CSE 250
Data Structures

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Lec 07: Proving Bounds

Announcements and Feedback

- PA1 out now
  - Testing due Sunday (no late submissions, no grace days)
  - Implementation due next Sunday (closes Tuesday, can use grace days)

Complexity Class

f and g are in the same complexity class, denoted $g(n) \in \Theta(f(n))$, iff:

g is bounded from above by something f-shaped

and

g is bounded from below by something f-shaped

Bounded from Above: Big O

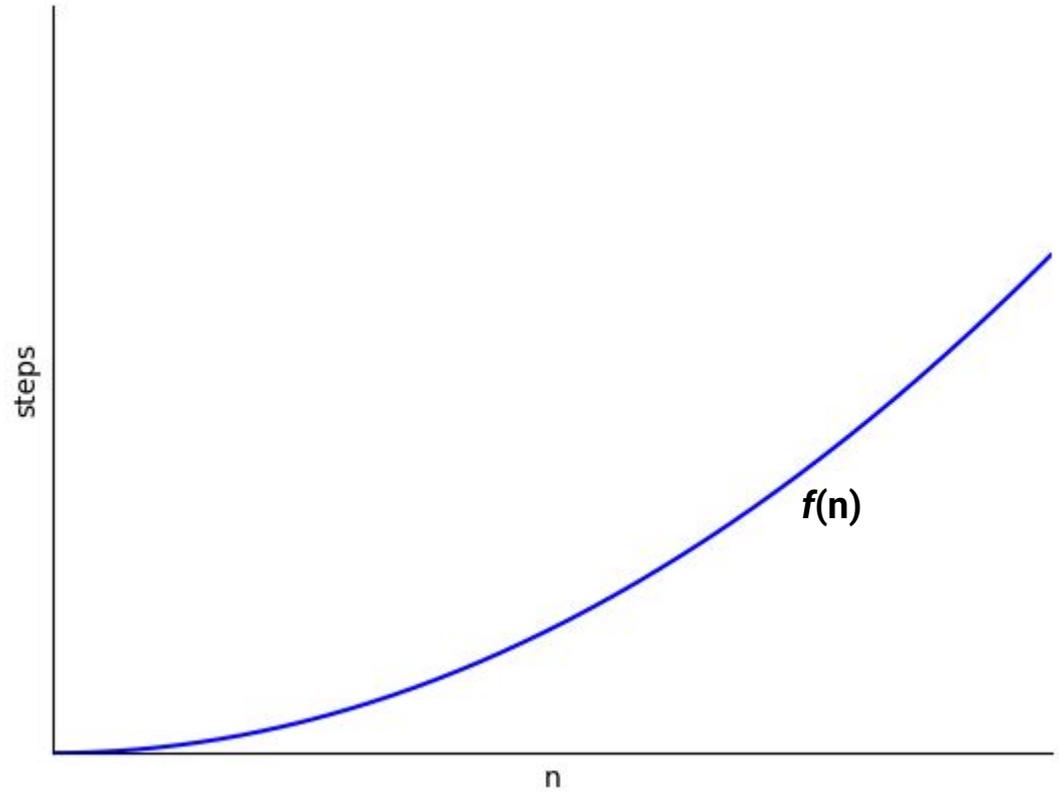$g(n)$ is bounded from above by $f(n)$ if:

There exists a constant $n_0 \geq 0$ and a constant $c > 0$ such that:
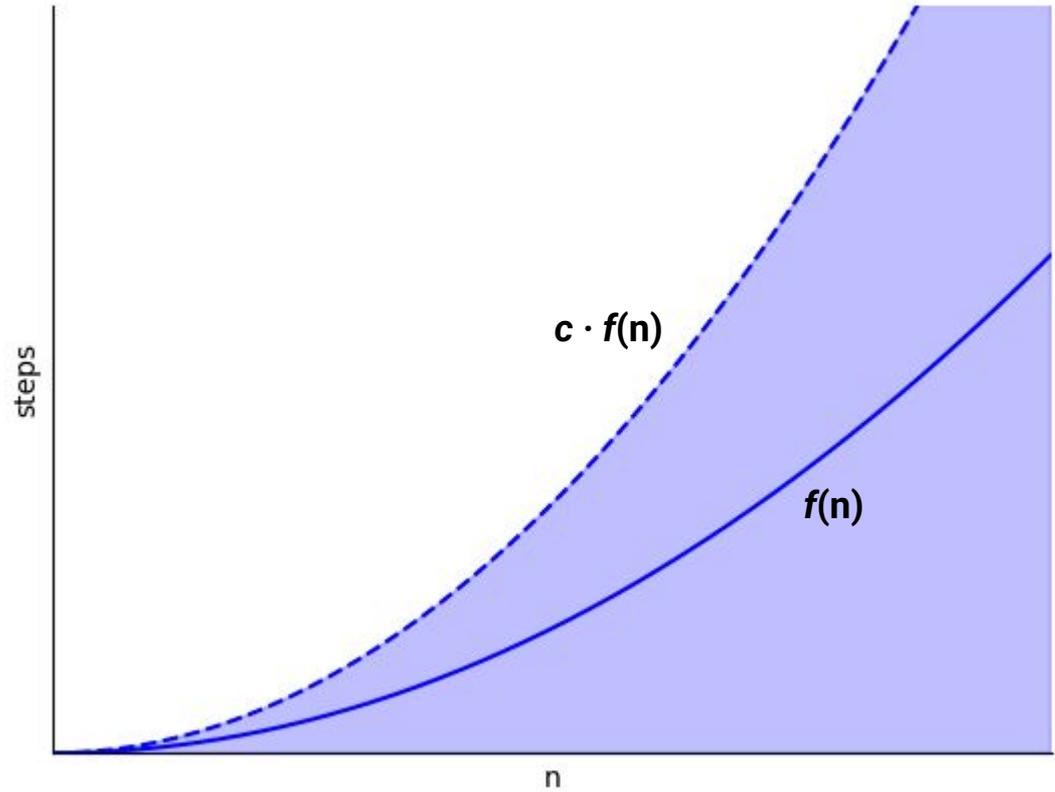
$$\text{For all } n \geq n_0, g(n) \leq c \cdot f(n)$$

In this case, we say that $g(n) \in O(f(n))$

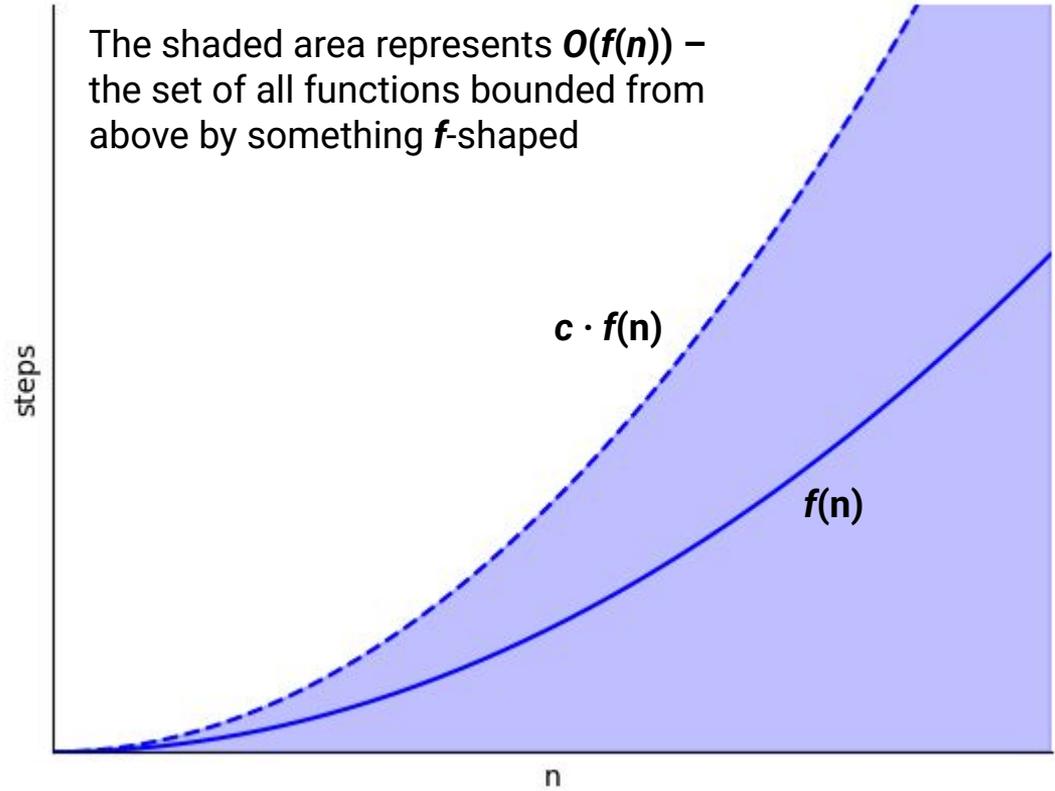$O(f(n))$ is the set of all functions bounded from above by $f(n)$

Bounded from Above: Big O

Bounded from Above: Big O

Bounded from Above: Big O

The shaded area represents **O(f(n))** – the set of all functions bounded from above by something **f**-shaped

c · f(n)

f(n)

steps

n

Bounded from Above: Big O

$$g(n) = \frac{n^2}{2} + 4n + 7$$

Prove that $g(n) \in O(n^2)$

$$\frac{n^2}{2} + 4n + 7 \leq c \cdot n^2$$

Inequality Tricks

1. $\forall\, a > 0,\ f(n){\cdot}a \geq g(n){\cdot}a \Rightarrow f(n) \geq g(n)$

2. $x \geq y$ and $a \geq b \Rightarrow x + a \geq y + b$

3. $x \geq a$ and $a \geq y \Rightarrow x \geq y$

4. $1 \leq \log(n) \leq n \leq n^2 \leq n^k\ (\text{for } k \geq 2) \leq 2^n$

Bounded from Above: Big O

$$\boxed{\frac{n^2}{2}} + \boxed{4n} + \boxed{7} \le c \cdot n^2$$

First show:

$$\frac{n^2}{2} \le c_1 \cdot n^2 \qquad 4n \le c_2 \cdot n^2 \qquad 7 \le c_3 \cdot n^2$$

Bounded from Above: Big O

$$\frac{n^2}{2} \leq c_1 \cdot n^2$$

Bounded from Above: Big O

$$\frac{n^2}{2} \leq c_1 \cdot n^2$$

This is true for all **$n \geq 0$** if we set **c_1** to **1/2**

Bounded from Above: Big O

$$4n \leq c_2 \cdot n^2$$

Bounded from Above: Big O

$$4n \leq c_2 \cdot n^2$$

This is true for all **$n \geq 1$** if we set **c_2** to **4**

Bounded from Above: Big O

$$7 \leq c_3 \cdot n^2$$

Bounded from Above: Big O

$$7 \le c_3 \cdot n^2$$

This is true for all **$n \ge 1$** if we set **c_3** to **7**

Bounded from Above: Big O

$$\boxed{\frac{n^2}{2}} + \boxed{4n} + \boxed{7} \le c \cdot n^2$$

First show:

$$\frac{n^2}{2} \le c_1 \cdot n^2 \qquad\qquad 4n \le c_2 \cdot n^2 \qquad\qquad 7 \le c_3 \cdot n^2$$

Bounded from Above: Big O

$$\boxed{\frac{n^2}{2}} + \boxed{4n} + \boxed{7} \leq c \cdot n^2$$

First show:

$$\frac{n^2}{2} \leq \frac{1}{2} \cdot n^2 \qquad 4n \leq 4 \cdot n^2 \qquad 7 \leq 7 \cdot n^2$$

Bounded from Above: Big O

$$\boxed{\frac{n^2}{2}} + \boxed{4n} + \boxed{7} \leq c \cdot n^2$$

First show:

$$\frac{n^2}{2} \leq \frac{1}{2} \cdot n^2 \qquad 4n \leq 4 \cdot n^2 \qquad 7 \leq 7 \cdot n^2$$

$$\frac{n^2}{2} + 4n + 7 \leq \frac{1}{2} \cdot n^2 + 4 \cdot n^2 + 7 \cdot n^2$$

Bounded from Above: Big O

$$\boxed{\frac{n^2}{2}} + \boxed{4n} + \boxed{7} \le c \cdot n^2$$

First show:

$$\frac{n^2}{2} \le \frac{1}{2} \cdot n^2 \qquad 4n \le 4 \cdot n^2 \qquad 7 \le 7 \cdot n^2$$

$$\frac{n^2}{2} + 4n + 7 \le \frac{1}{2} \cdot n^2 + 4 \cdot n^2 + 7 \cdot n^2$$

Bounded from Above: Big O

$$\frac{n^2}{2} + 4n + 7 \le c \cdot n^2$$

First show:

$$\frac{n^2}{2} \le \frac{1}{2} \cdot n^2 \qquad 4n \le 4 \cdot n^2 \qquad 7 \le 7 \cdot n^2$$

$$\frac{n^2}{2} + 4n + 7 \le \frac{1}{2} \cdot n^2 + 4 \cdot n^2 + 7 \cdot n^2$$

Bounded from Above: Big O

$$\frac{n^2}{2} + 4n + 7 \leq c \cdot n^2$$

First show:

$$\frac{n^2}{2} \leq \frac{1}{2} \cdot n^2 \qquad\qquad 4n \leq 4 \cdot n^2 \qquad\qquad 7 \leq 7 \cdot n^2$$

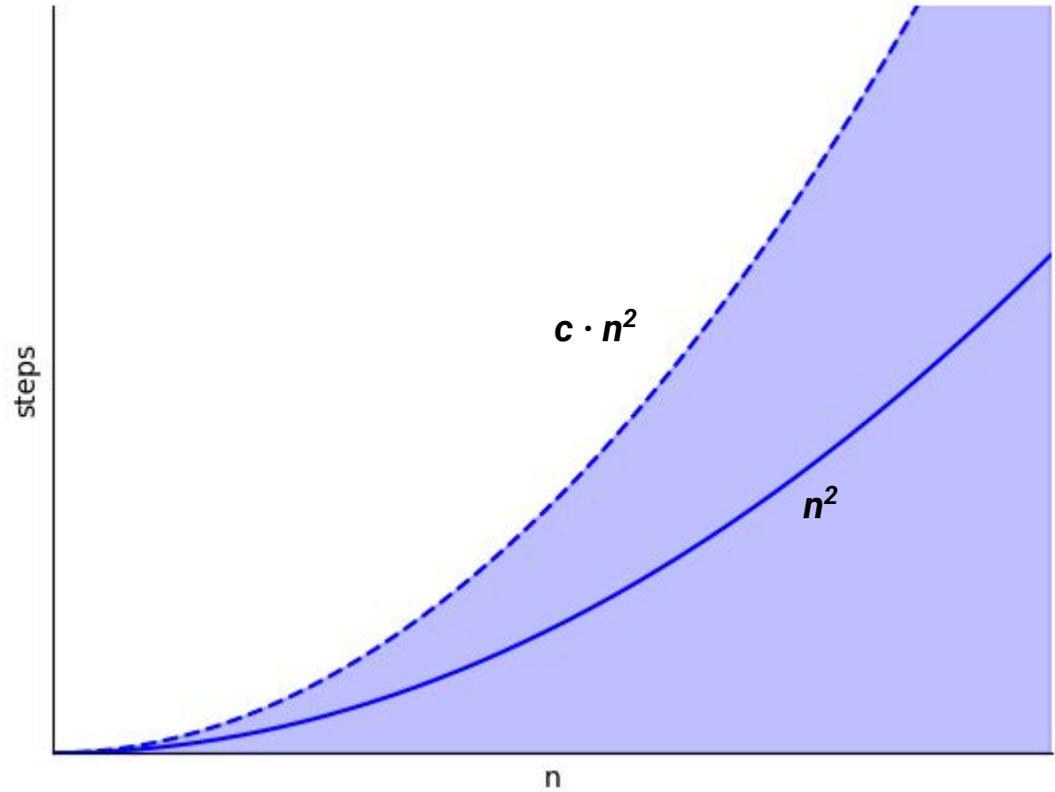$$\frac{n^2}{2} + 4n + 7 \leq \frac{1}{2} \cdot n^2 + 4 \cdot n^2 + 7 \cdot n^2 = (\frac{1}{2} + 4 + 7) \cdot n^2$$

Bounded from Above: Big O

$$\frac{n^2}{2} + 4n + 7 \leq c \cdot n^2$$

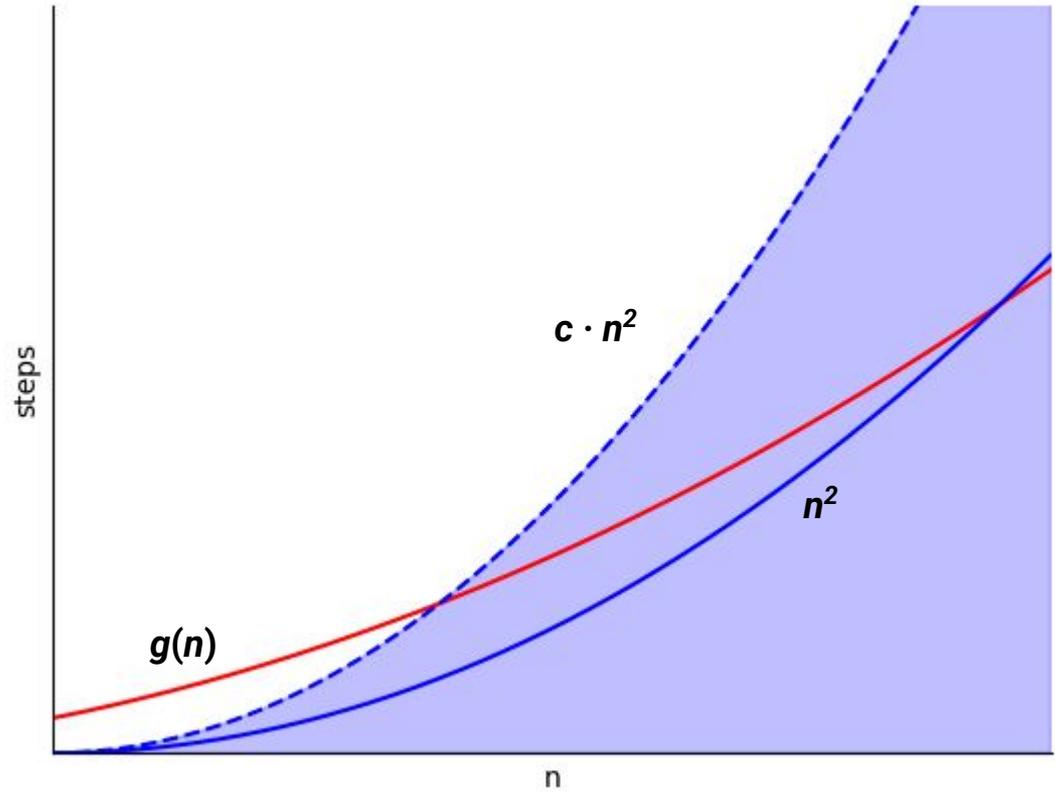Therefore if we let **c = 11.5**, then for all **n ≥ 1** the above holds true
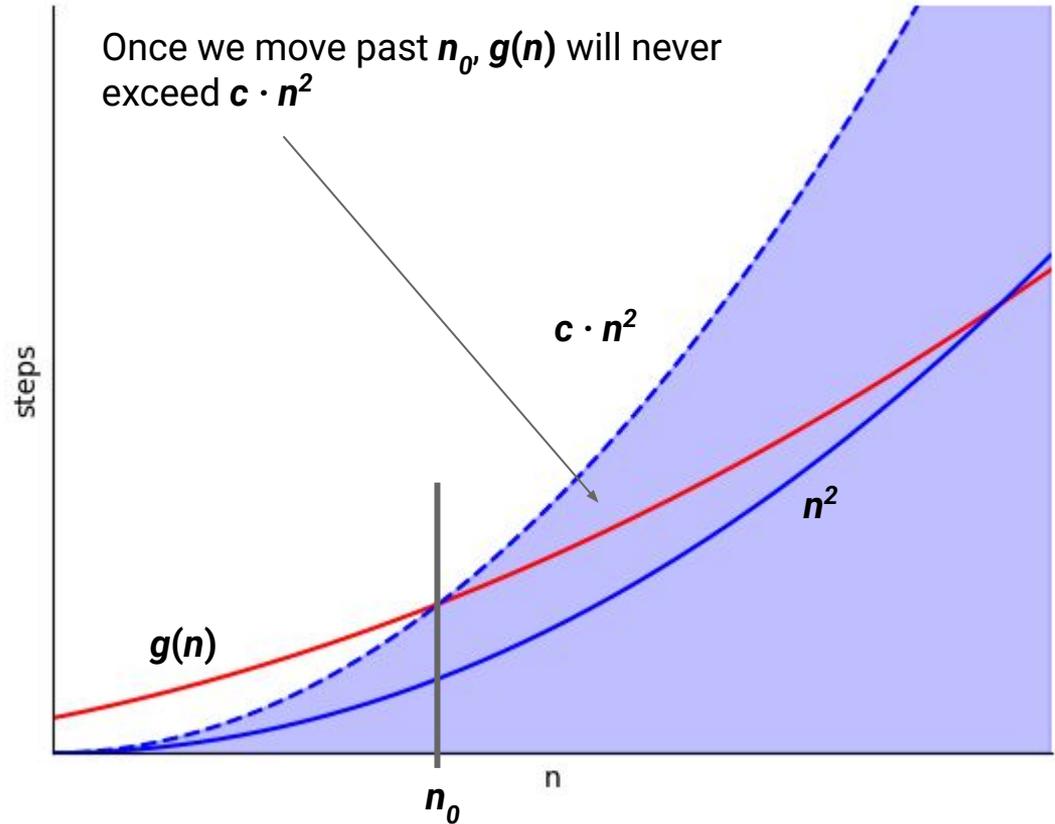
Therfore **g(n) ∈ O(n²)**

Bounded from Above: Big O

$c \cdot n^2$

n^2

steps

n

Bounded from Above: Big O

Bounded from Above: Big O

Once we move past n_0, $g(n)$ will never exceed $c \cdot n^2$

$c \cdot n^2$

n^2

$g(n)$

steps

n_0

n

Bounded from Below: Big Ω

$g(n)$ is bounded from below by $f(n)$ if:

There exists a constant $n_0 \geq 0$ and a constant $c > 0$ such that:

$$\text{For all } n \geq n_0, g(n) \geq c \cdot f(n)$$

In this case, we say that $g(n) \in \Omega(f(n))$

$\Omega(f(n))$ is the set of all functions bounded from below by $f(n)$

Bounded from Below: Big Ω

$$g(n) = \frac{n^2}{2} + 4n + 7$$

Prove that $g(n) \in \Omega(n^2)$

$$\frac{n^2}{2} + 4n + 7 \geq c \cdot n^2$$

Bounded from Below: Big Ω

$$\boxed{\frac{n^2}{2}} + 4n + 7 \geq c \cdot n^2$$

We'll start with a similar approach...

$$\frac{n^2}{2} \geq c_1 \cdot n^2$$

Bounded from Below: Big Ω

$$\frac{n^2}{2} \geq c_1 \cdot n^2$$

Bounded from Below: Big Ω

$$\frac{n^2}{2} \geq c_1 \cdot n^2$$

This is true for all $n \geq 0$ if we set c_1 to **1/2**

Bounded from Below: Big Ω

$$\frac{n^2}{2} + 4n + 7 \geq c \cdot n^2$$

Now that we've shown this…what else do we need to show for the overall equation to be true?

$$\frac{n^2}{2} \geq c_1 \cdot n^2$$

Bounded from Below: Big Ω
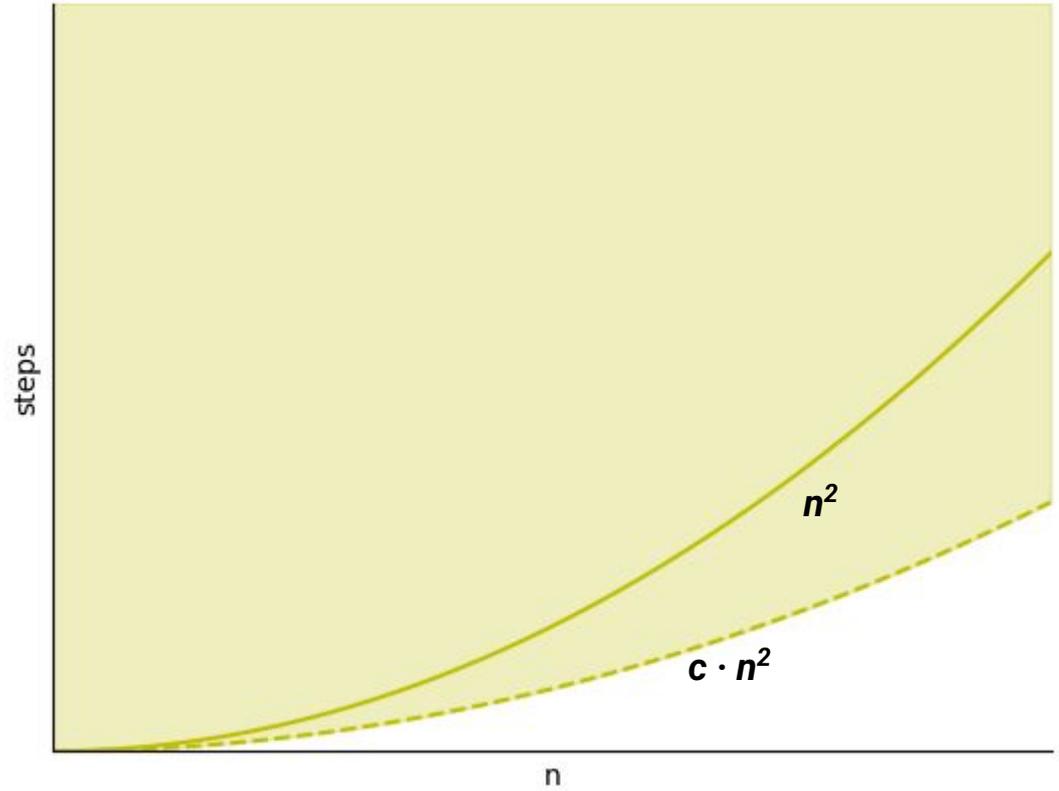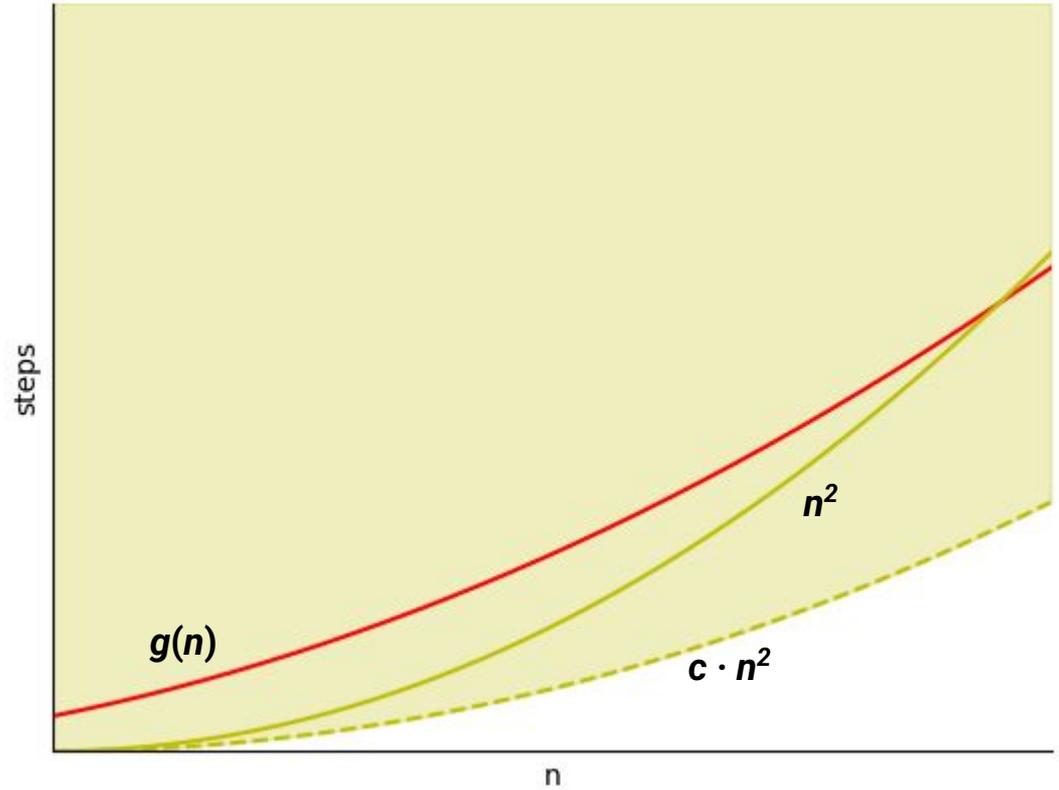
$$\frac{n^2}{2} + 4n + 7 \geq c \cdot n^2$$

$$\frac{n^2}{2} \geq c_1 \cdot n^2$$

Now that we've shown this...what else do we need to show for the overall equation to be true?

Just need to show that 4n and 7 are ≥ 0

By adding non-negative things to the first term we can only make it bigger!

Bounded from Below: Big Ω
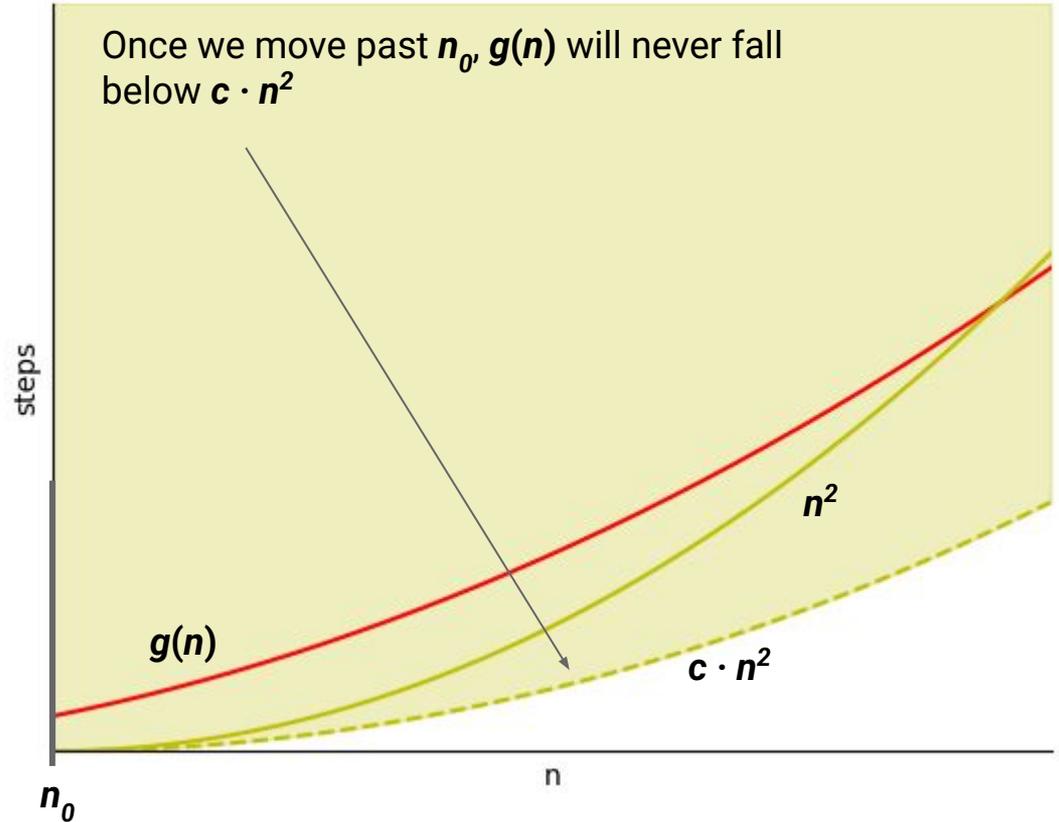
Bounded from Below: Big Ω

Bounded from Below: Big Ω

Once we move past n_0, $g(n)$ will never fall below $c \cdot n^2$

steps

n^2

$g(n)$

$c \cdot n^2$

n

n_0

Complexity Class: Big Θ

f and *g* are in the same complexity class, denoted *g(n)* ∈ Θ(*f(n)*), iff:

    g is bounded from above by something *f*-shaped

    and

    g is bounded from below by something *f*-shaped

Complexity Class: Big Θ

f and *g* are in the same complexity class, denoted *g(n)* ∈ **Θ**(*f(n)*), iff:

$g(n) \in O(f(n))$

and

$g(n) \in \Omega(f(n))$

Bounded from Below: Big Θ

$$g(n) = \frac{n^2}{2} + 4n + 7$$

Prove that $g(n) \in \Theta(n^2)$
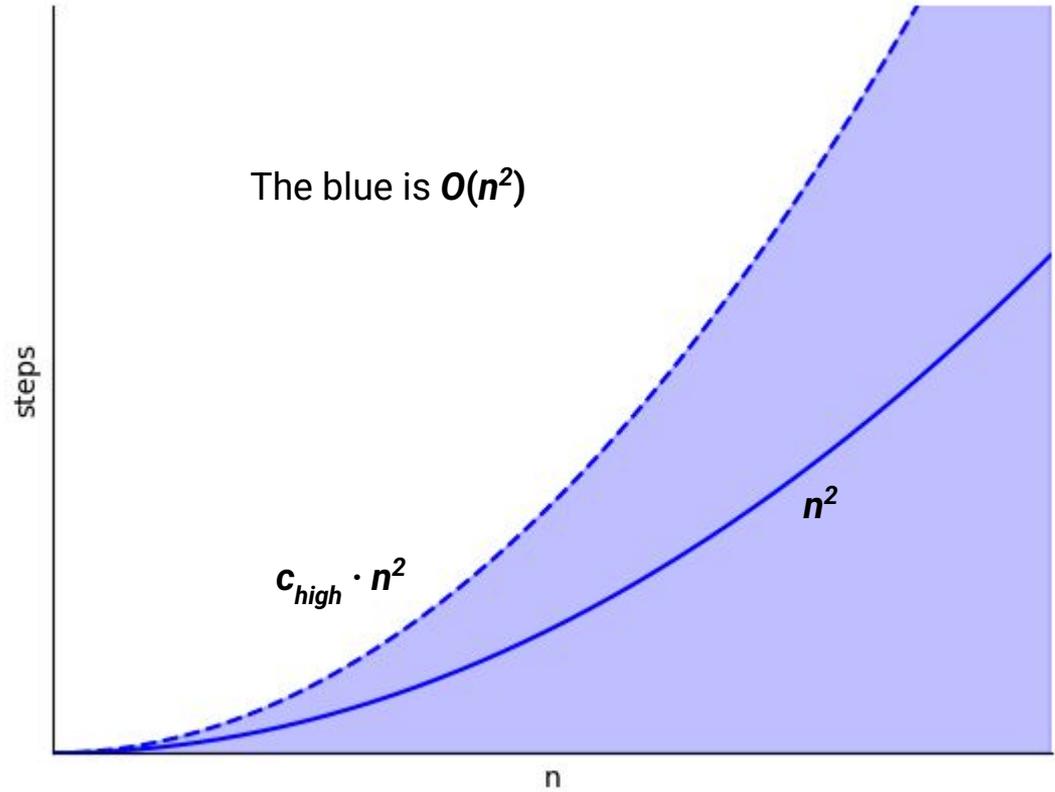
Bounded from Below: Big Θ

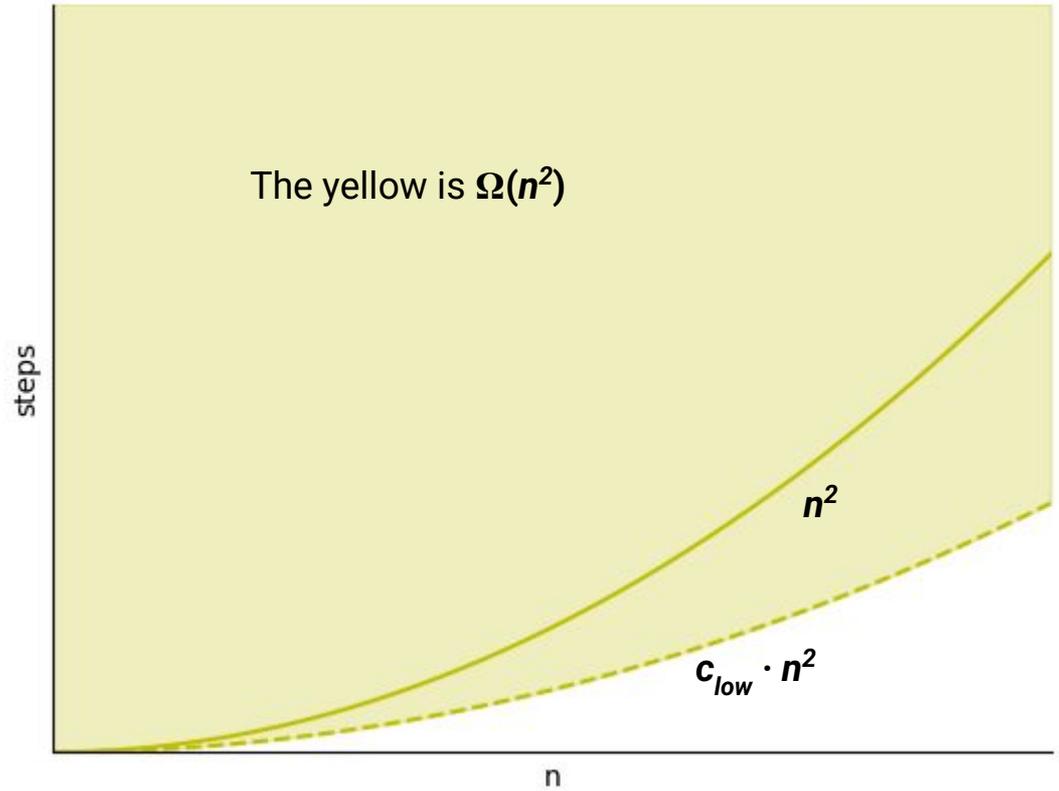$$g(n) = \frac{n^2}{2} + 4n + 7$$

Prove that $g(n) \in \Theta(n^2)$

We just proved that $g(n) \in O(n^2)$ and $g(n) \in \Omega(n^2)$

Therefore we have proven that $g(n) \in \Theta(n^2)$

Complexity Class: Big Θ

The blue is $O(n^2)$

$c_{high} \cdot n^2$

n^2

steps

n

Complexity Class: Big Θ

The yellow is $\Omega(n^2)$

steps

n^2

$c_{low} \cdot n^2$

n

Complexity Class: Big Θ

The overlap (green) is $\Theta(n^2)$

$c_{high} \cdot n^2$

n^2

$c_{low} \cdot n^2$

steps

n

43
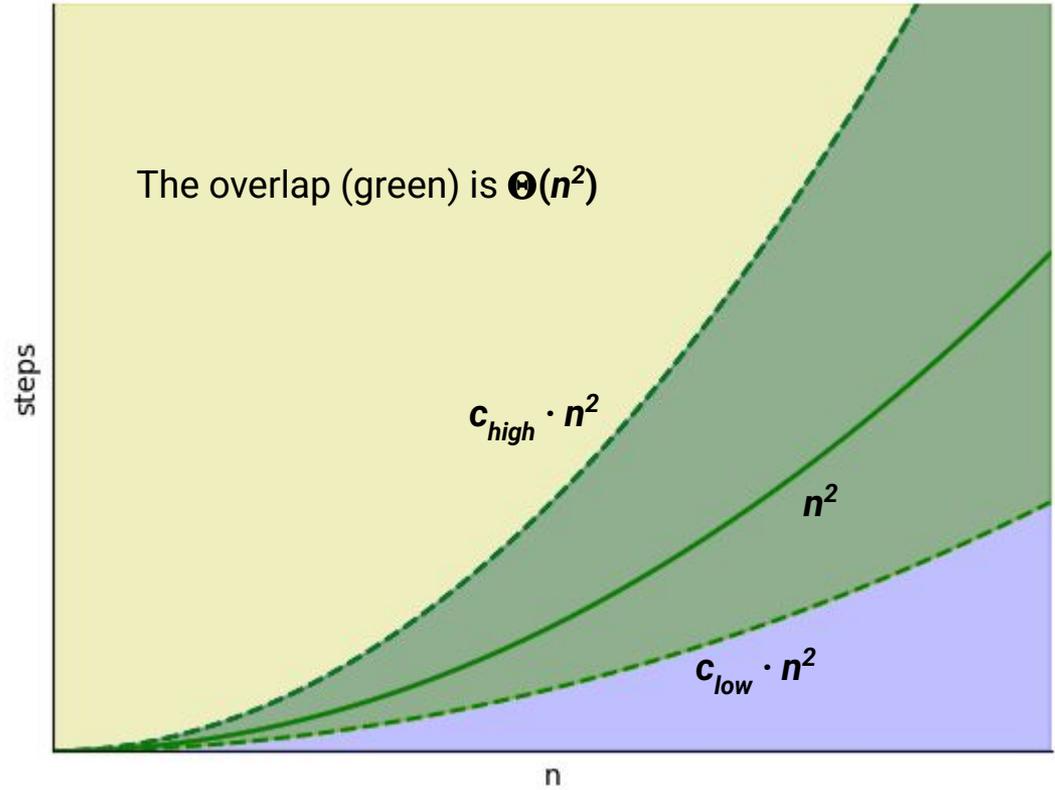
Complexity Class: Big Θ
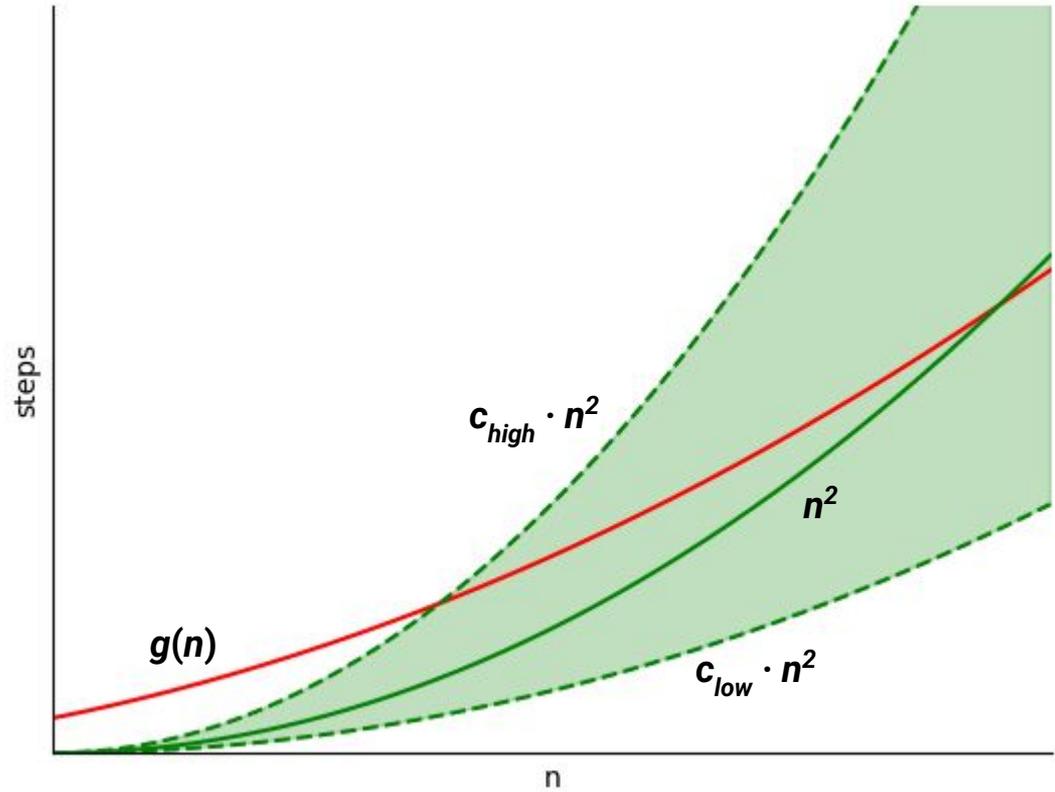
Complexity Class: Big Θ

Once we move past n_0, $g(n)$ will never leave the green region.

It is therefore in the quadratic complexity class.

Remember: the constants are *specific* constants that we solved for

steps

$c_{high} \cdot n^2$

n^2

$g(n)$

$c_{low} \cdot n^2$

n_0

n
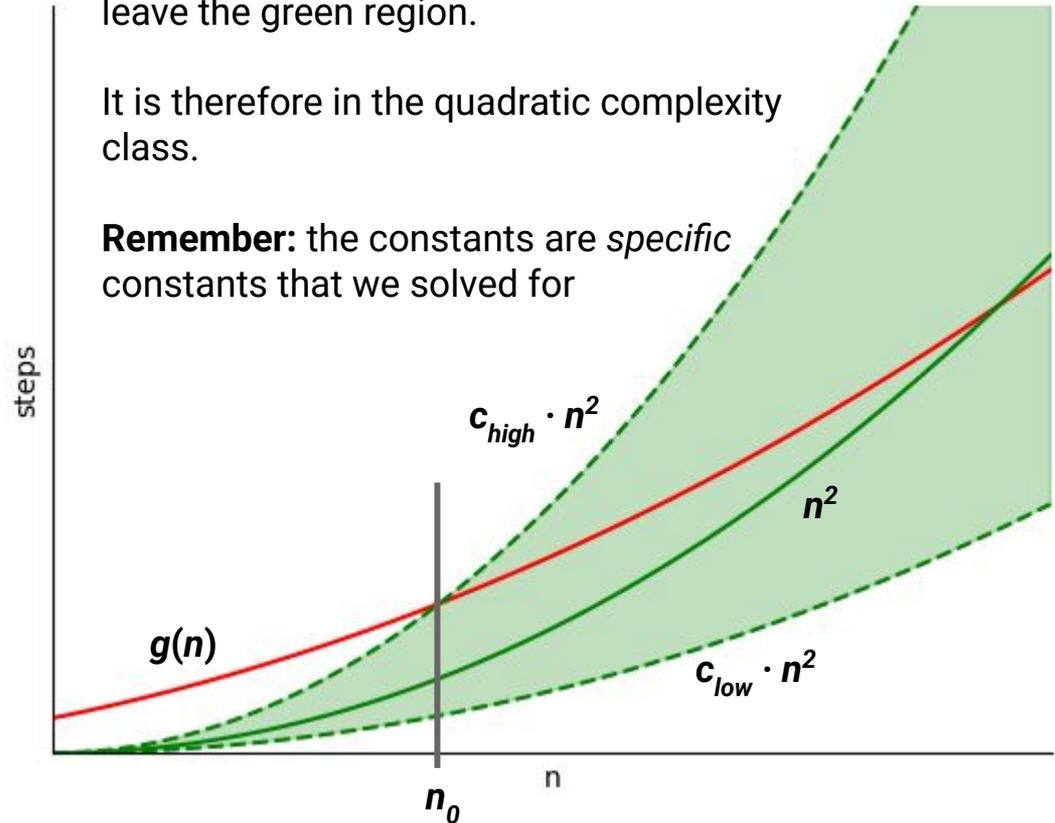
Complexity Class Ranking
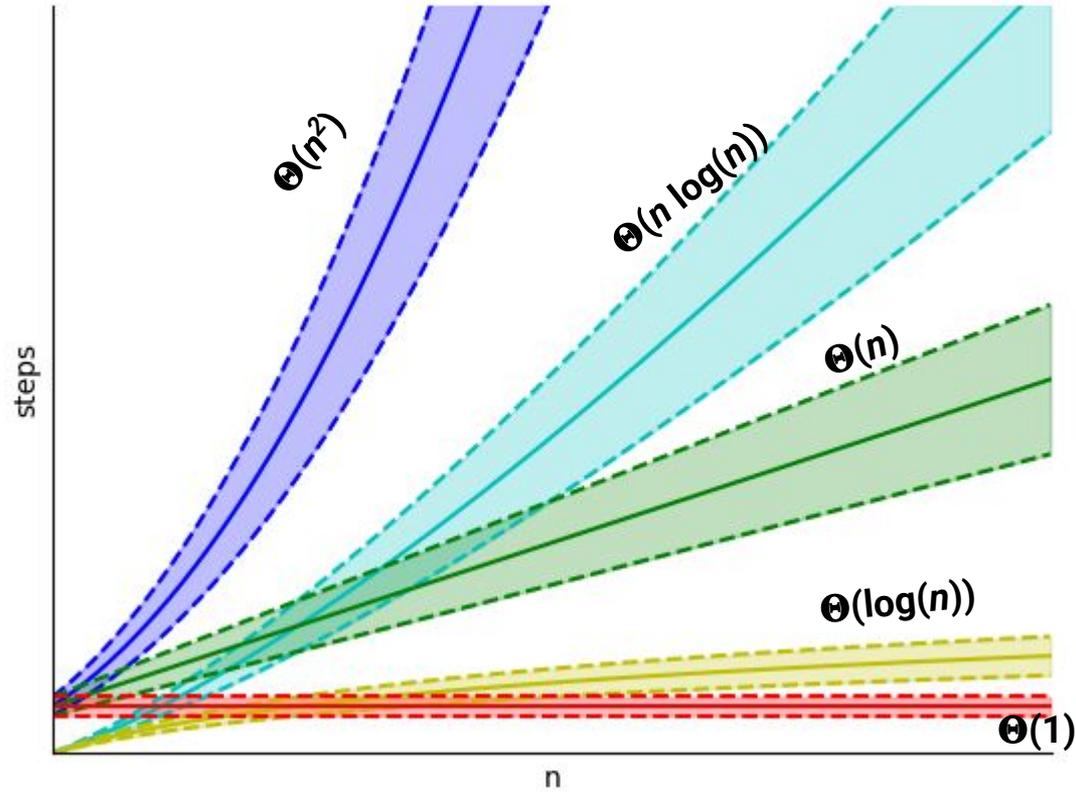
$$\Theta(1) < \Theta(\log(n)) < \Theta(n) < \Theta(n\log(n)) < \Theta(n^2) < \Theta(n^3) < \Theta(2^n)$$

Big O Subsets
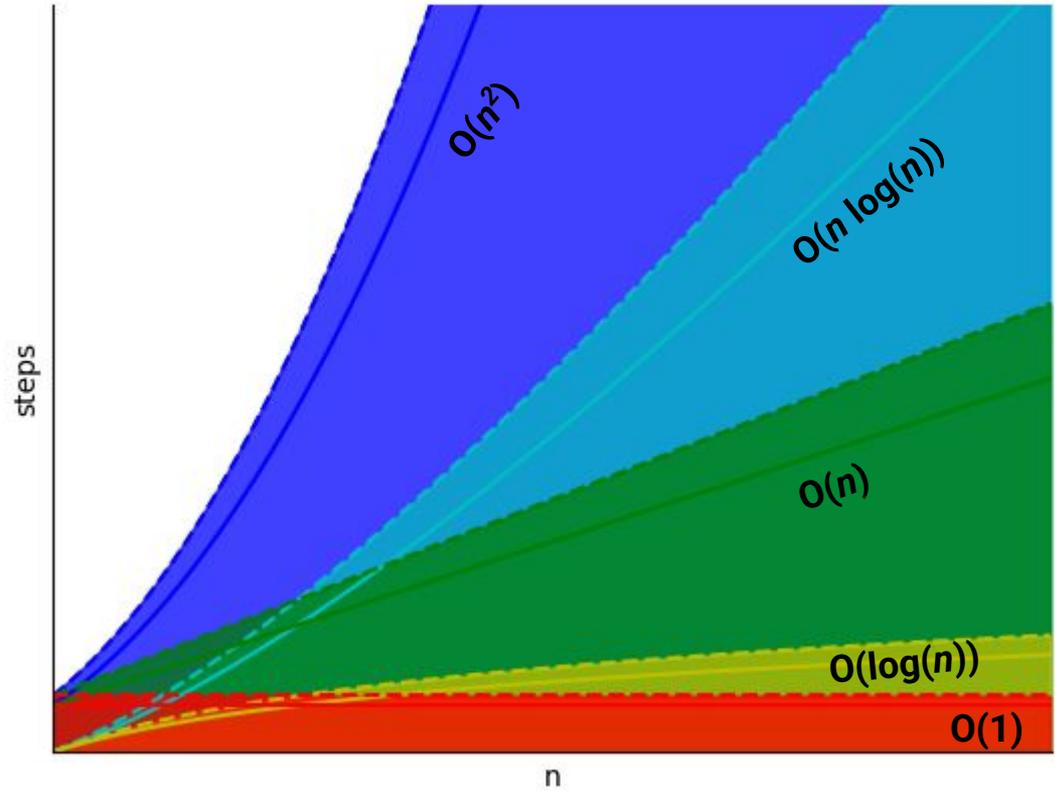
$$O(1) \subset O(\log(n)) \subset O(n) \subset O(n \log(n)) \subset O(n^2) \subset O(n^3) \subset O(2^n)$$

Shortcut

What complexity class do each of the following belong to:

$f(n) = 4n + n^2$

$g(n) = 2^n + 4n$

$h(n) = 100\ n\ \log(n) + 73n$

Shortcut

What complexity class do each of the following belong to:

$f(n) = 4n + n^2 \in \Theta(n^2)$

$g(n) = 2^n + 4n \in \Theta(2^n)$

$h(n) = 100\,n\log(n) + 73n \in \Theta(n\log(n))$

Shortcut

What complexity class do each of the following belong to:

$f(n) = 4n + n^2 \in \Theta(n^2)$

$g(n) = 2^n + 4n \in \Theta(2^n)$

$h(n) = 100\, n \log(n) + 73n \in \Theta(n \log(n))$

Shortcut: Just consider the complexity of the most dominant term

Why Focus on Dominating Terms?

$f(n)$	10	20	50	100	1000
$log(log(n))$	0.43 ns	0.52 ns	0.62 ns	0.68 ns	0.82 ns
$log(n)$	0.83 ns	1.01 ns	1.41 ns	1.66 ns	2.49 ns
n	2.5 ns	5 ns	12.5 ns	25 ns	0.25 μs
$nlog(n)$	8.3 ns	22 ns	71 ns	0.17 μs	2.49 μs
n^2	25 ns	0.1 μs	0.63 μs	2.5 μs	0.25 ms
n^5	25 μs	0.8 ms	78 ms	2.5 s	**2.9 days**
2^n	0.25 μs	0.26 ms	**3.26 days**	**10^{13} years**	**10^{284} years**
$n!$	0.91 ms	**19 years**	**10^{47} years**	**10^{141} years**	🤯

Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in O(n^2)$ ✓

Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in O(n^2)$ ✓

Is $f(n)$ in $O(n^3)$?

Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in O(n^2)$ ✓

Is $f(n)$ in $O(n^3)$? ✓

Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in O(n^2)$ ✓

Is $f(n)$ in $O(n^3)$? ✓

Is $f(n)$ in $O(2^n)$?

Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in O(n^2)$ ✓

Is $f(n)$ in $O(n^3)$? ✓

Is $f(n)$ in $O(2^n)$? ✓

Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in O(n^2)$ ✓

Is $f(n)$ in $O(n^3)$? ✓

Is $f(n)$ in $O(2^n)$? ✓

Is $f(n)$ in $O(n)$?

Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in O(n^2)$ ✓

Is $f(n)$ in $O(n^3)$? ✓

Is $f(n)$ in $O(2^n)$? ✓

Is $f(n)$ in $O(n)$? ✗

Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in O(n^2)$ ✔

Is $f(n)$ in $O(n^3)$? ✔

Is $f(n)$ in $O(2^n)$? ✔

Is $f(n)$ in $O(n)$? ✗

n^2, n^3, and 2^n all bound $f(n)$ from above
n^2 is a **<u>tight</u>** upper bound of $f(n)$
(there is no smaller upper bound for $f(n)$)

Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in \Omega(n^2)$ ✓

Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in \Omega(n^2)$ ✓

Is $f(n)$ in $\Omega(n^3)$?

Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in \Omega(n^2)$ ✓

Is $f(n)$ in $\Omega(n^3)$? ✗

Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in \Omega(n^2)$ ✔

Is $f(n)$ in $\Omega(n^3)$? ✗

Is $f(n)$ in $\Omega(\log(n))$?

Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in \Omega(n^2)$ ✓

Is $f(n)$ in $\Omega(n^3)$? ✗

Is $f(n)$ in $\Omega(\log(n))$? ✓

Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in \Omega(n^2)$ ✓

Is $f(n)$ in $\Omega(n^3)$? ✗

Is $f(n)$ in $\Omega(\log(n))$? ✓

Is $f(n)$ in $\Omega(n)$?

Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in \Omega(n^2)$ ✓

Is $f(n)$ in $\Omega(n^3)$? ✗

Is $f(n)$ in $\Omega(\log(n))$? ✓

Is $f(n)$ in $\Omega(n)$? ✓

Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in \Omega(n^2)$ ✔

Is $f(n)$ in $\Omega(n^3)$? ✘

Is $f(n)$ in $\Omega(\log(n))$? ✔

Is $f(n)$ in $\Omega(n)$? ✔

n^2, n, and **log(n)** all bound $f(n)$ from below
n^2 is a <u>**tight**</u> lower bound of $f(n)$
(there is no larger lower bound for $f(n)$)

Tight Bounds

If $g(n) \in \Theta(f(n))$, then:
- $g(n) \in O(f(n))$ is a tight upper bound
- $g(n) \in \Omega(f(n))$ is a tight lower bound

Tight Bounds

If $g(n) \in \Theta(f(n))$, then:
- $g(n) \in O(f(n))$ is a tight upper bound
- $g(n) \in \Omega(f(n))$ is a tight lower bound

But what if the tight upper bound and tight lower bound are not the same?

Multi-class Functions

$$T(n) = \begin{cases} n^2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

What is the tight upper bound of this function?

Multi-class Functions

$$T(n) = \begin{cases} n^2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

What is the tight upper bound of this function? $T(n) \in O(n^2)$

Multi-class Functions

$$T(n) = \begin{cases} n^2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

What is the tight upper bound of this function? **$T(n) \in O(n^2)$**

What is the tight lower bound of this function?

Multi-class Functions

$$T(n) = \begin{cases} n^2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

What is the tight upper bound of this function? $T(n) \in O(n^2)$

What is the tight lower bound of this function? $T(n) \in \Omega(n)$

Multi-class Functions

$$T(n) = \begin{cases} n^2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

What is the tight upper bound of this function? **$T(n) \in O(n^2)$**

What is the tight lower bound of this function? **$T(n) \in \Omega(n)$**

What is the complexity class of this function?

Multi-class Functions

$$T(n) = \begin{cases} n^2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

What is the tight upper bound of this function? **$T(n) \in O(n^2)$**

What is the tight lower bound of this function? **$T(n) \in \Omega(n)$**

What is the complexity class of this function? It does not have one!

Multi-class Functions

$$T(n) = \begin{cases} n^2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

It is not bounded from above by **n**, therefore it cannot be in Θ(**n**)

It is not bounded from below by **n²**, therefore it cannot be in Θ(**n²**)

What is the tight upper bound of this function? **T(n) ∈ O(n²)**

What is the tight lower bound of this function? **T(n) ∈ Ω(n)**

What is the complexity class of this function? It does not have one!