CSE 250
Data Structures

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Capen 208

Lecture 11
Amortized Runtime

Announcements

- PA1 Implementation AutoLab is now open!
  - Don’t rely on AutoLab to tell you what’s wrong – that’s what **your** tests are for
  - Add more tests as needed
- WA2 releases next week

`List` Summary (so far...)

	`Array`	`LinkedList` *(by index)*	`LinkedList` *(by reference)*
`get(i)`	$\Theta(1)$	$\Theta(i) \subset O(n)$	$\Theta(1)$
`set(i, v)`	$\Theta(1)$	$\Theta(i) \subset O(n)$	$\Theta(1)$
`size()`	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
`add(v)`	$O(n)$	$\Theta(1)$	$\Theta(1)$
`add(i, v)`	*???*	$\Theta(i) \subset O(n)$	$\Theta(1)$
`remove(idx)`	$O(n)$	$\Theta(i) \subset O(n)$	$\Theta(1)$

`ArrayList.add(elem)` - Analysis

$$T_{\text{add}}(n) = \begin{cases} 1 \text{ if used} < \text{data.length} \\ n \text{ if used} = \text{data.length} \end{cases}$$

What are the bounds of $T_{\text{add}}(n)$?

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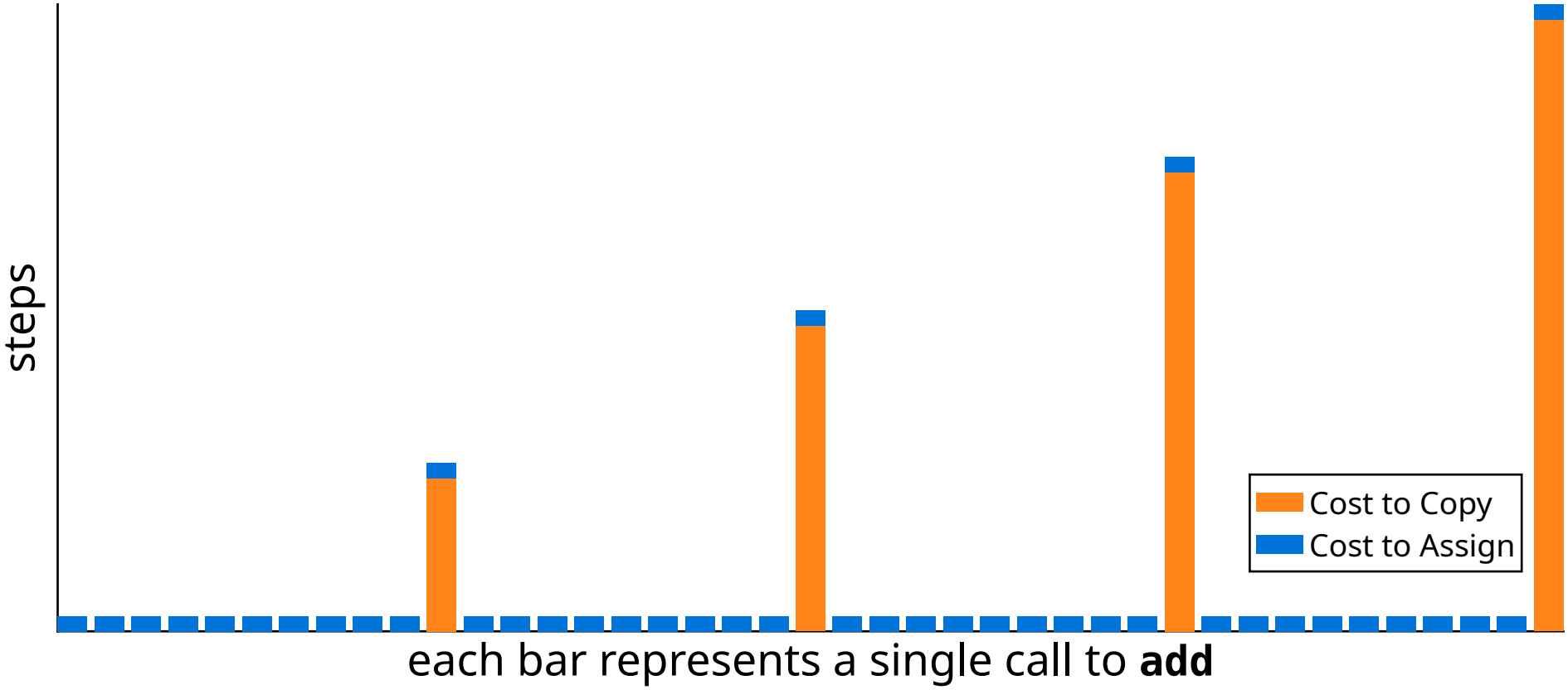
How often do our add calls require n steps?

It depends on how we calculate **`newLength`**!

`ArrayList.add(elem)`: +10 Each Resize

Initial Size = 10, **`newLength = data.length + 10`**



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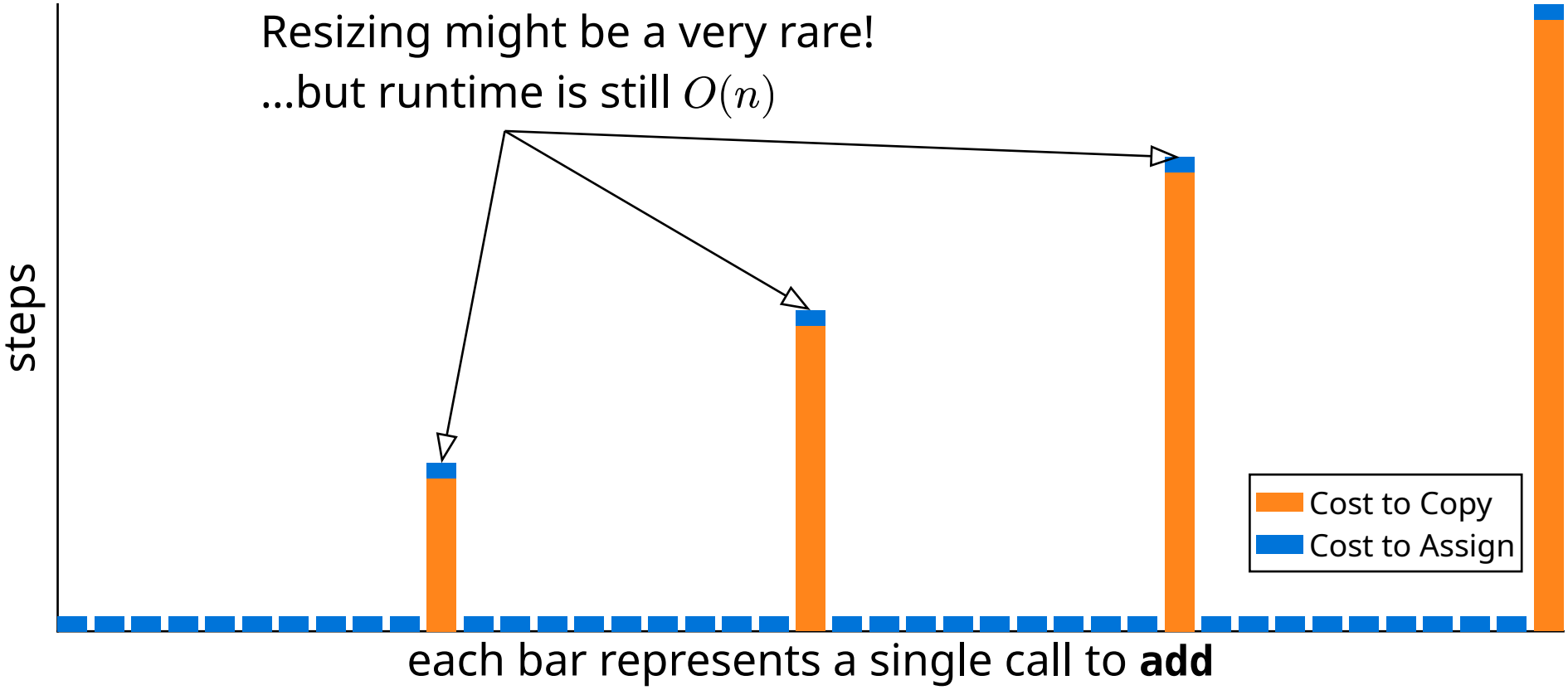

A Note on Runtime Complexity

So far, when discussing runtime bounds, we have done so without taking any extra information/context into account

For example:

- The worst-case runtime of **`ArrayList.add`** is $O(n)$
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But sometimes extra context is relevant...how can we include it in our analysis?

Common Pattern: Repeated Calls

Oftentimes, we will call a function many times in a row

- ie read through a CSV file and **`add`** all records to a **`List`**

What can we say about the runtime in this case?

Appending n Elements to a `LinkedList`

```java
List<Integer> list = new LinkedList<Integer>();
for (int i = 0; i < n; i++) {
    list.add(i);
}
```

What is the unqualified runtime of this code snippet?

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But is this a tight bound? Not every iteration takes n steps...

We need to take a more detailed look to be sure.

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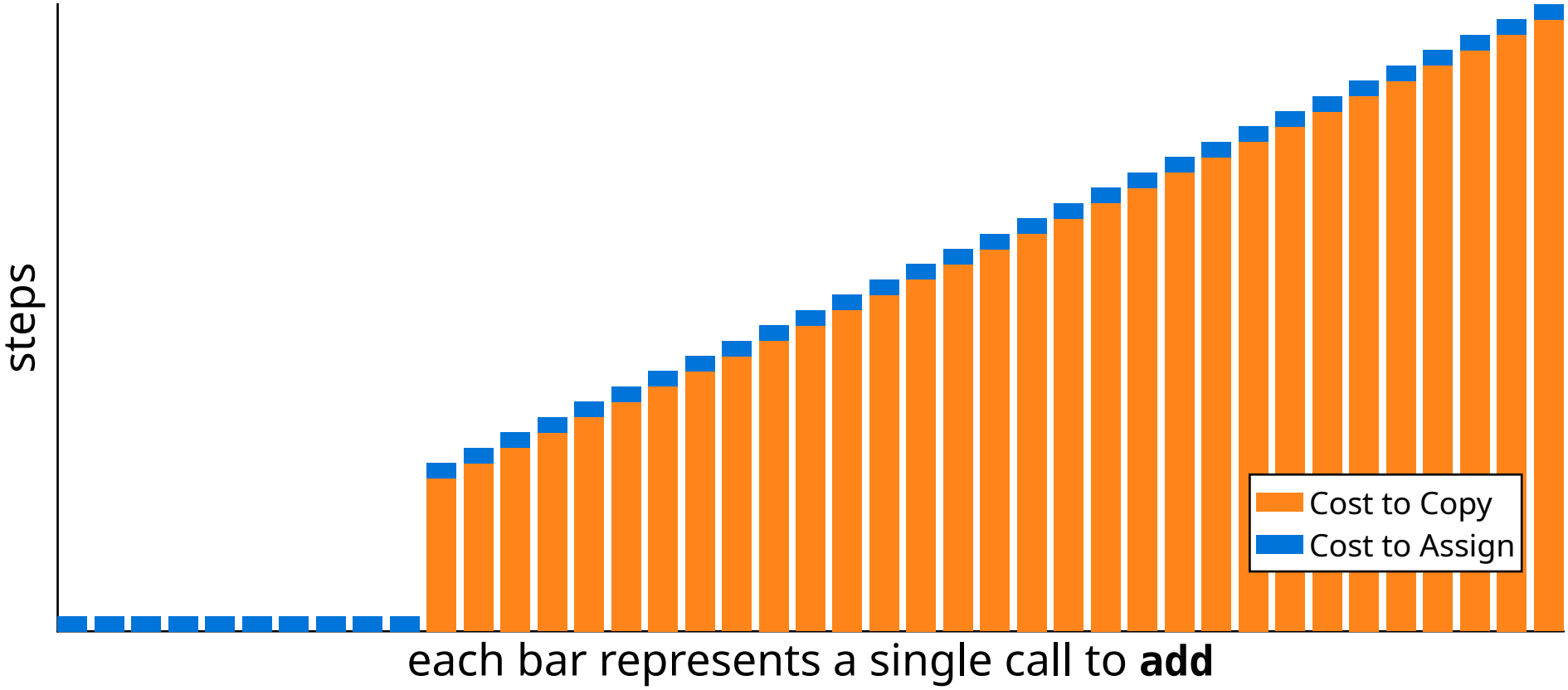

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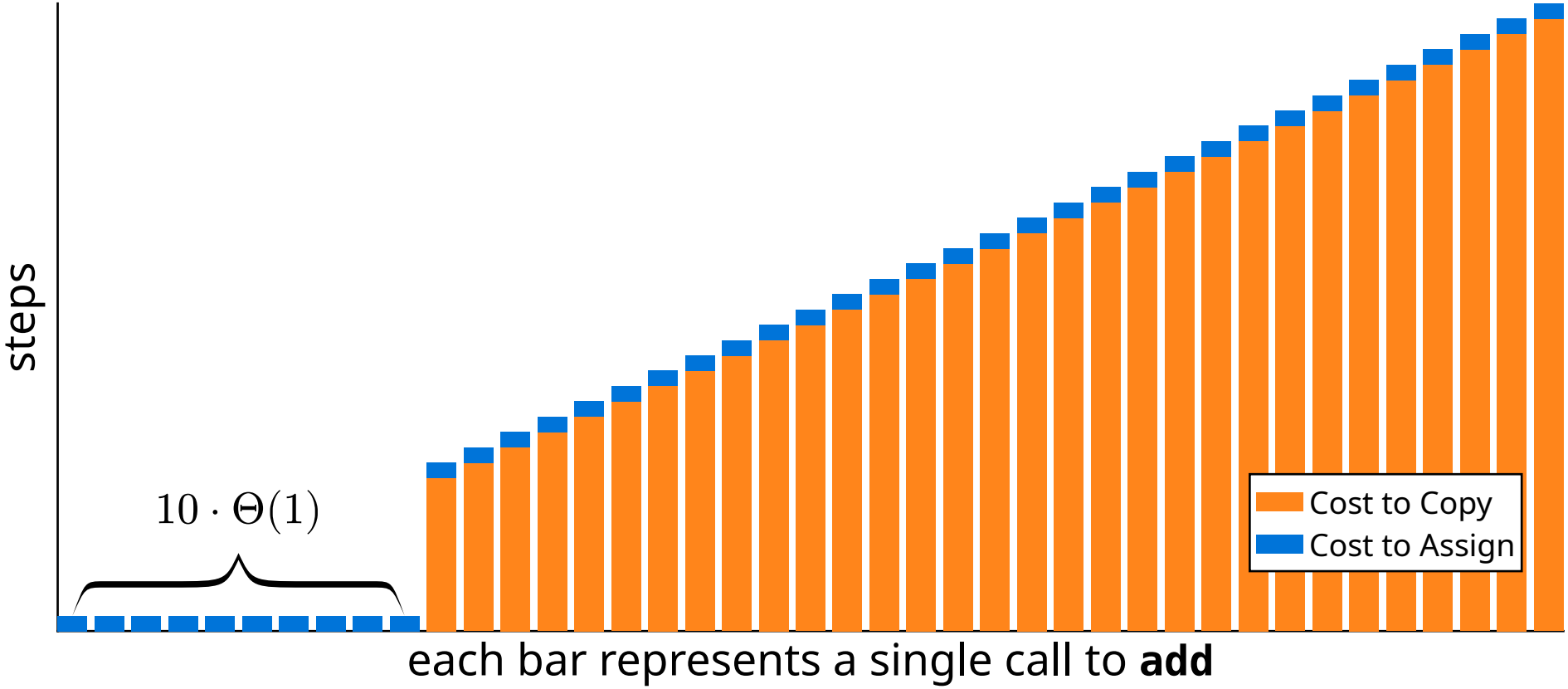

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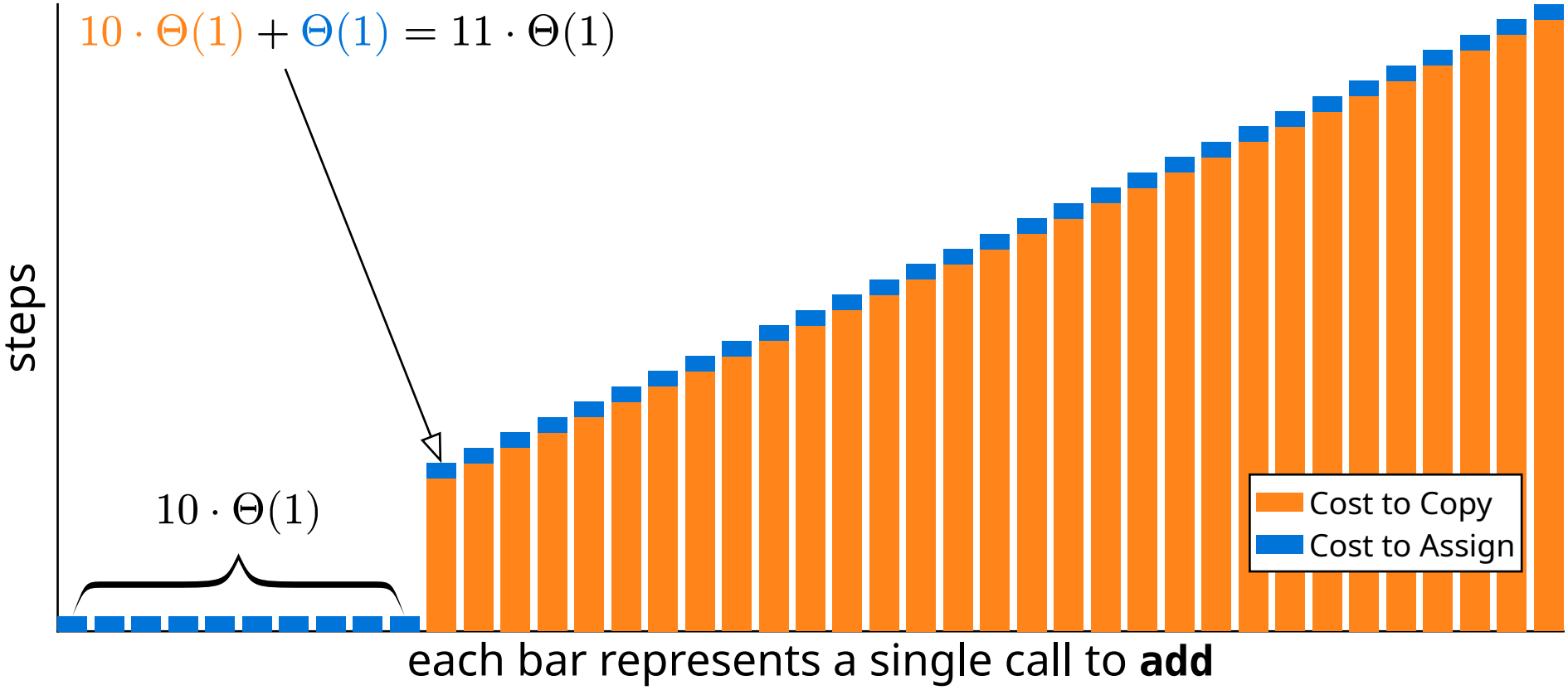

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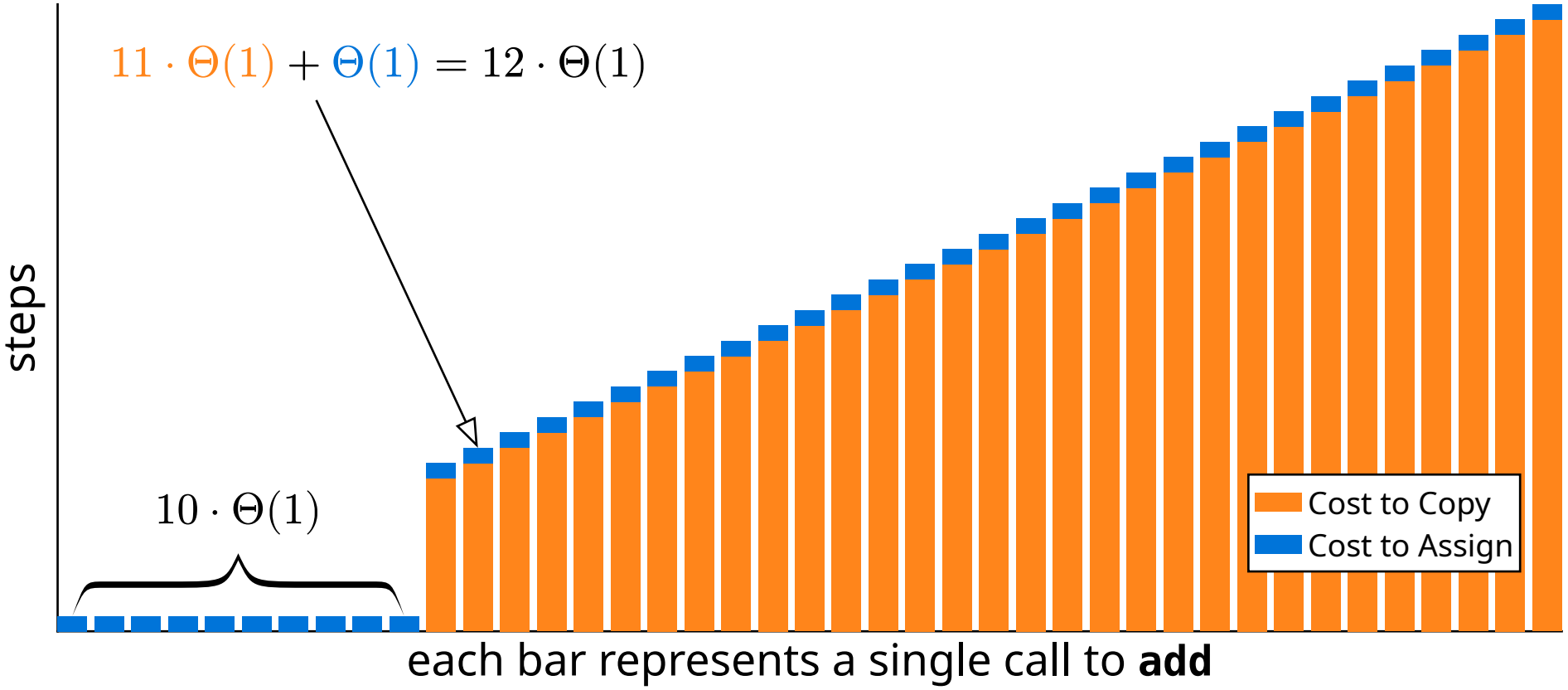

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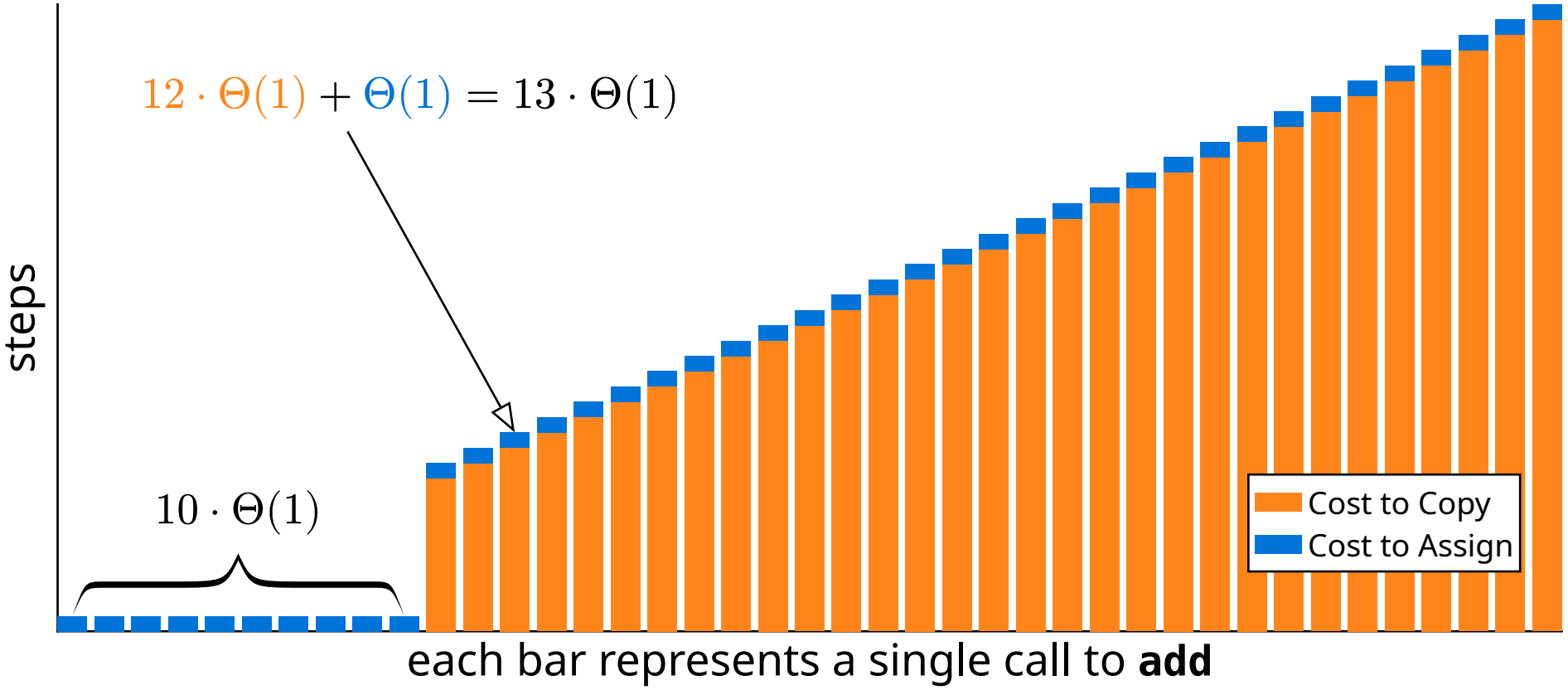

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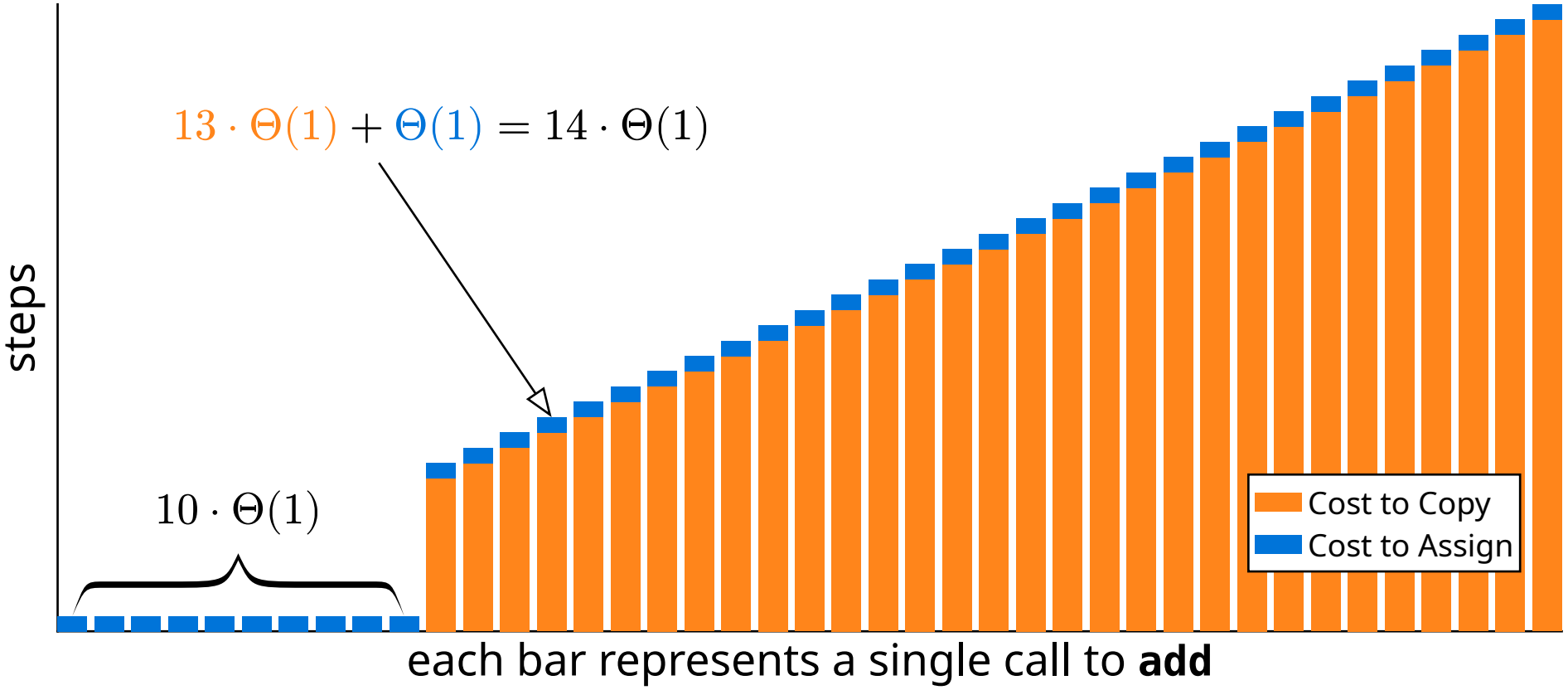

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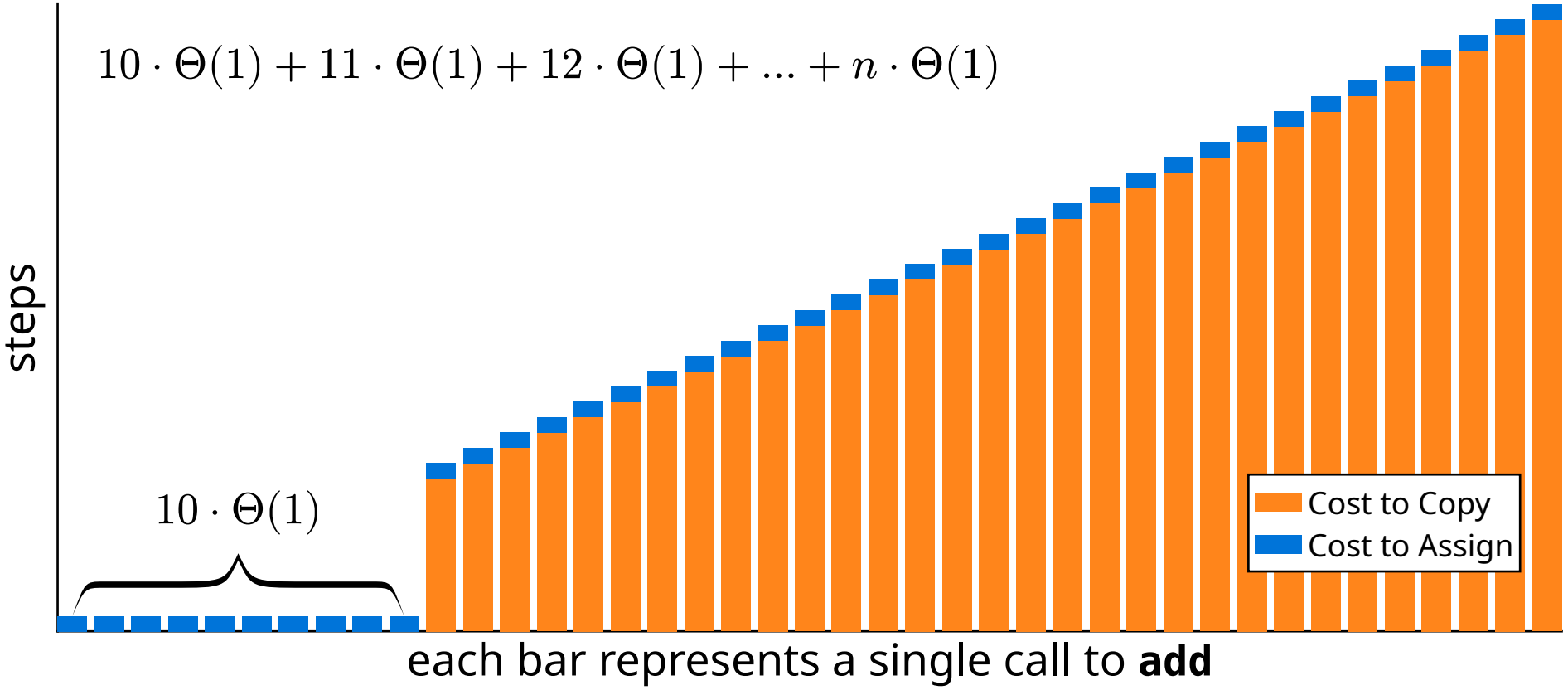

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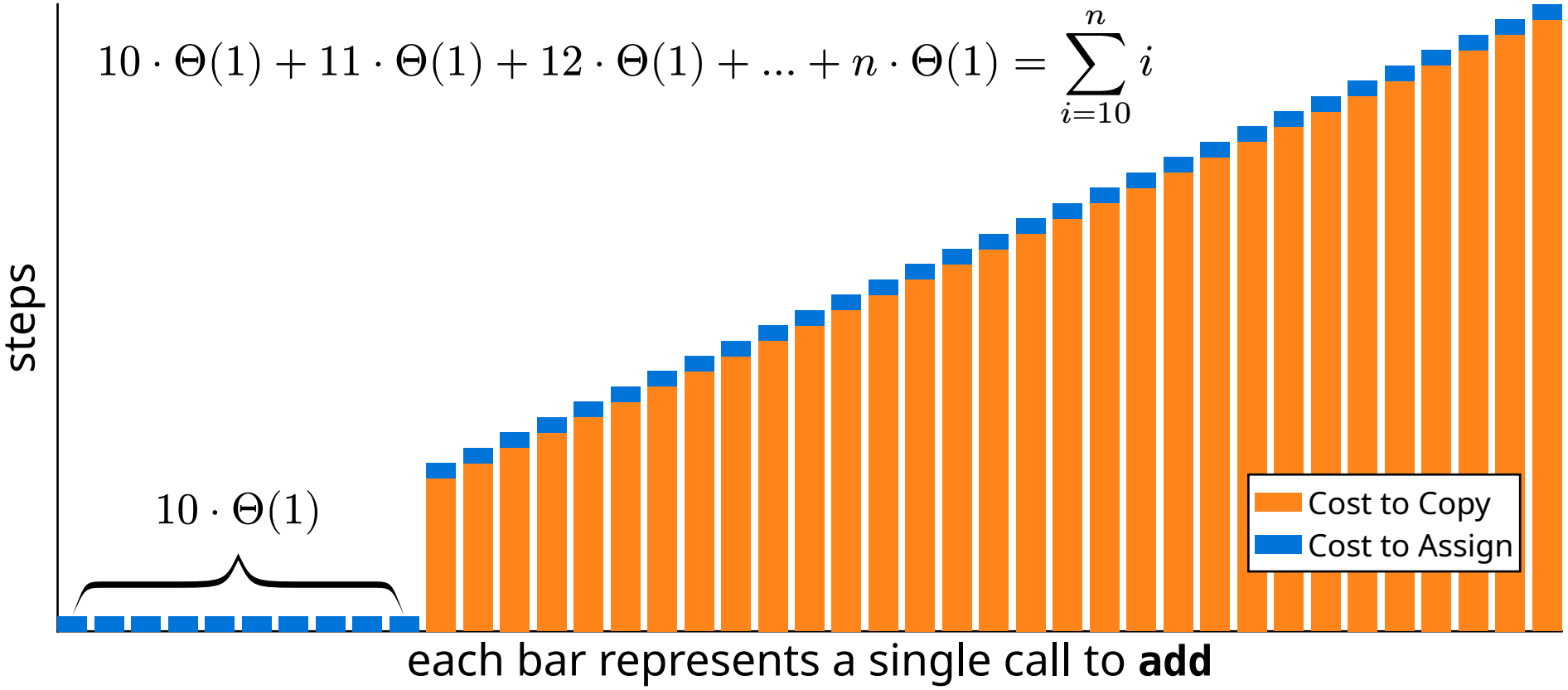

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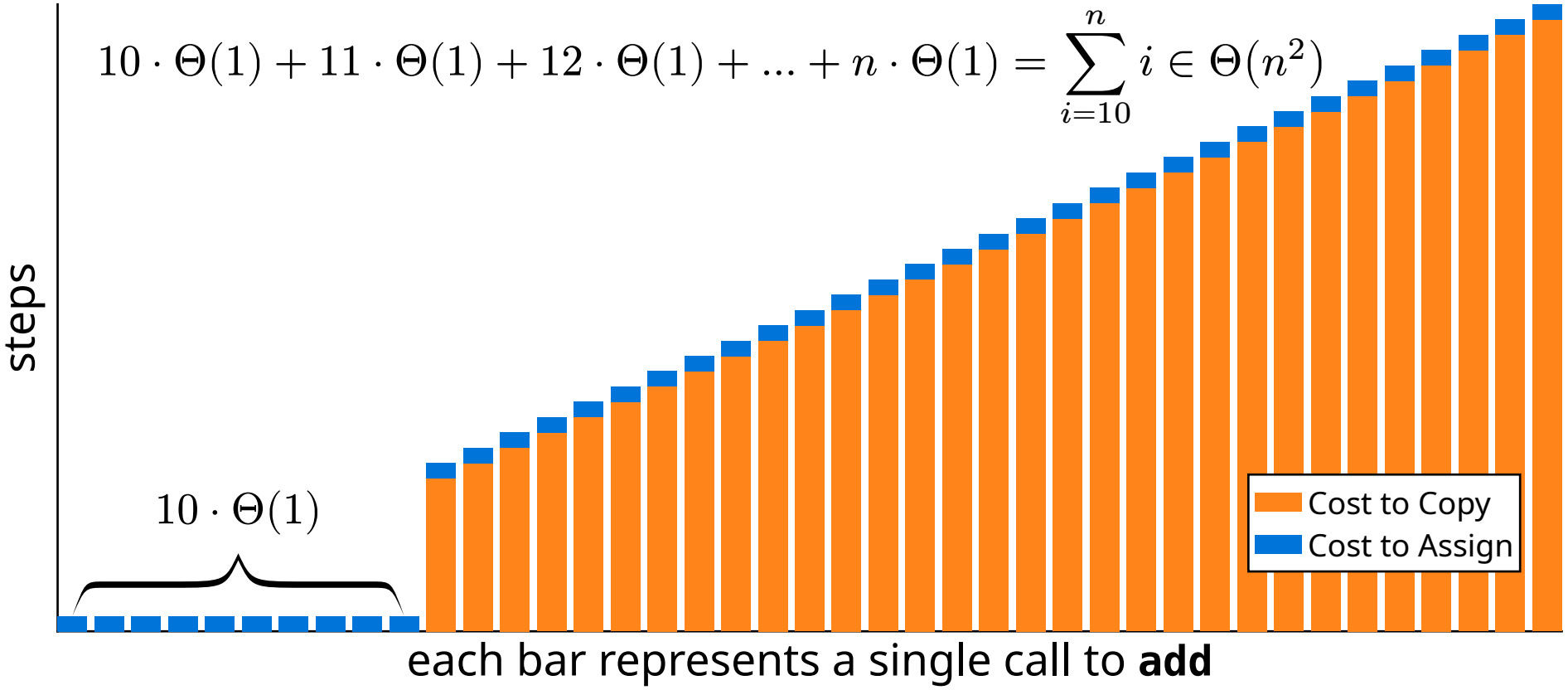

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`ArrayList.add(elem)`: +2 Each Resize

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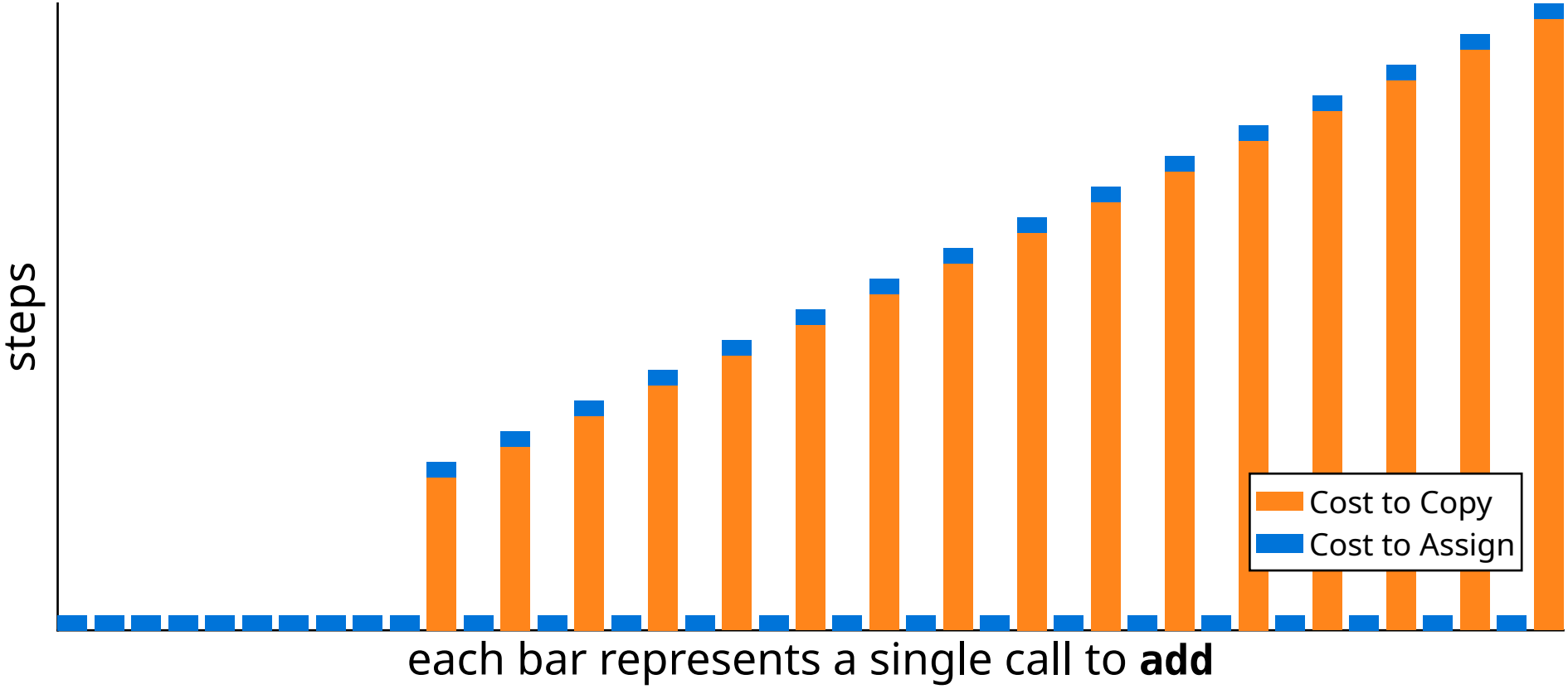

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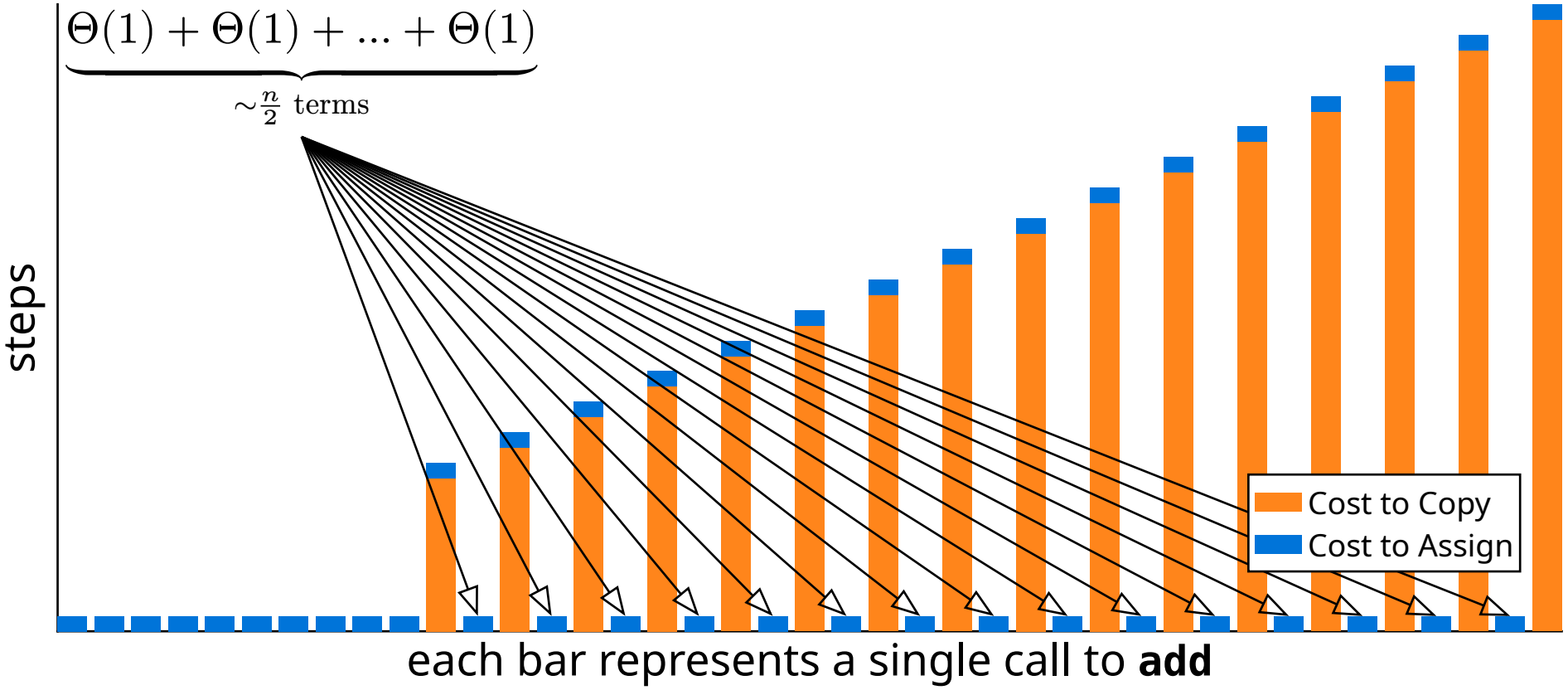

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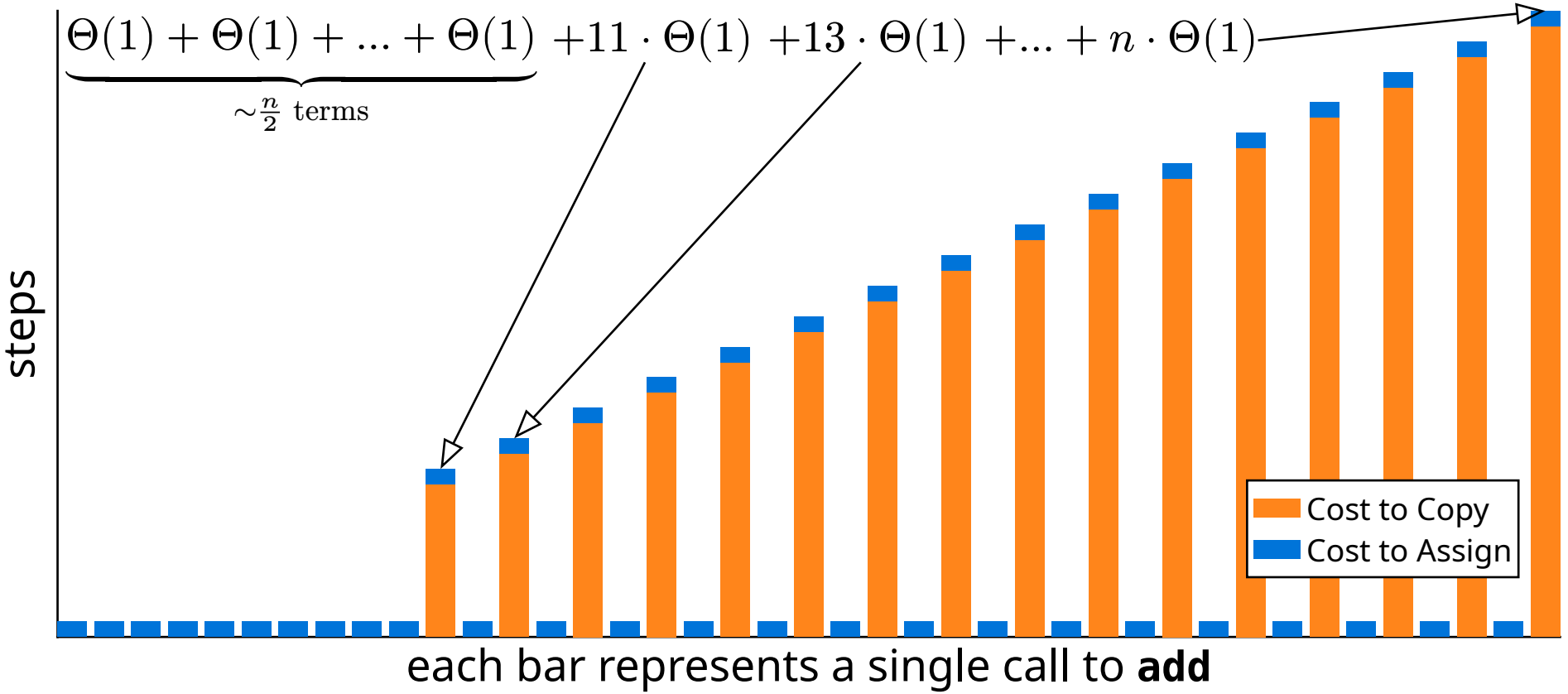

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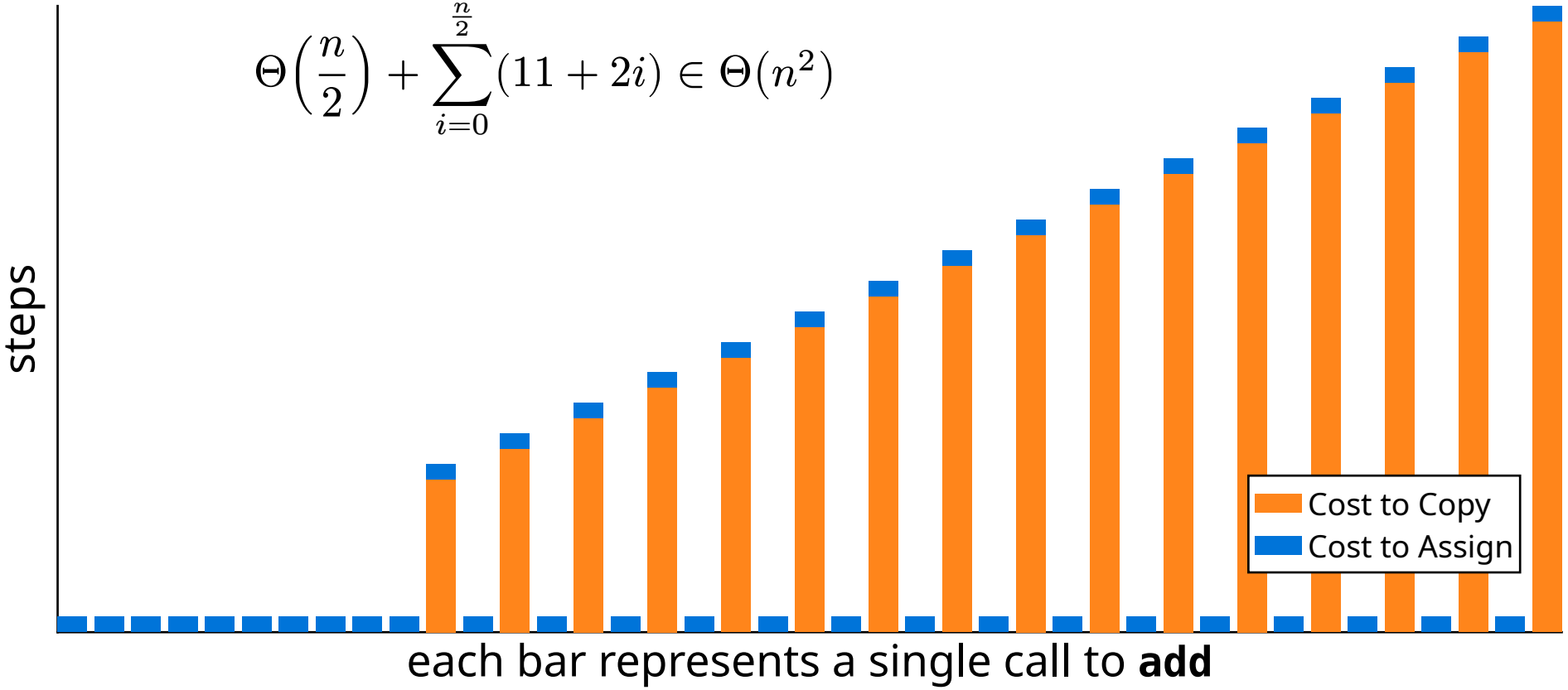

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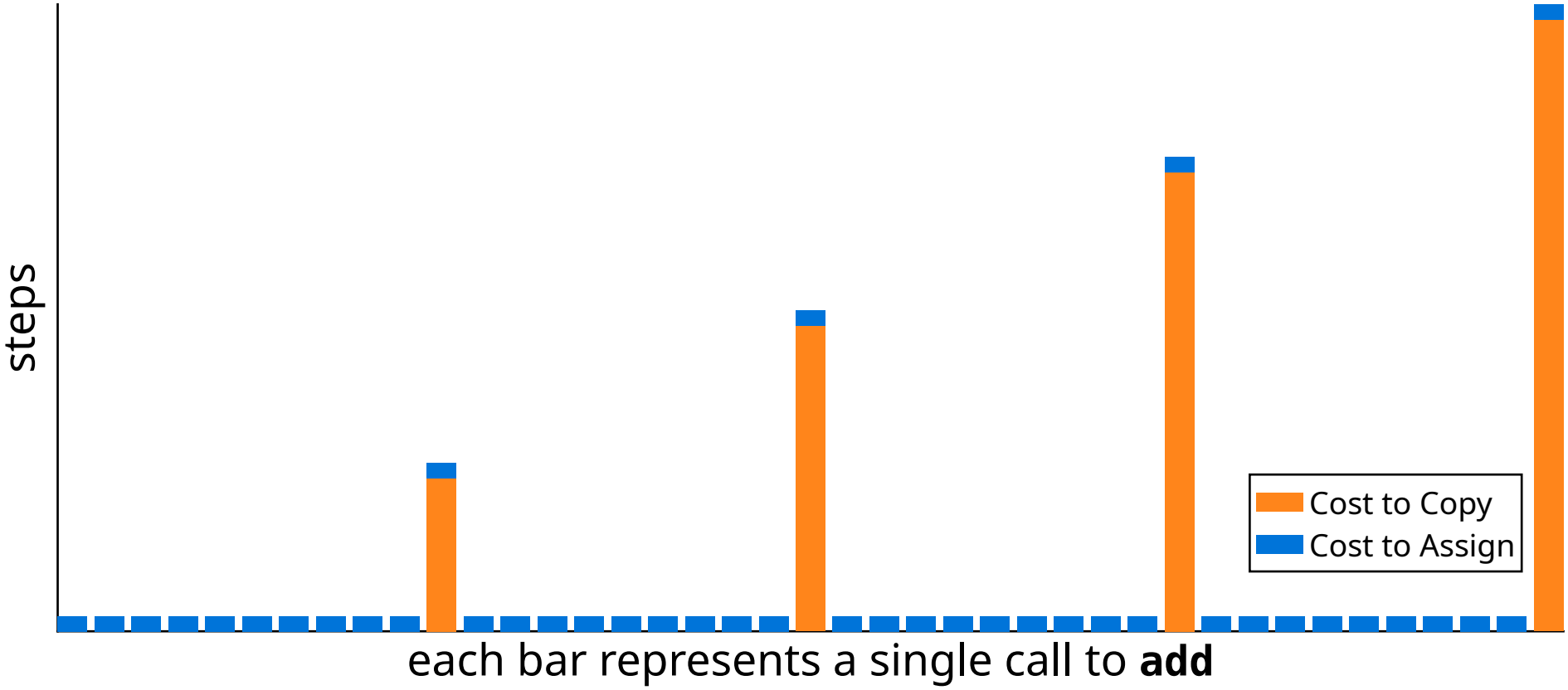

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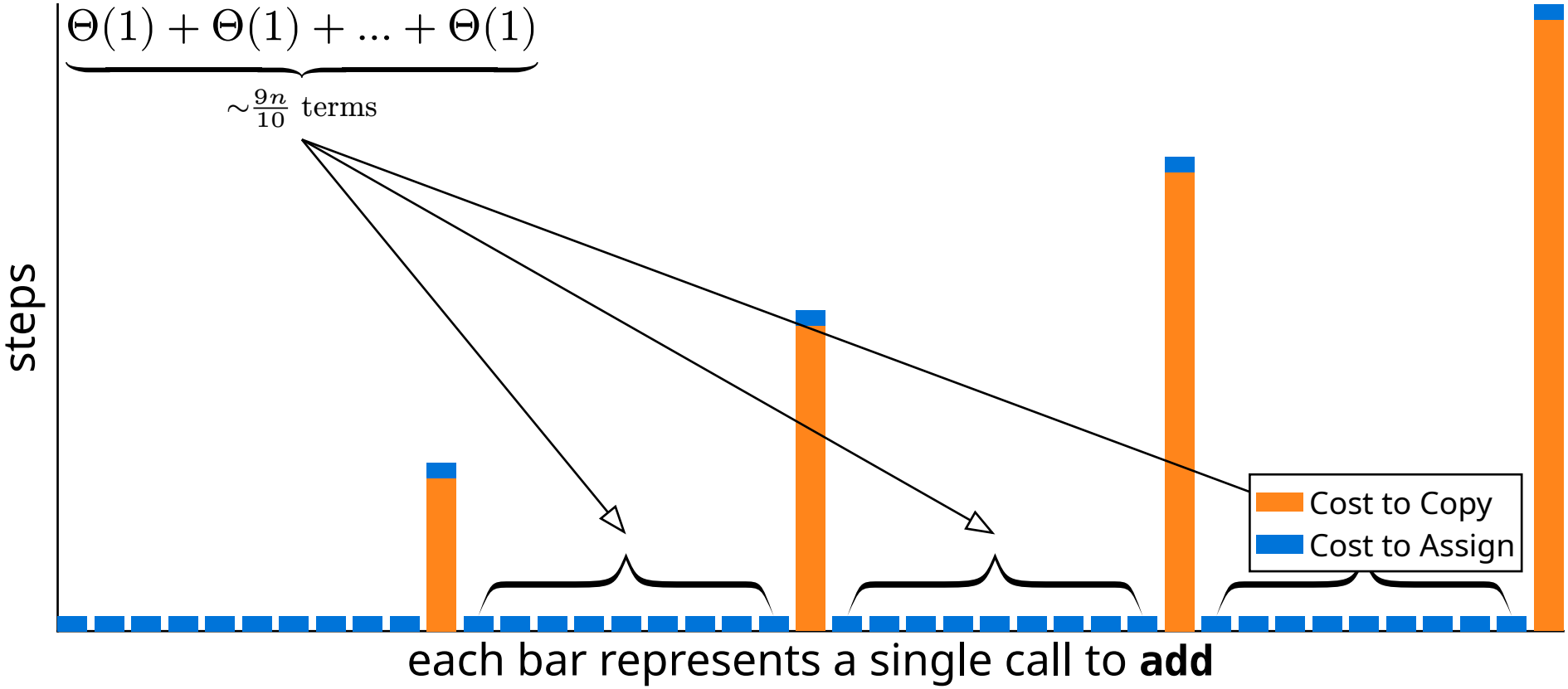

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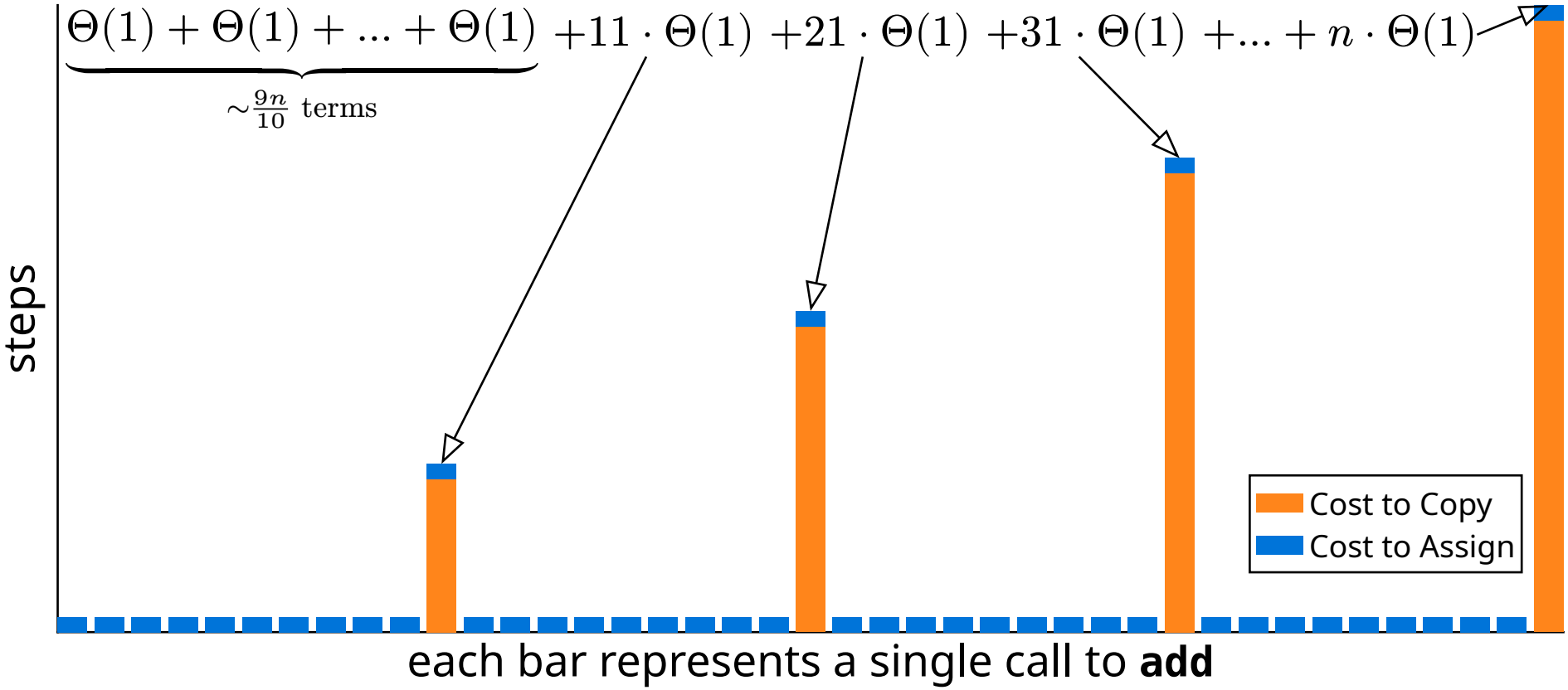

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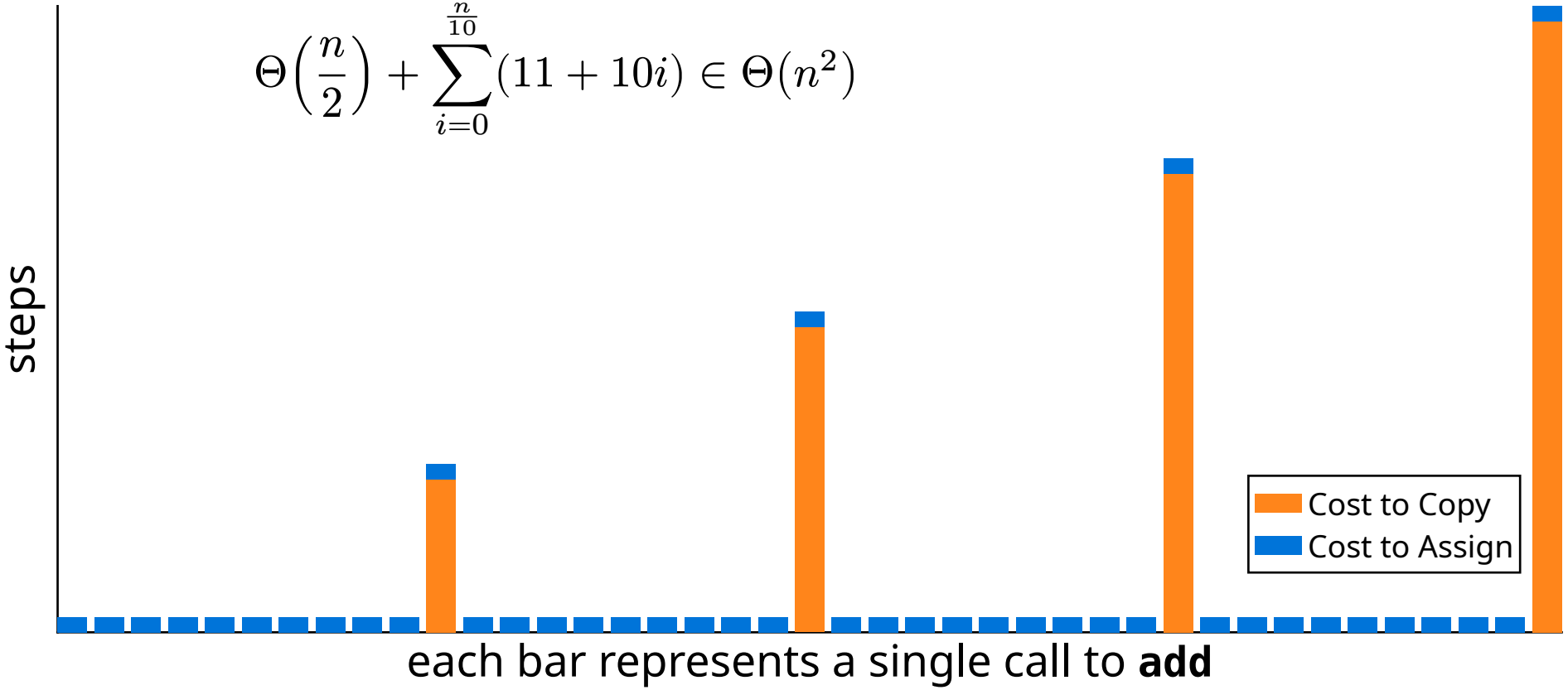

A Different Approach

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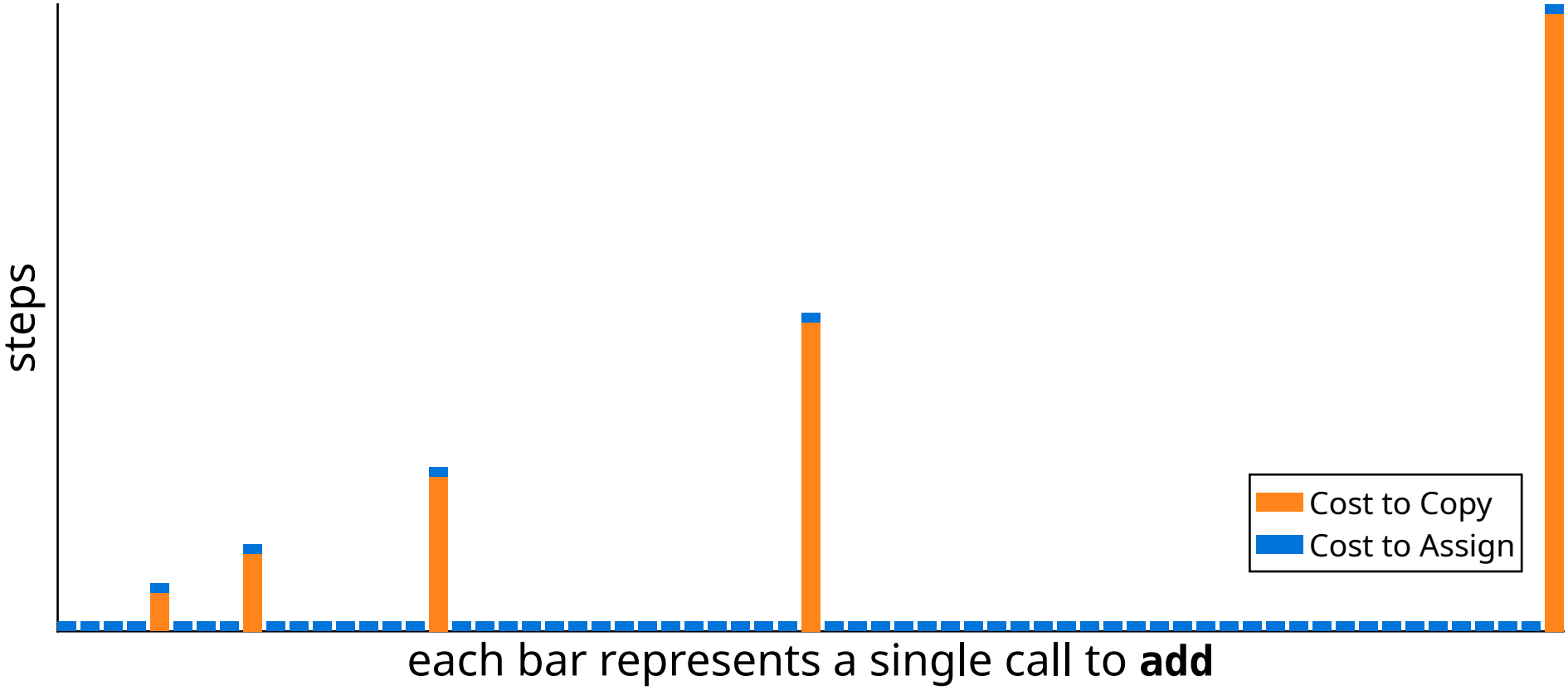
Double it!

Doubling the Length Each Time

Initial Size = 4, **`newLength = data.length * 2`**



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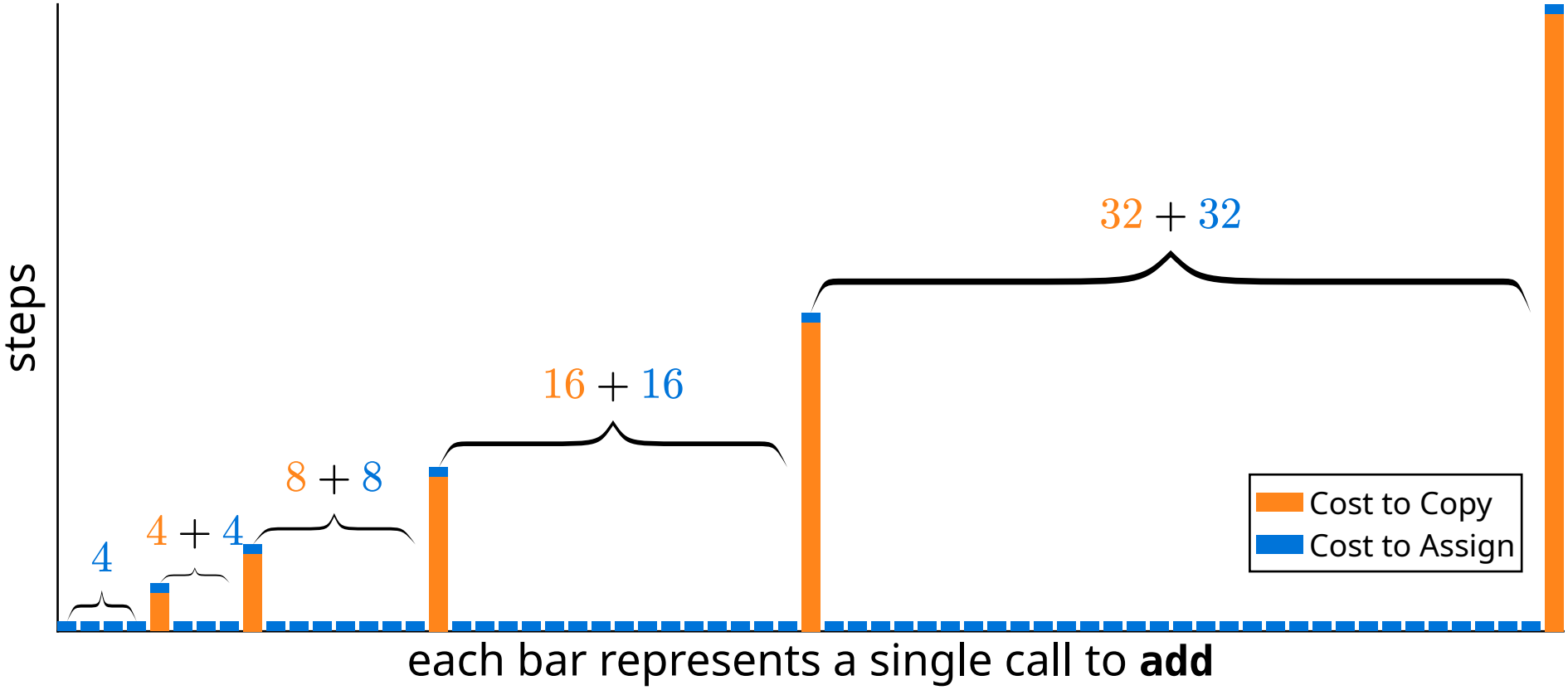

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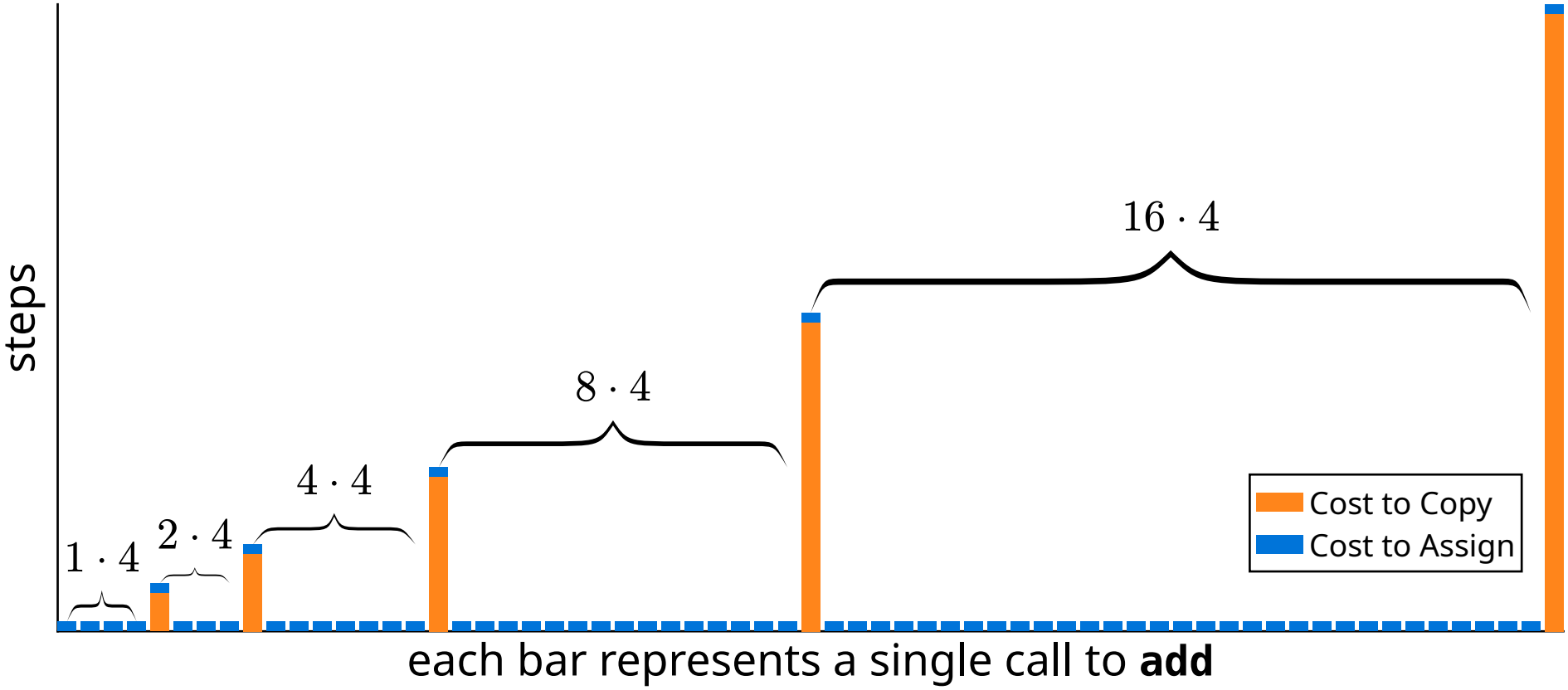

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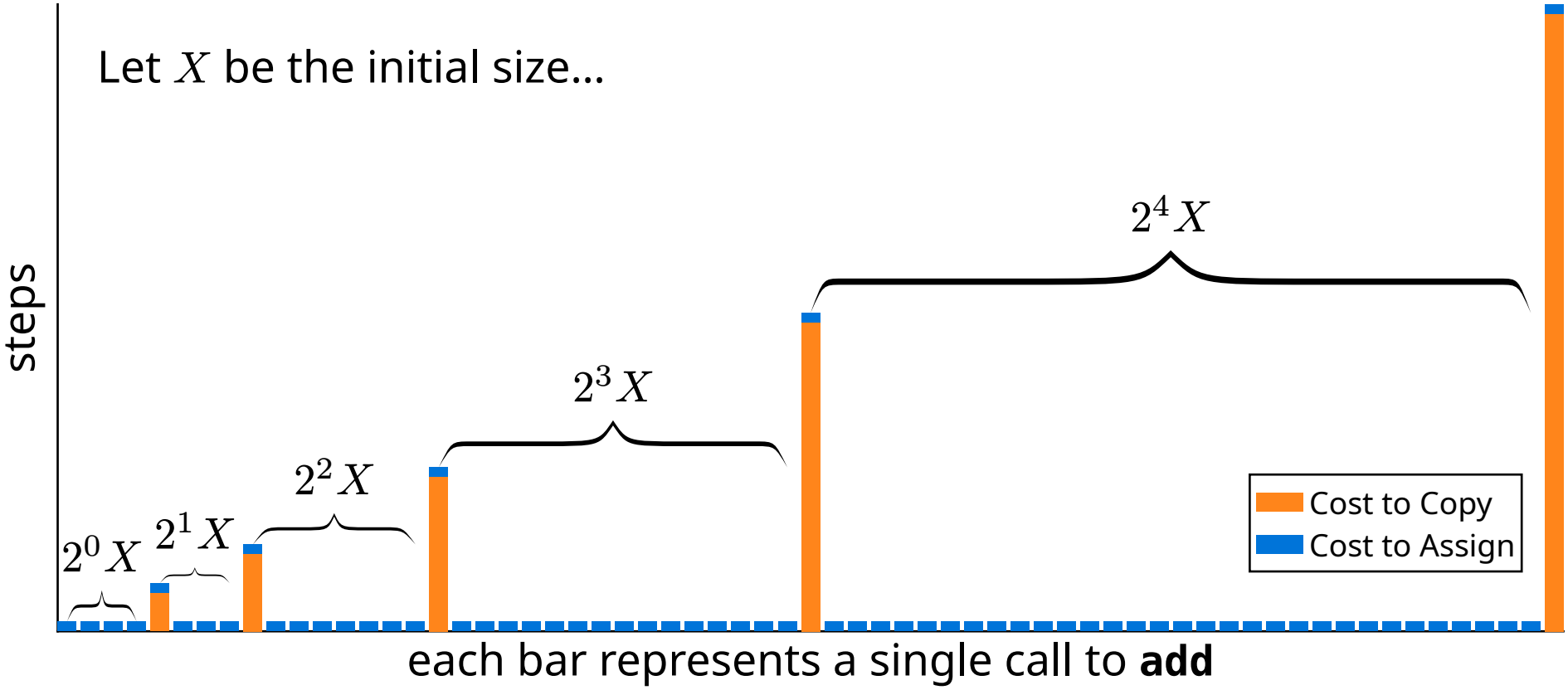

Analysis

Cost of the i^{th} region of **add** calls: $2^i X + 2^i X \in \Theta(2^i)$

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$$\begin{aligned}\sum_{i=0}^{\log(n)} 2^i &= 2^{\log(n)+1} - 1 \\ &= 2 \cdot 2^{\log(n)} - 1 \\ &= 2n - 1 \in \Theta(n)\end{aligned}$$

Another Perspective

Imagine a completely full **`ArrayList`** with 16 elements

What do the next 16 calls to `add(e)` look like?

1. Allocate a single new array of size 32	1 step
2. Copy 16 elements to the new array	16 steps
3. Insert 16 elements into the open spaces	16 steps

Total: $2 \cdot 16 + 1$ steps

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Imagine a completely full **`ArrayList`** with 32 elements

What do the next 32 calls to `add(e)` look like?

1. Allocate a single new array of size 64	1 step
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Total: $2 \cdot 32 + 1$ steps

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Imagine a completely full **`ArrayList`** with 64 elements

What do the next 64 calls to `add(e)` look like?

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2. Copy 64 elements to the new array	64 steps
3. Insert 64 elements into the open spaces	64 steps

Total: $2 \cdot 64 + 1$ steps

Another Perspective

Imagine a completely full **`ArrayList`** with n elements

What do the next n calls to `add(e)` look like?

1. Allocate a single new array of size $2n$	1 step
2. Copy n elements to the new array	n steps
3. Insert n elements into the open spaces	n steps

Total: $2 \cdot n + 1$ steps

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Runtime of 1 call to **add**: $\Theta(1)$

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`size()`	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
`add(v)`	$O(n)$, Amortized $\Theta(1)$	$\Theta(1)$	$\Theta(1)$
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They also sell a reusable cup for **$3.50**, and filling it only costs **$0.50**

How much does it cost to buy coffee for a week?

- Without the reusable cup? **$7**
- With the reusable cup? **$7**

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*The **amortized cost** of buying coffee with the reusable up still ends up being $1 per cup, even though you spend a lot more on the first day ($4).*

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These ideas can also be applied to choosing between ***`LinkedList`*** *and* ***`ArrayList`*** *for appending*

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...but is that bound tight? We don't know!

When all you have is a tight upper bound on unqualified runtime, the shortcut of $(\#\ \text{iterations} \times \text{cost})$ is not guaranteed to give a tight bound!

You would need to do further analysis

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...and by definition this is a tight bound!

Knowing the amortized runtime makes analyzing loops easy!

Just use the shortcut $(\#\text{iterations} \times \text{amortized cost})$