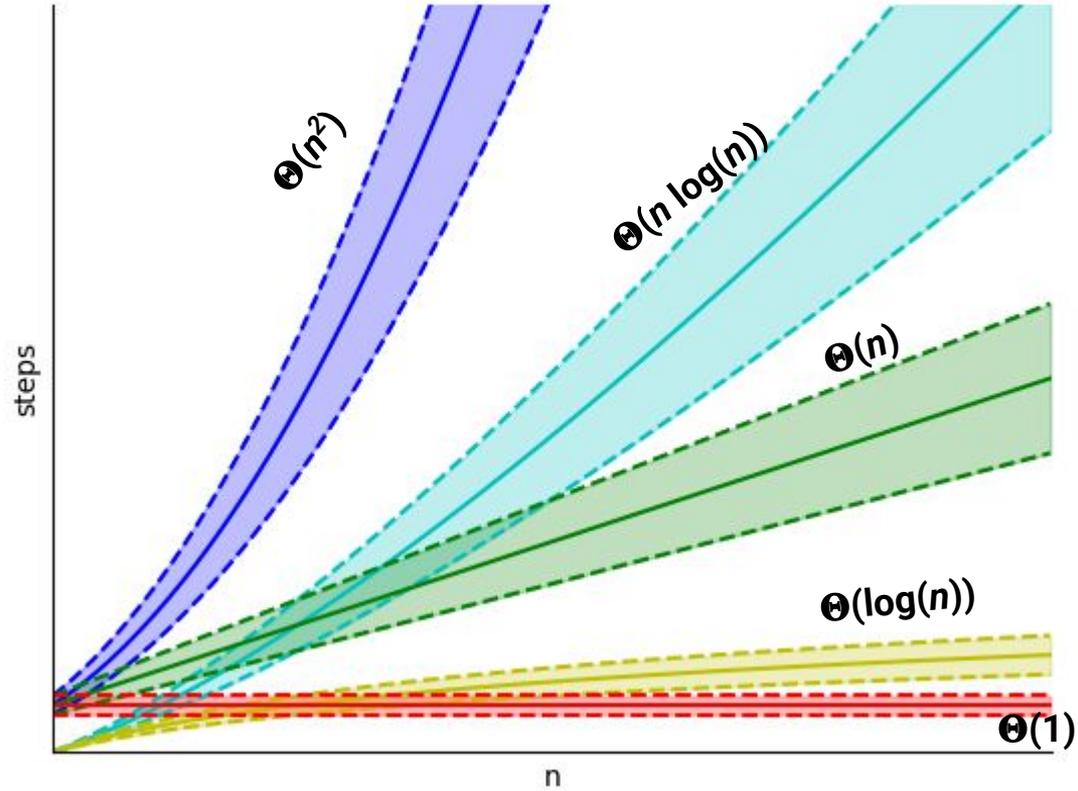
$$\Theta(1) < \Theta(\log(n)) < \Theta(n) < \Theta(n \log(n)) < \Theta(n^2) < \Theta(n^3) < \Theta(2^n)$$

Rules of Thumb

$$\Theta(1) < \Theta(\log(n)) < \Theta(n) < \Theta(n \log(n)) < \Theta(n^2) < \Theta(n^3) < \Theta(2^n)$$

Rules of Thumb

$$O(1) \subset O(\log(n)) \subset O(n) \subset O(n\log(n)) \subset O(n^2) \subset O(n^3) \subset O(2^n)$$

$$\Omega(2^n) \subset \Omega(n^3) \subset \Omega(n^2) \subset \Omega(n\log(n)) \subset \Omega(n) \subset \Omega(\log(n)) \subset \Omega(1)$$

Rules of Thumb

If something is bounded from above by **log(*n*)**, it's also bounded from above by ***n***

$$O(1) \subset O(\log(n)) \subset O(n) \subset O(n \log(n)) \subset O(n^2) \subset O(n^3) \subset O(2^n)$$

$$\Omega(2^n) \subset \Omega(n^3) \subset \Omega(n^2) \subset \Omega(n \log(n)) \subset \Omega(n) \subset \Omega(\log(n)) \subset \Omega(1)$$

Rules of Thumb

$$O(1) \subset O(\log(n)) \subset O(n) \subset O(n \log(n)) \subset O(n^2) \subset O(n^3) \subset O(2^n)$$

$$\Omega(2^n) \subset \Omega(n^3) \subset \Omega(n^2) \subset \Omega(n \log(n)) \subset \Omega(n) \subset \Omega(\log(n)) \subset \Omega(1)$$

If something is bounded from below by n^2, it's also bounded from below by n

Rules of Thumb

O(*f*(*n*)) (**Big-O**): The complexity class of *f*(*n*) and every lesser class

Θ(*f*(*n*)) (**Big-Θ**): The complexity class of *f*(*n*)

Ω(*f*(*n*)) (**Big-Ω**): The complexity class of *f*(*n*) and every greater class