CSE 331: Algorithms & Complexity

"BFS"

Prof. Charlie Anne Carlson (She/Her)

Lecture 11

Monday September 22nd, 2025

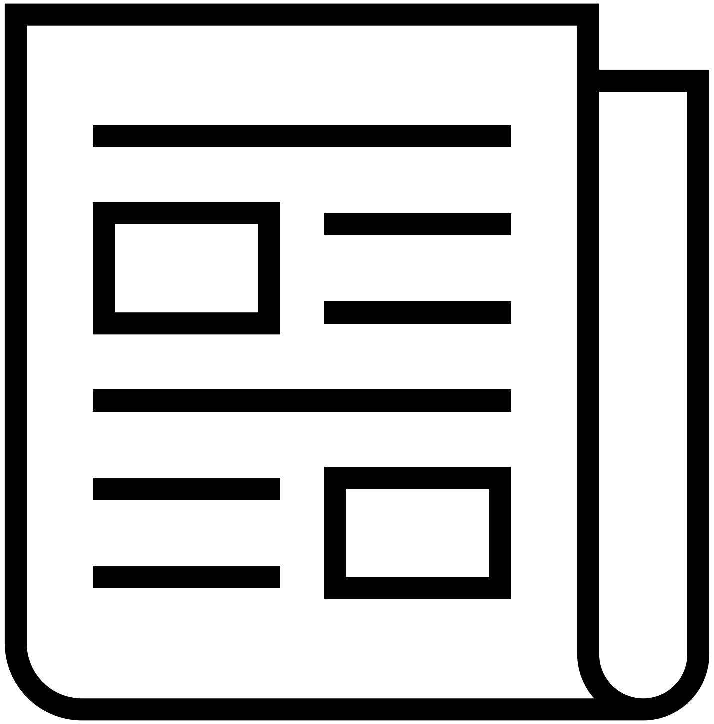
University at Buffalo

Schedule

1. Course Updates
2. Graph Connectivity
3. Graph Traversal
   1. BFS
   2. DFS
   3. WFS

Course Updates

- HW 1 Grading Out Tomorrow
- HW 2 Due Tomorrow
- HW 3 Out Tomorrow
- Group Project
  - Team Emails Soon
  - No Autolab Registration
- First Quiz NEXT Monday!



Connectivity Problem(s)

s-t Connectivity Problem:

Input: Graph $G = (V, E)$, source s, and destination t.

Output: True if there exists a path and False otherwise.

s-t Shortest Path Problem:

Input: Graph $G = (V, E)$, source s, and destination t.

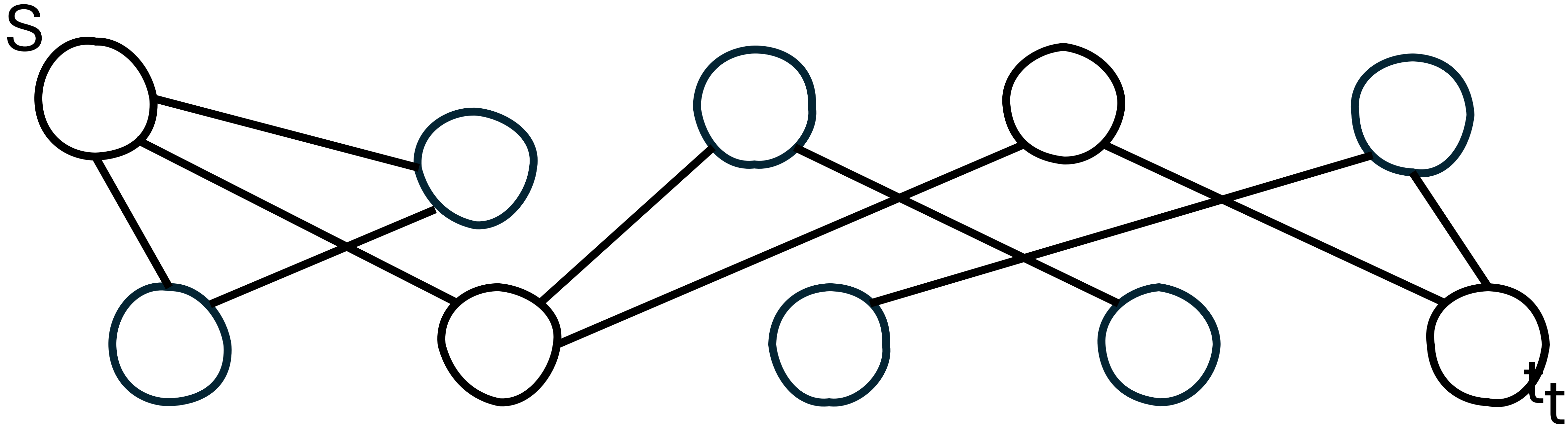
Output: The length of the shortest path from s to t (∞ if there is no path).

Another Connectivity Problem

Graph Connectivity:

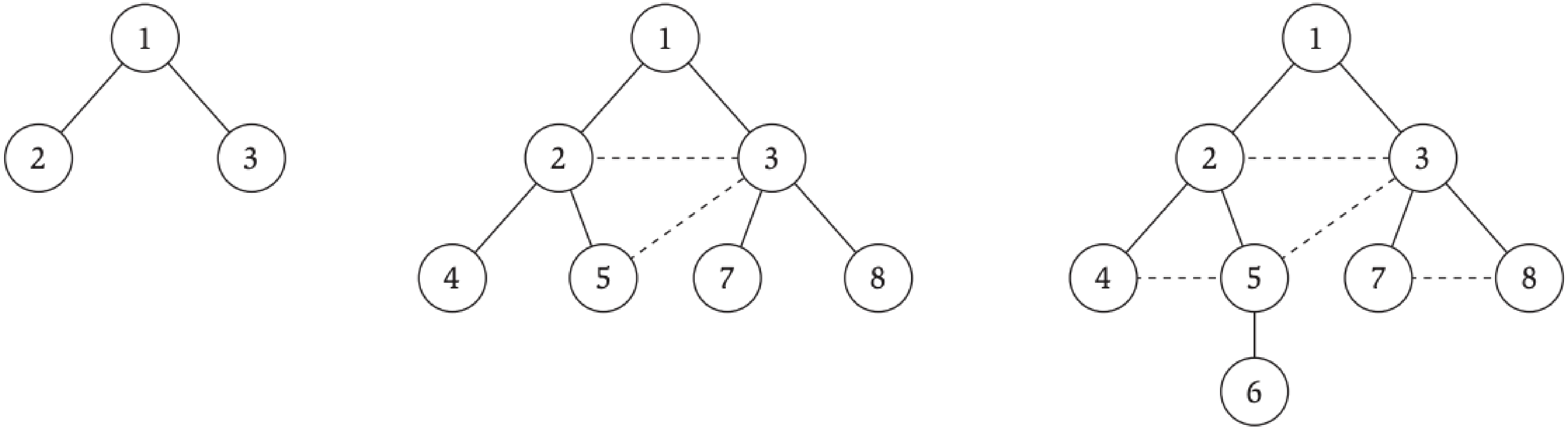
Input: $G = (V, E)$ and $s \in V$

Output: $\mathrm{T} = \{\mathrm{u} \in \mathrm{T}\text{: there exists a path from s to u in G}\}$



Breadth First Search (Layers)

- $L_0 = s$
- $L_1 =$ neighbors of L_0.
- $L_2 =$ neighbors of L_1 that are not in L_0.
- $L_\mathrm{i} =$ neighbors of $L_{\mathrm{i}-1}$ that are not in previous layer.



Breadth First Search ("Algorithm")

- Input: $G = (V, E)$ and $s \in V$
- Let $L_0 = \{s\}$
- Assume $L_0, \ldots, L_i$ have been constructed:
  - Let $L_{\mathrm{i}+1}$ be nodes do not appear in $L_0, \ldots, L_i$ and have an edge to L_i.
  - If $L_{\mathrm{i}+1}$ is empty, stop.
- Return all layers.

Breadth First Search (Properties)

- For each j ≥ 0, layer L_j produced by BFS consists of all nodes at distance j from s.
- There is a path from s to t if and only if t appears in some layer.
- For any $\{\mathrm{u}, \mathrm{v}\} \in \mathrm{E}$, if $u \in L_i$ and $v \in L_j$ then i and j differ by at most 1.
- You can think of the output as a tree! We call this the **BFS (discovery) Tree**.

Breadth First Search (Properties)

Claim: For any $\{\mathrm{u}, \mathrm{v}\} \in \mathrm{E}$, if $u \in L_i$ and $v \in L_j$ then i and j differ by at most 1.

Breadth First Search (Properties)

Claim: For any $\{u, v\} \in E$, if $u \in L_i$ and $v \in L_j$ then i and j differ by at most 1.

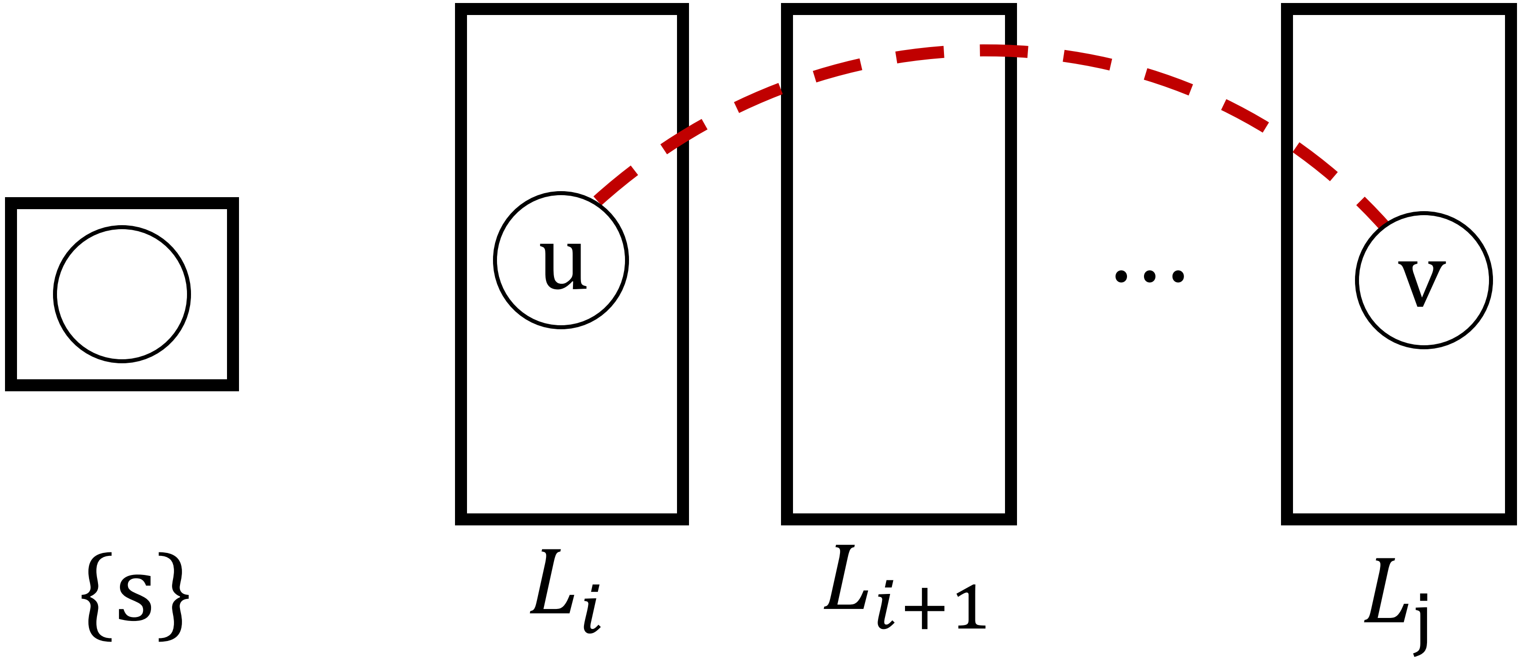
Proof Outline:

- Without loss of generality, we assume that $i \leq j$.
- Suppose for the sake of contradiction that $d(i, j) > 1$.

Breadth First Search (Properties)

Proof Outline:

- Without loss of generality, we assume that $i < j$.
- Suppose for the sake of contradiction that $d(i, j) > 1$.



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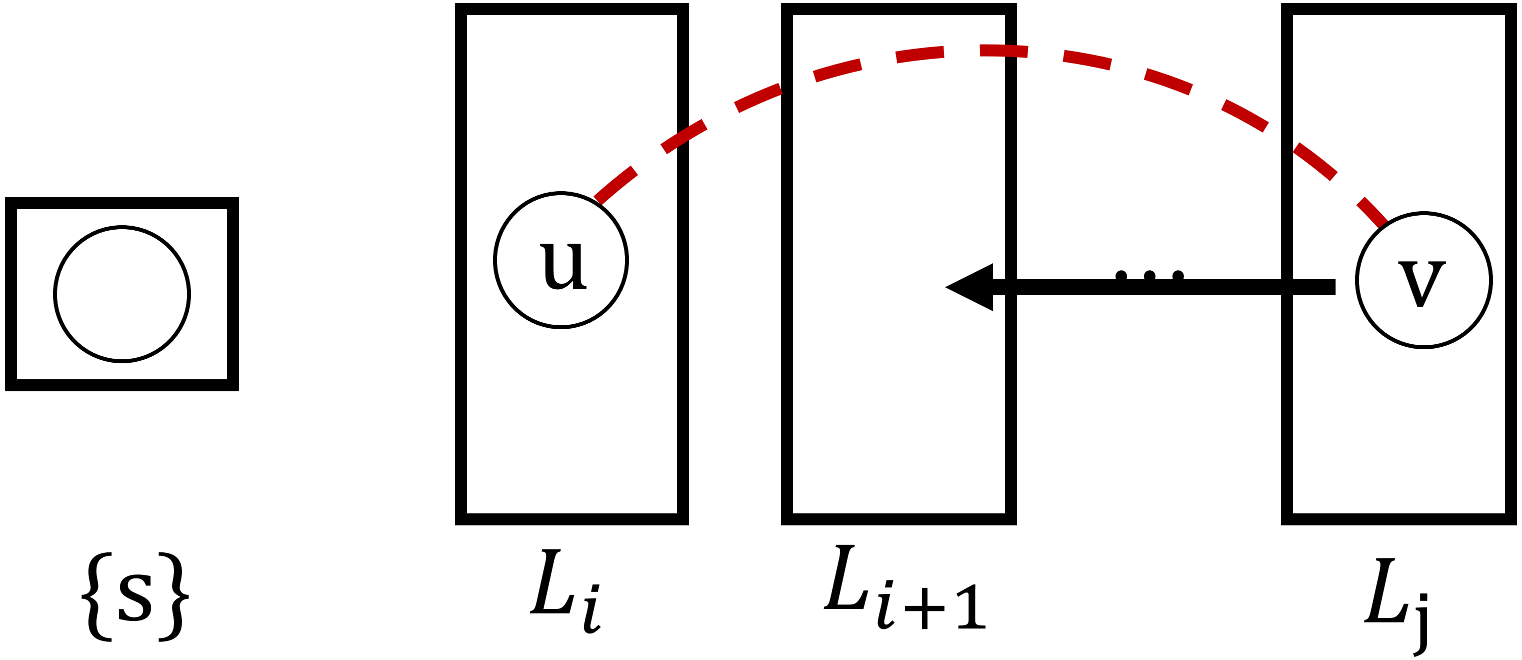
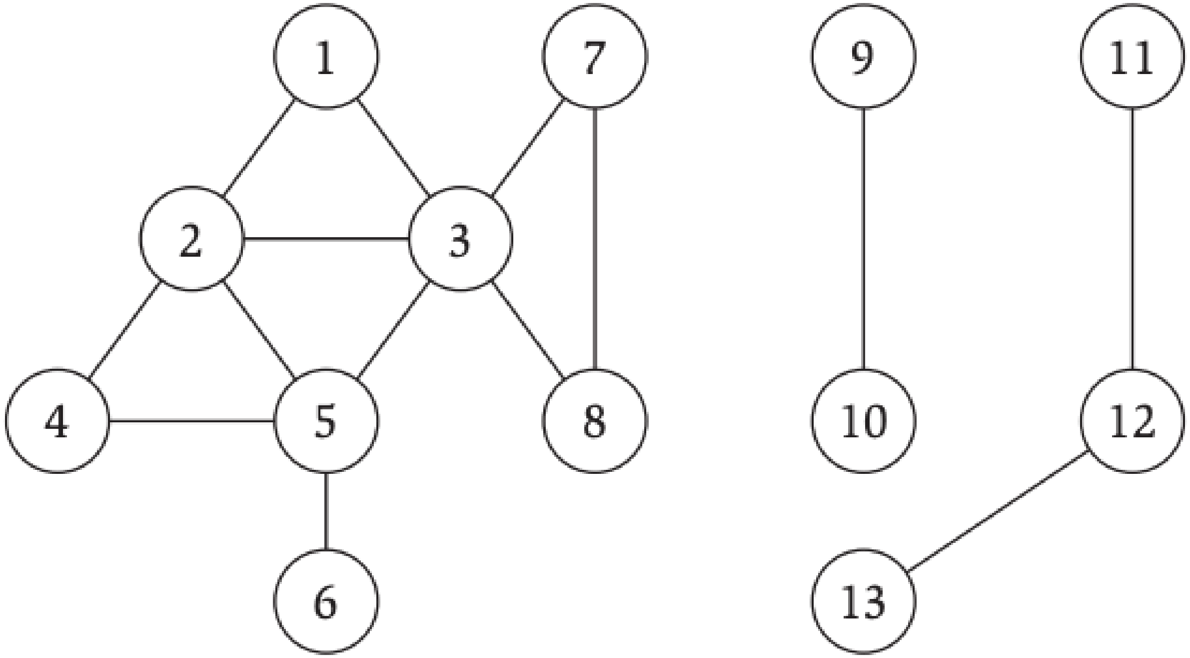


Breadth First Search (Properties)

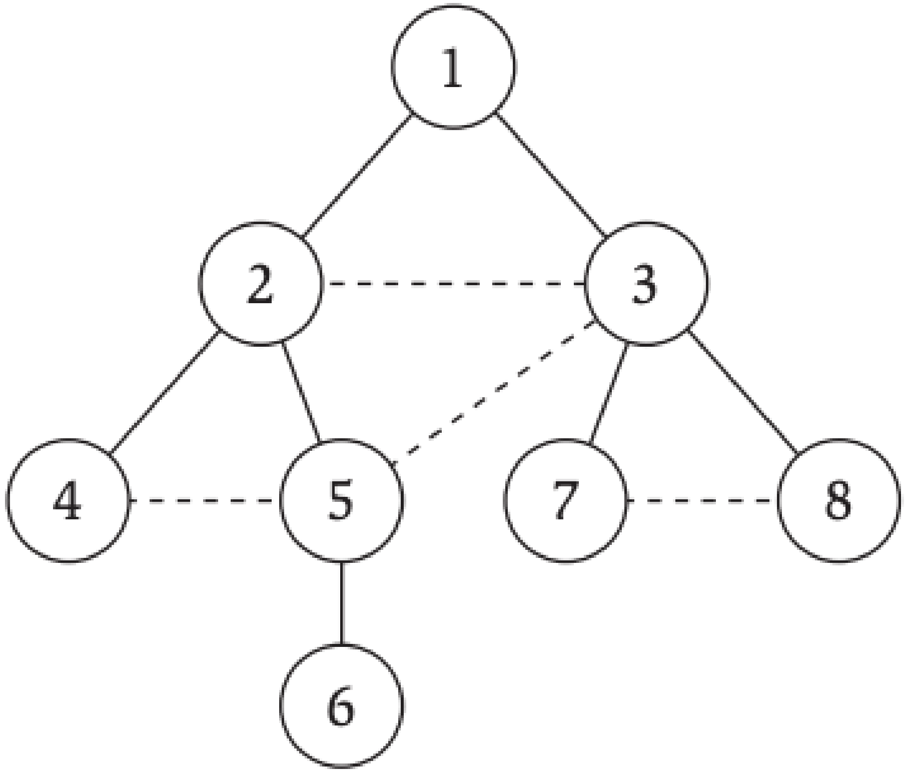
Proof:

- Without loss of generality, we assume that $\mathrm{i} < \mathrm{j}$.
- Suppose for the sake of contradiction that $\mathrm{d}(\mathrm{i}, \mathrm{j}) > 1$.
  - This implies $\mathrm{i} + 1 < \mathrm{j}$.
- Then $u \in L_i, v \notin L_0, L_1, \dots, L_i$, and $\{\mathrm{u}, \mathrm{v}\} \in \mathrm{E}$.
  - Hence, BFS would have put $v \in L_{i+1}$.
  - This contradicts initial assumption that $v \in L_{\mathrm{j}}$ and $\mathrm{j} > \mathrm{i} + 1$.

Breadth First Search (Tree)

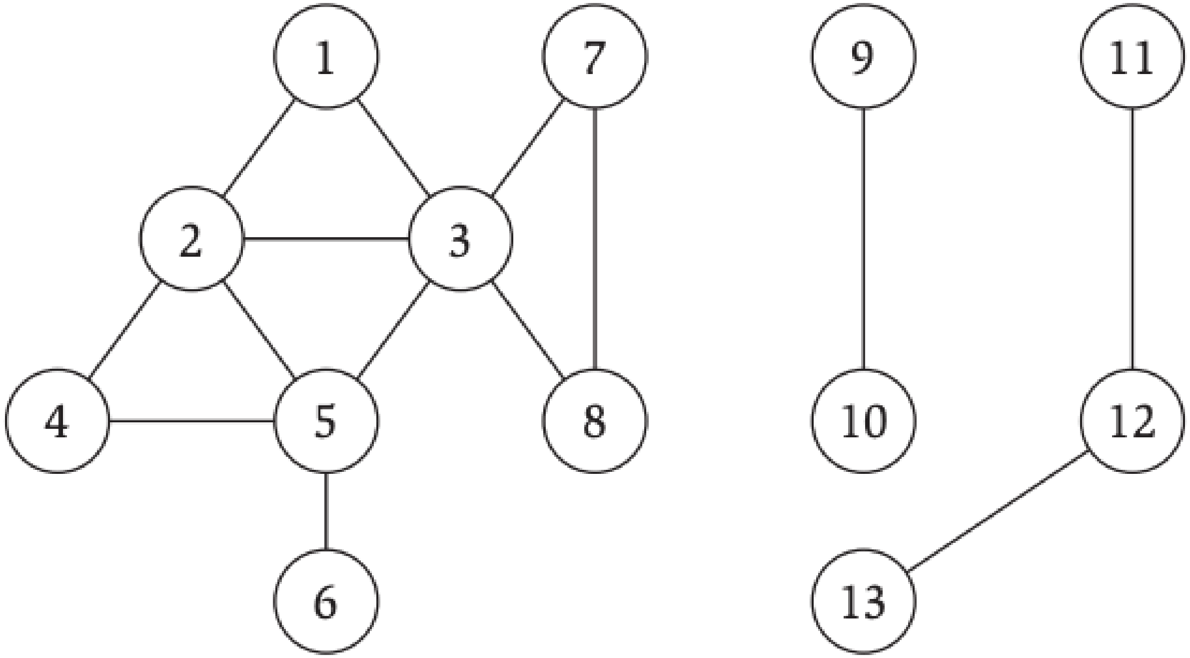


Original Graph



Search Tree

Breadth First Search (Tree)



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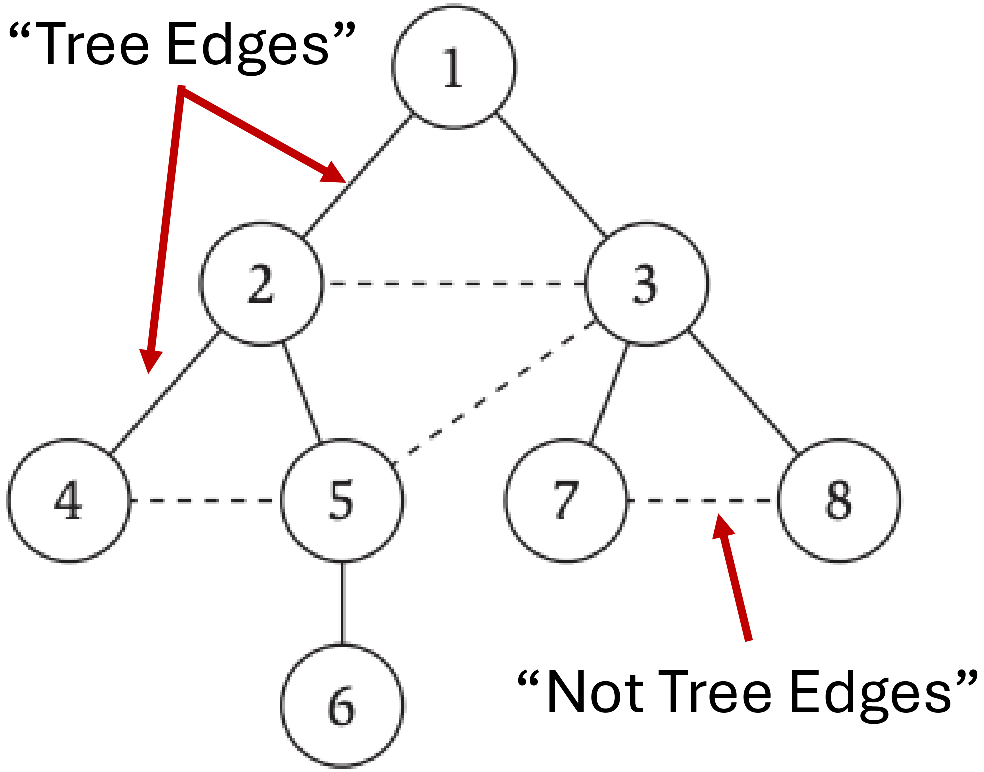
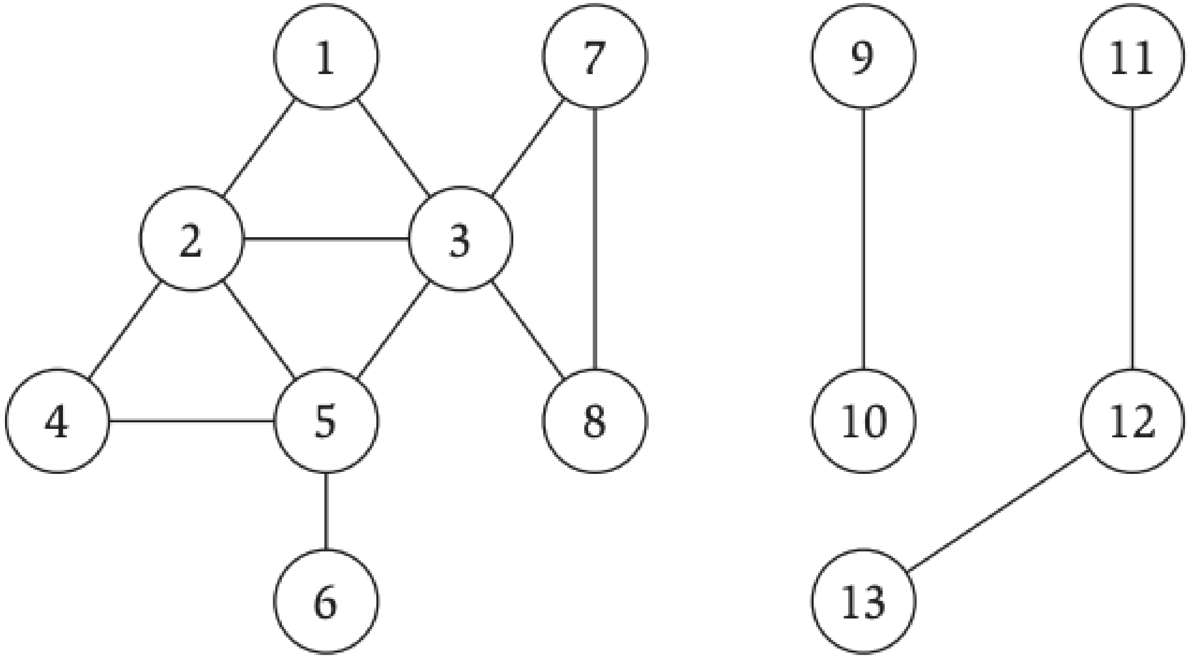


Search Tree

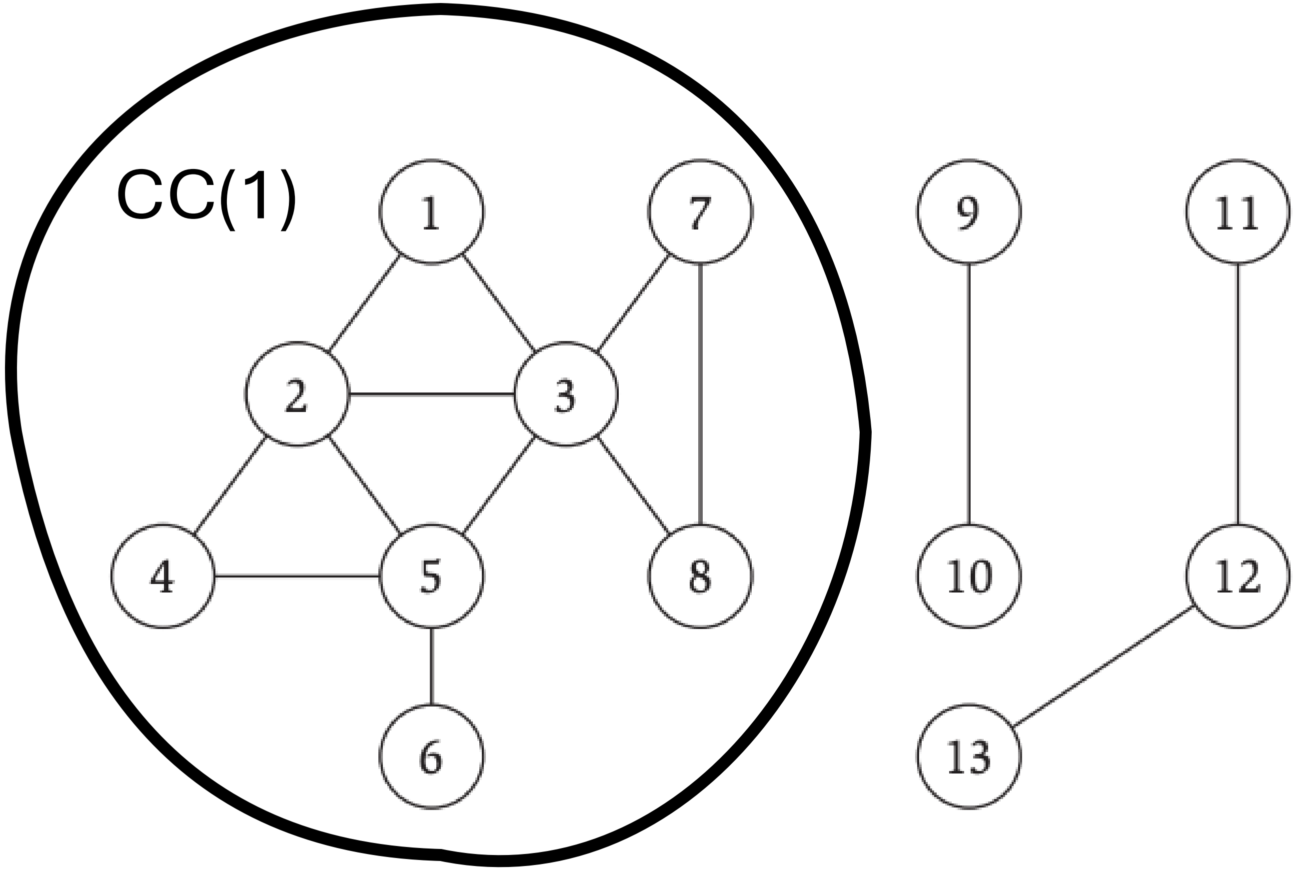
Connected Component of (s) (CC(s))



The CC(s) is the set of all vertices that are connected to s by a path.

"The set of vertices that you can reach from s using a simple path."

Connected Component of (s) (CC(s))



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Explore Algorithm

- Input: $G = (V, E)$ and $s \in V$
- Output: CC(s)
- Let $\mathrm{R} = \{\mathrm{s}\}$
- While there exists $\{\mathrm{u}, \mathrm{v}\} \in \mathrm{E}$ such that $\mathrm{u} \in \mathrm{R}$ and $v \notin R$:
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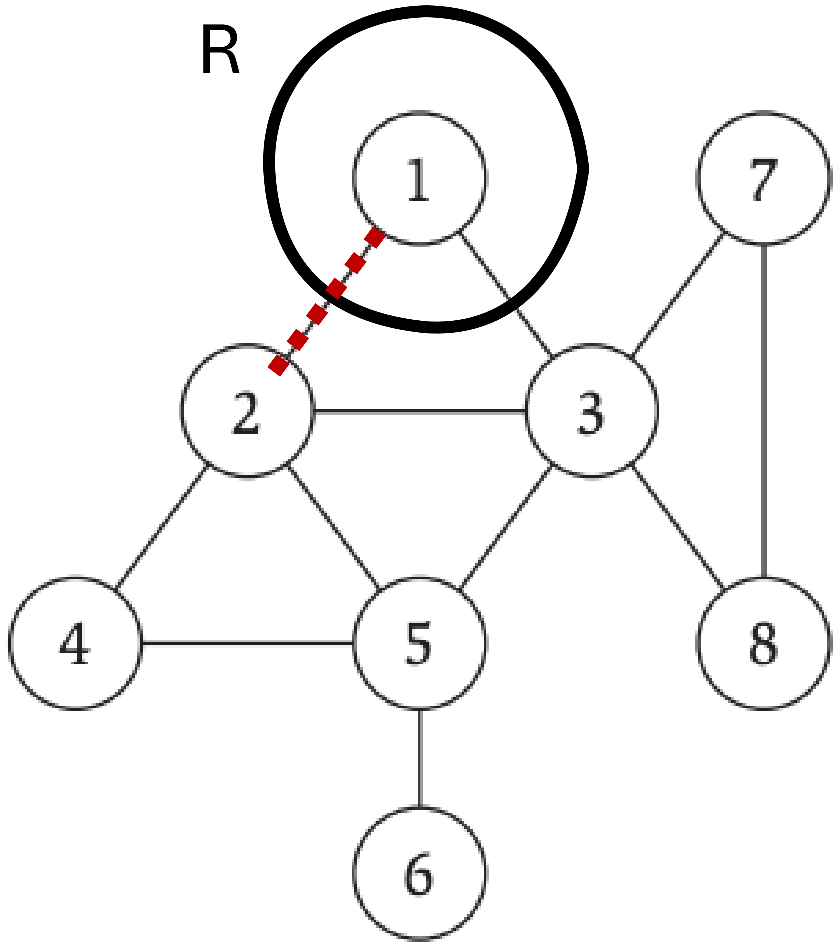

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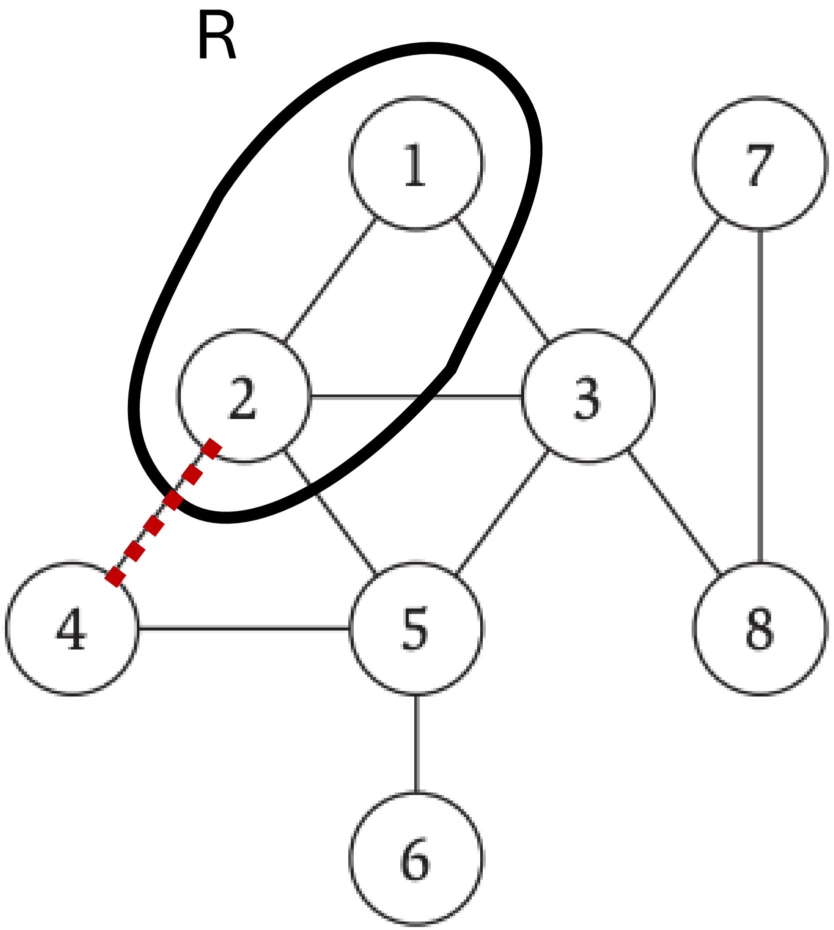

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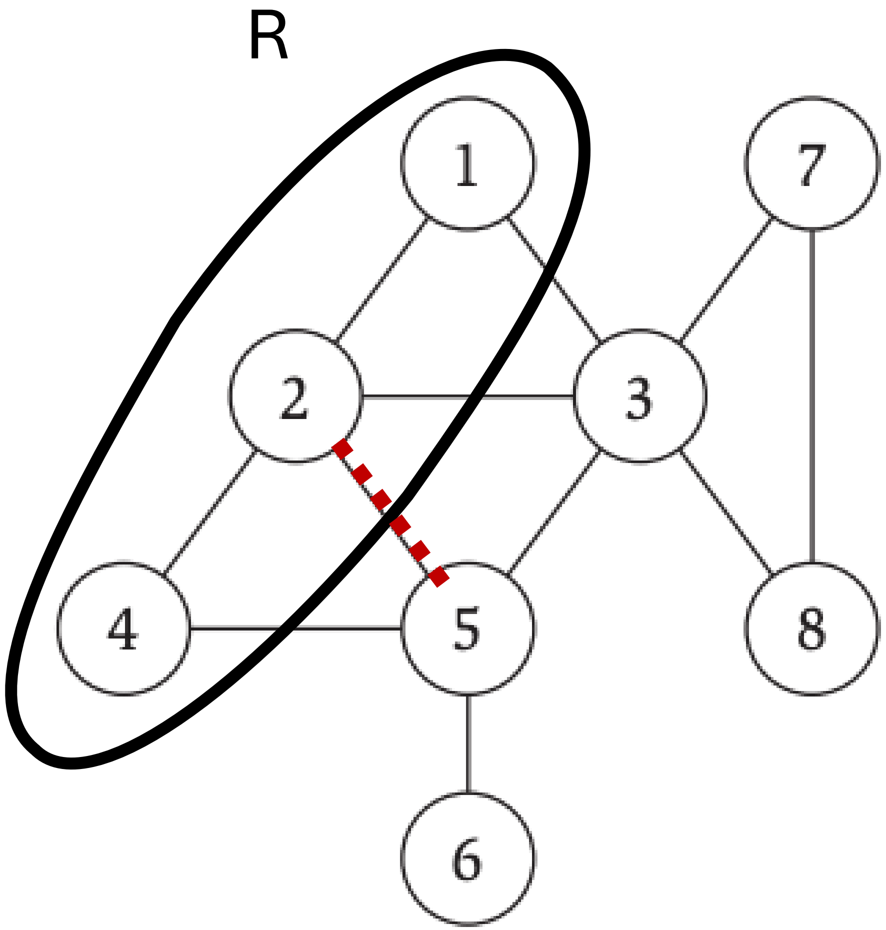

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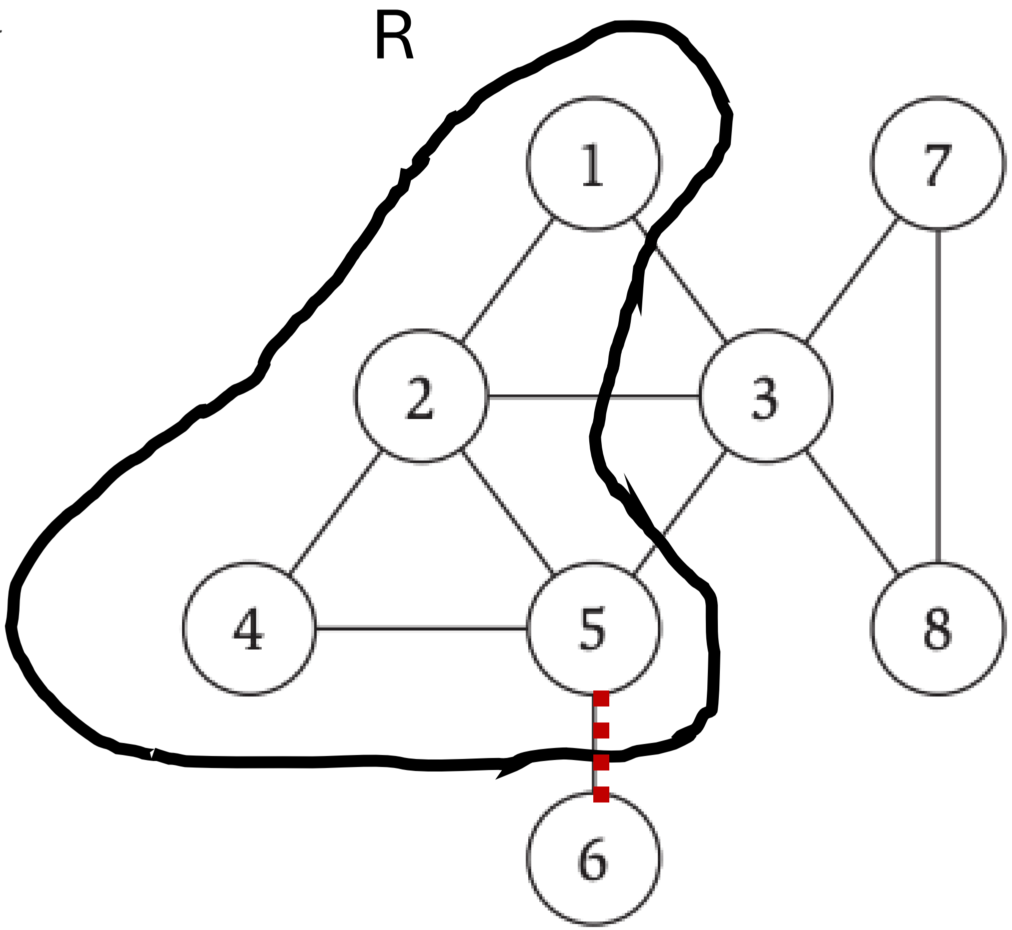

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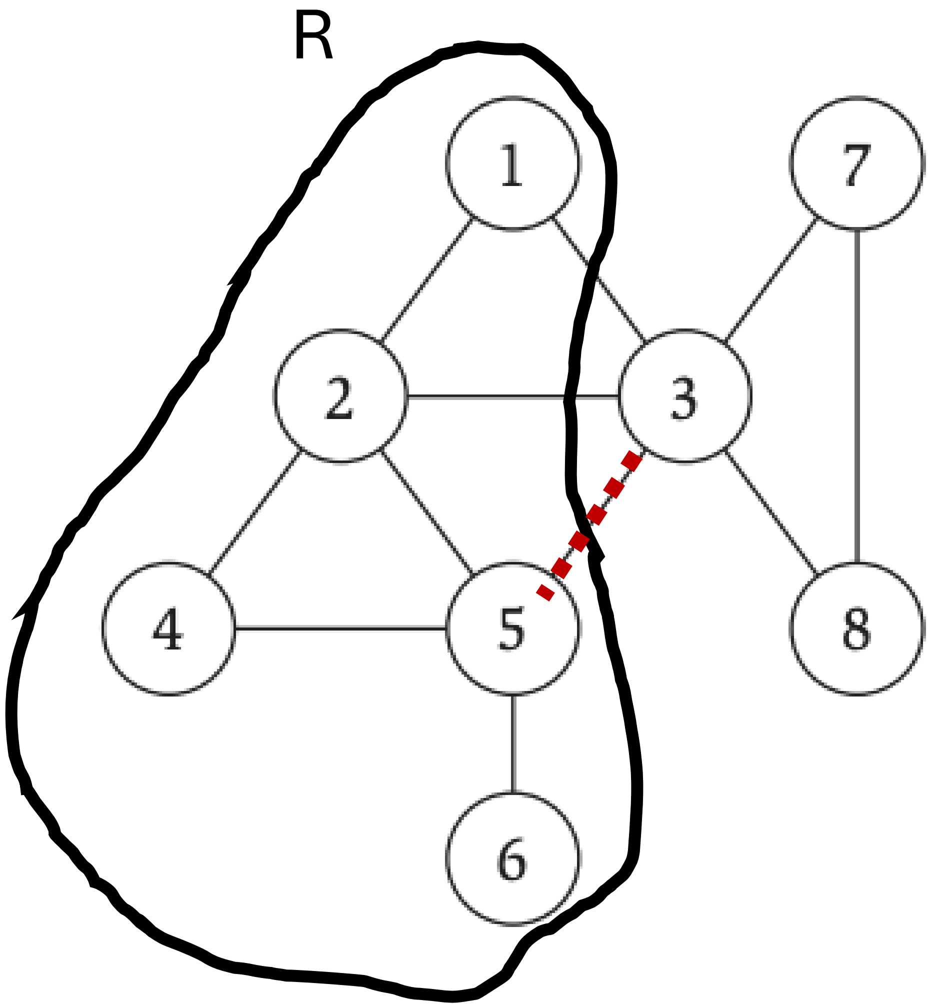

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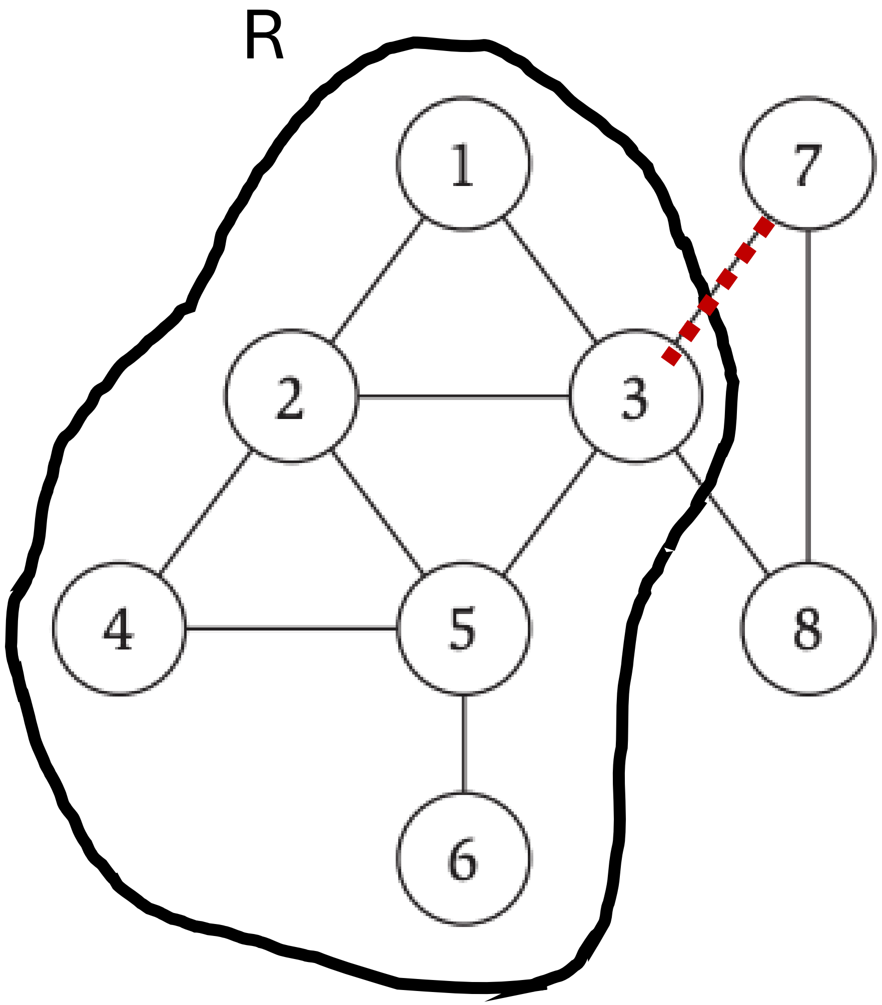

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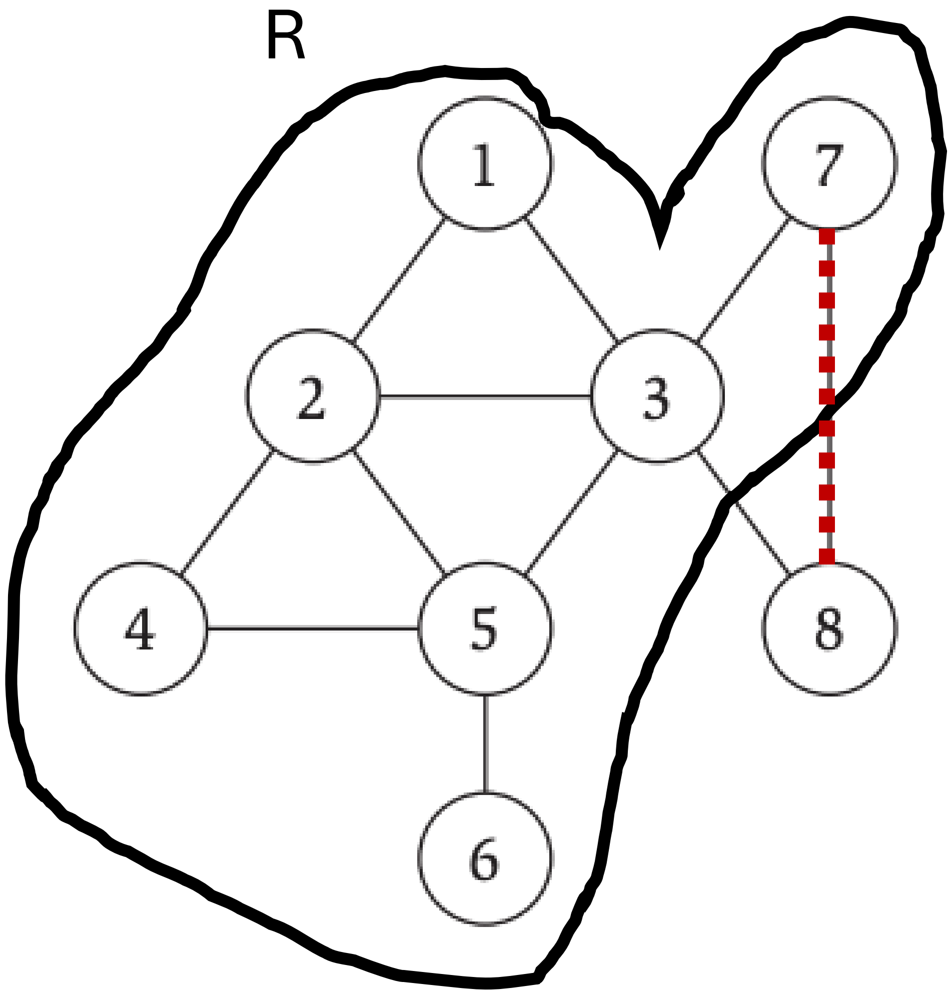

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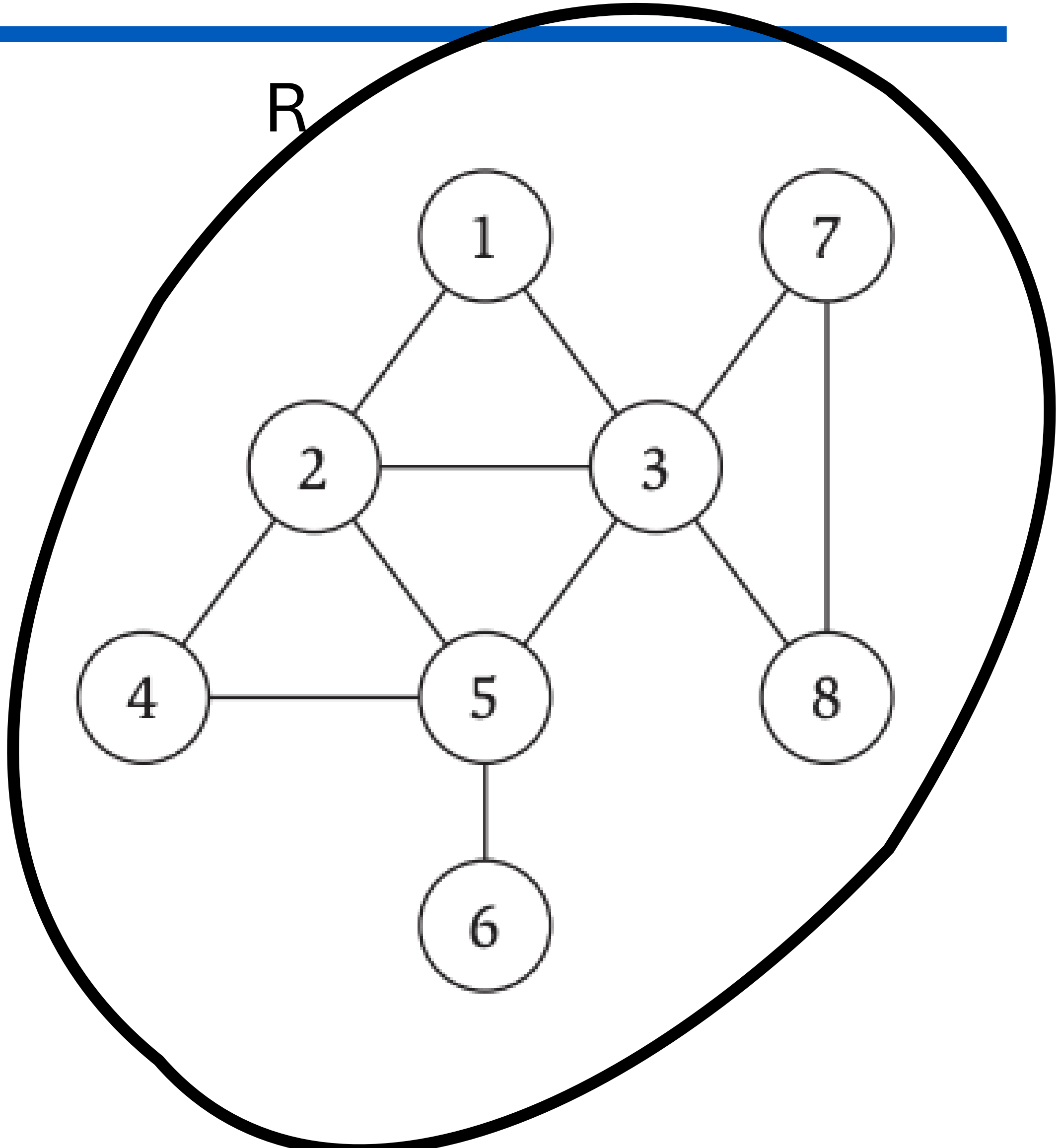

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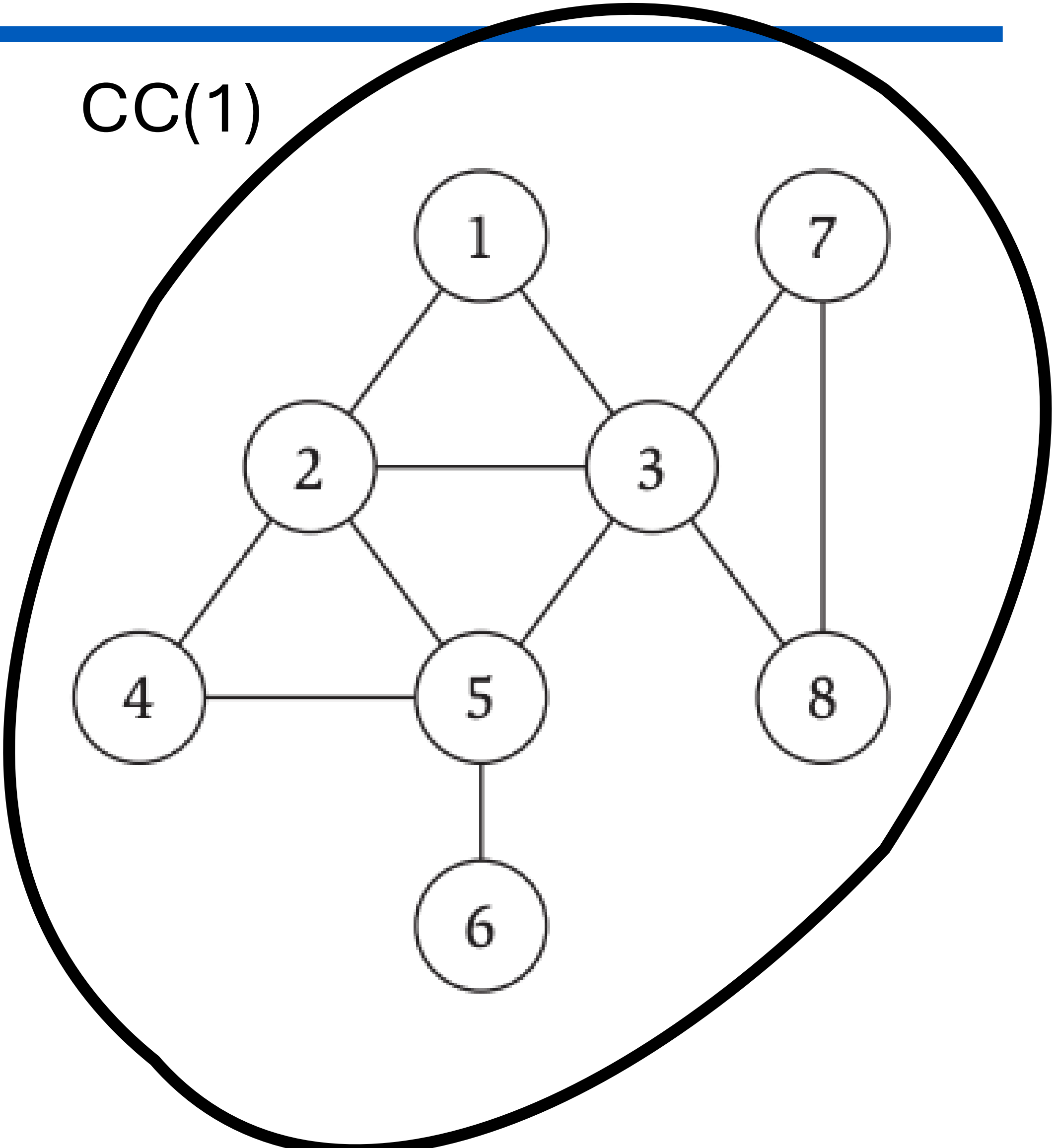

Q: What is the difference? (BFS vs Explore)

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BFS is Explore but Explore isn't necessarily BFS!

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Q: How do we show this solves Connectivity?

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  - Q: How do we do this?

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- Argue that R = CC(s)!
  - Show $\mathrm{R} \subseteq \mathrm{CC}(s)$
  - Show $\mathrm{CC}(s) \subseteq \mathrm{R}$

Breadth First Search (Properties)

Claim: $\mathrm{R} \subseteq \mathrm{CC}(s)$

Proof Idea:

- This wants us to show that everything reached by Explore is in the connected component of s.
- Let's do induction on iteration of the algorithm.
  - Do you believe the first iteration.
  - Given any iteration is true, how do you feel about the next iteration?

Q: Does this always terminate?

- Input: $G = (V, E)$ and $s \in V$
- Output: CC(s)
- Let $\mathrm{R} = \{s\}$
- While there exists $\{\mathrm{u}, \mathrm{v}\} \in \mathrm{E}$ such that $\mathrm{u} \in \mathrm{R}$ and $v \notin R$:
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- Well, we must be adding a vertex in each iteration and there are only so many vertices, right?

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Explore Proofs

Claim: $\mathrm{CC}(s) \subseteq \mathrm{R}$

Proof:

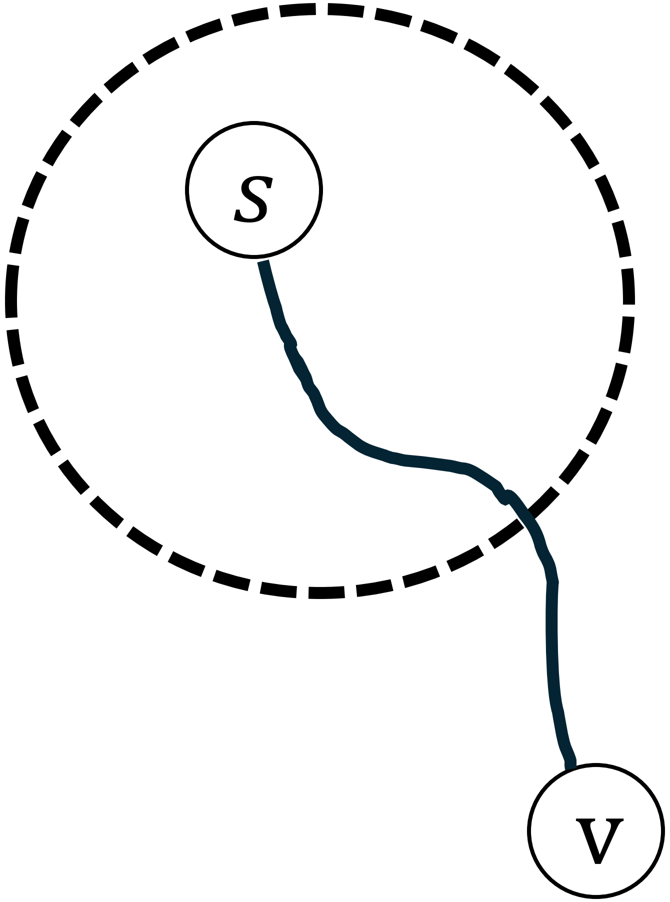
- This saying that every vertex in the connected component is added to R by Explore.
- This is saying that for every vertex v such that there is a path from s to v, v is added to R by Explore.

Explore Proofs

Claim: $\mathrm{CC}(s) \subseteq \mathrm{R}$

Proof:

- Suppose to the contrary that there exists $\mathrm{v} \in \mathrm{CC}(s)$ such that $\mathrm{v} \notin \mathrm{R}$.
  - Then there must exist a path that starts at s (inside R) and ends at v (outside R).

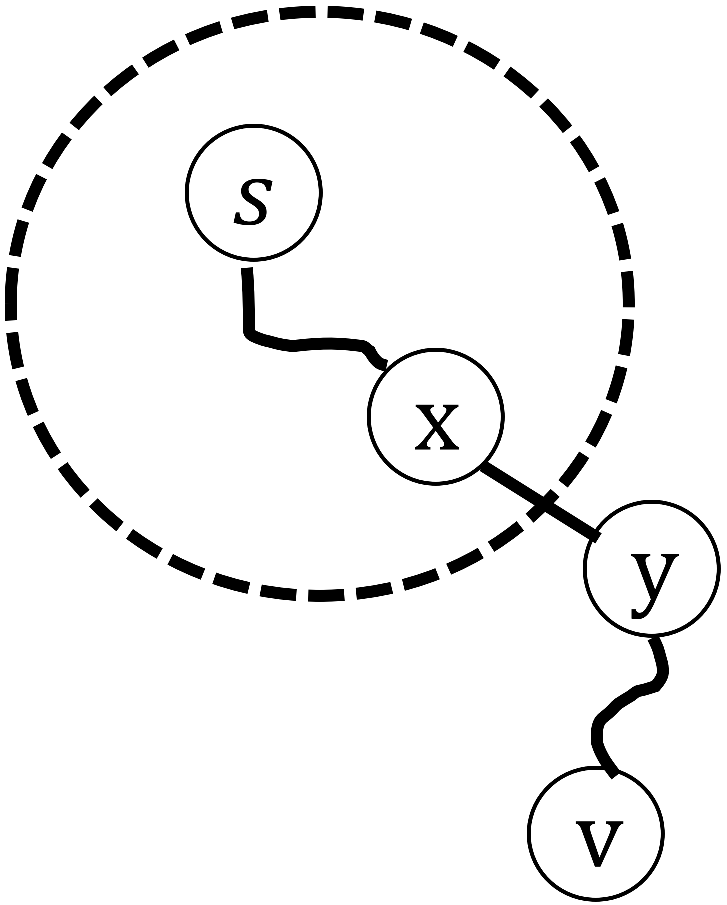


Explore Proofs

Claim: $\mathrm{CC}(s) \subseteq \mathrm{R}$

Proof:

- There then must exist $\{x, y\} \in E$ such that $x \in R$ and $y \notin R$.



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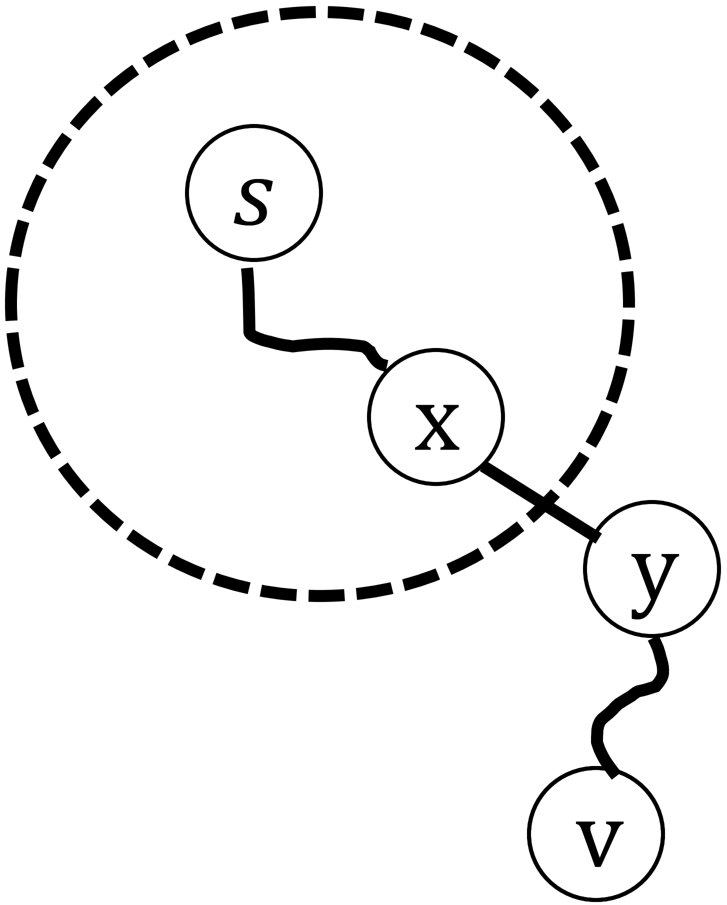
- There then must exist $\{\mathrm{x}, \mathrm{y}\} \in \mathrm{E}$ such that $\mathrm{x} \in \mathrm{R}$ and $\mathrm{y} \notin \mathrm{R}$.
- If this the case, then the algorithm wouldn’t have terminated and would have instead added y. =><=



Q: How would you describe BFS?

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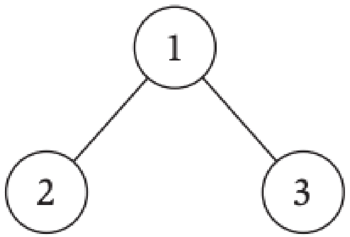

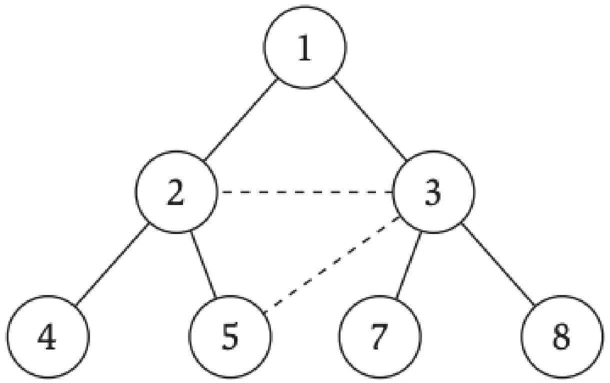

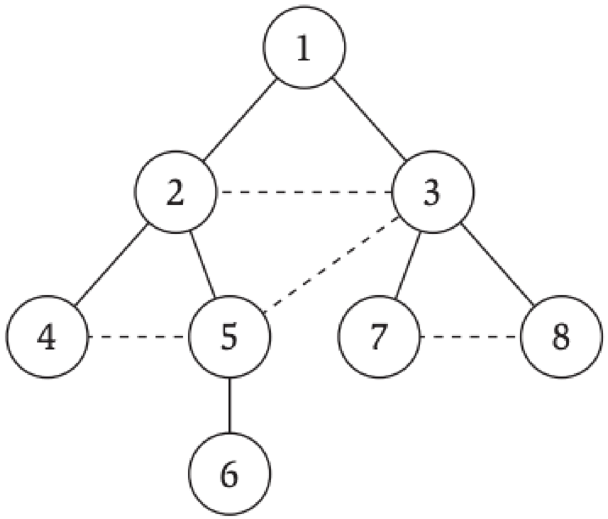

Depth First Search

- **Input**: The current vertex $\mathrm{u} \ \in V$
- **Global**: An array of exploration $A \in \{0,1\}^V$
- Mark current vertex as explored ($\mathrm{A[u]} \ = \ 1$).
- For each $\{\mathrm{u}, \mathrm{v}\} \ \in \mathrm{E}$:
  - If v is not explored ($\mathrm{A[v]} \ == \ 0$):
    - DFS(v,A)

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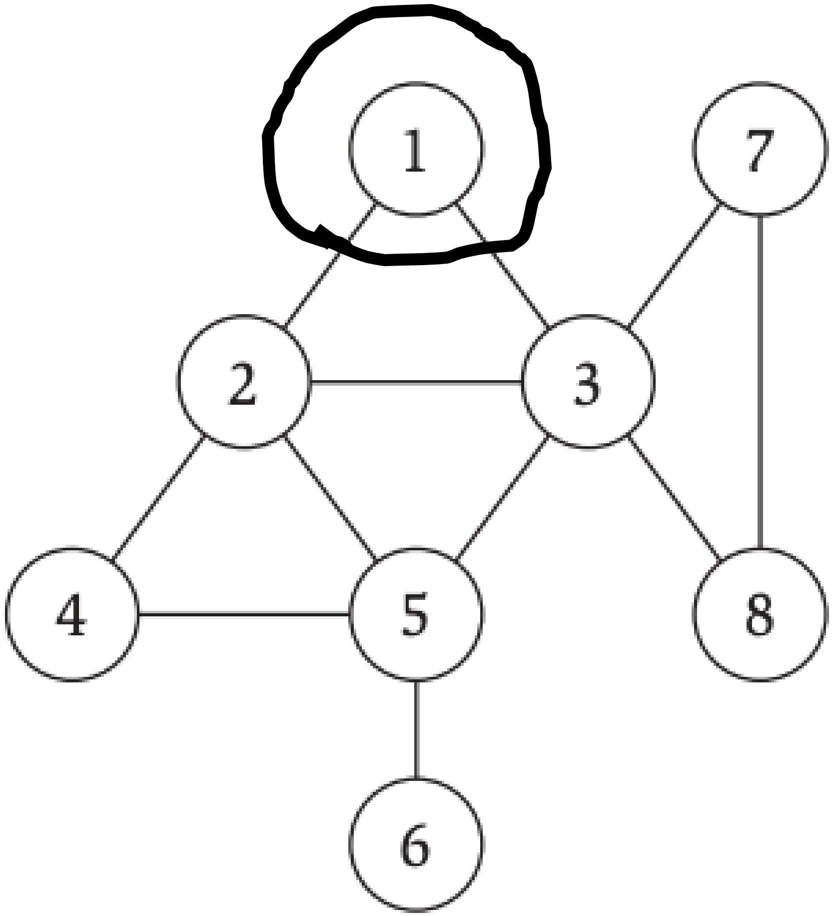
- **Idea**: You are recursing or “drilling down”. If you get stuck, you go up a step and try the next choice.

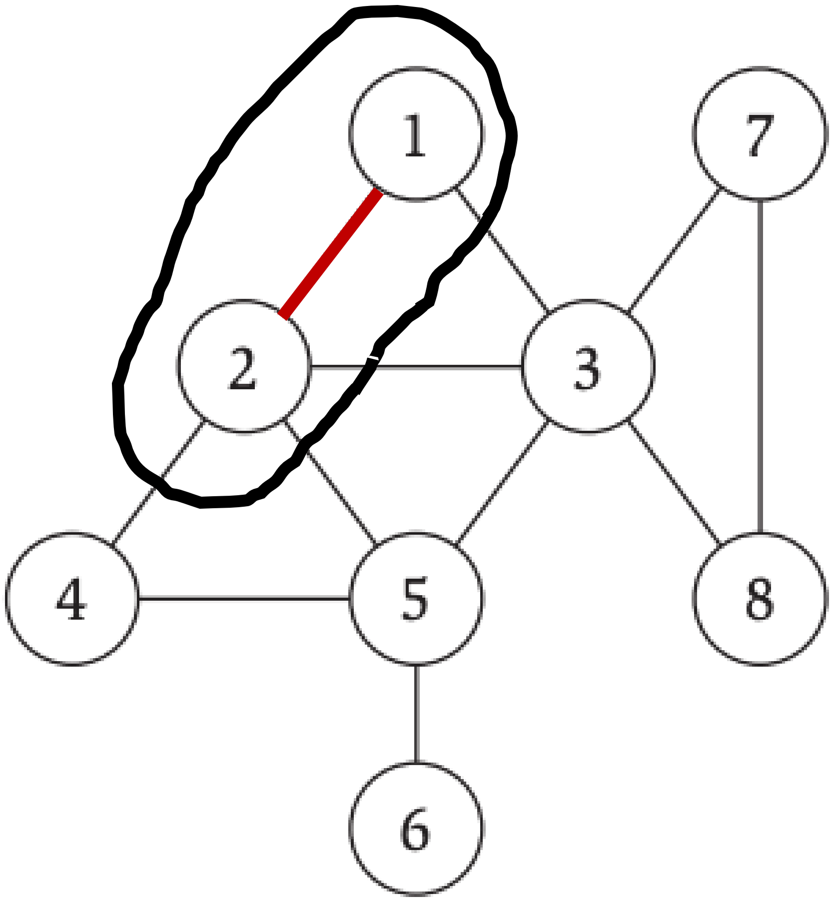
Depth First Search

A = {1}



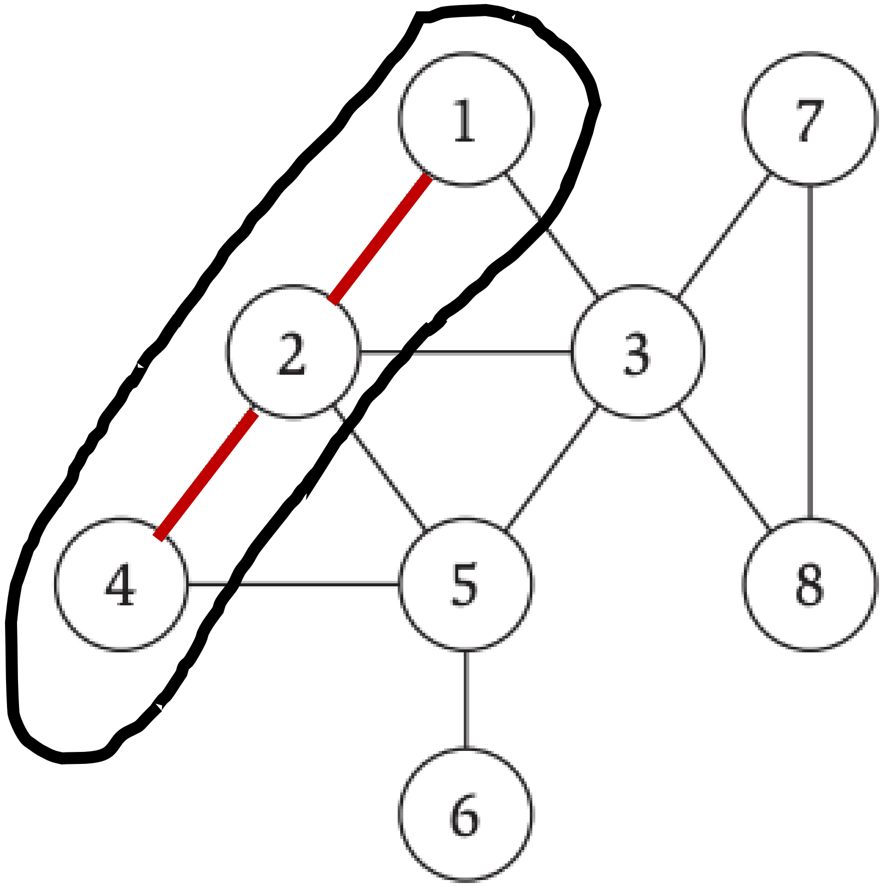
Depth First Search

A = {1,2}



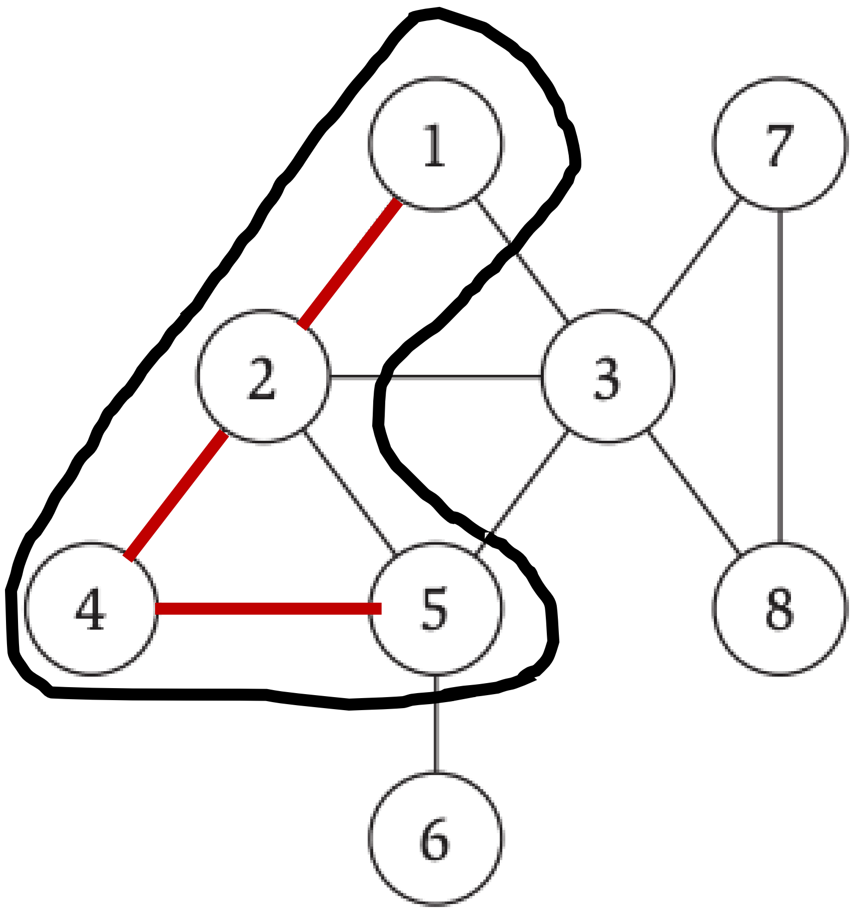
Depth First Search

A = {1,2,4}



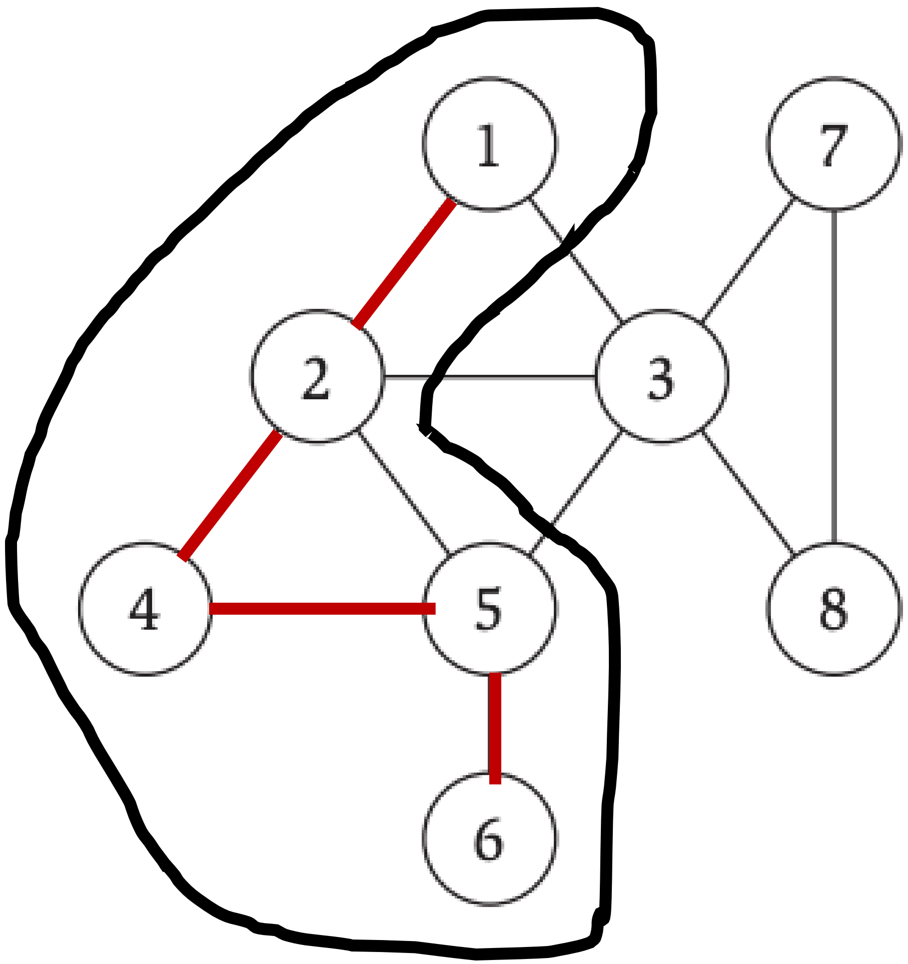
Depth First Search

A = {1,2,4,5}



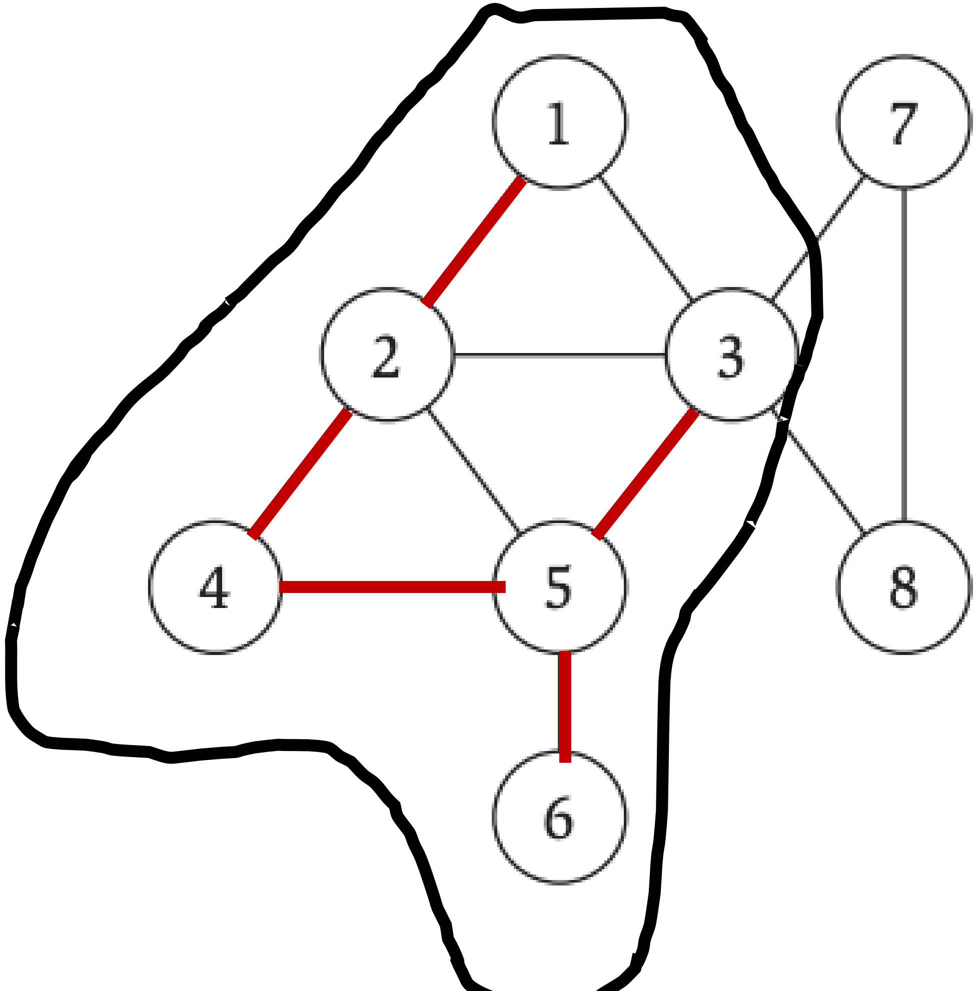
Depth First Search

A = {1,2,4,5,6}



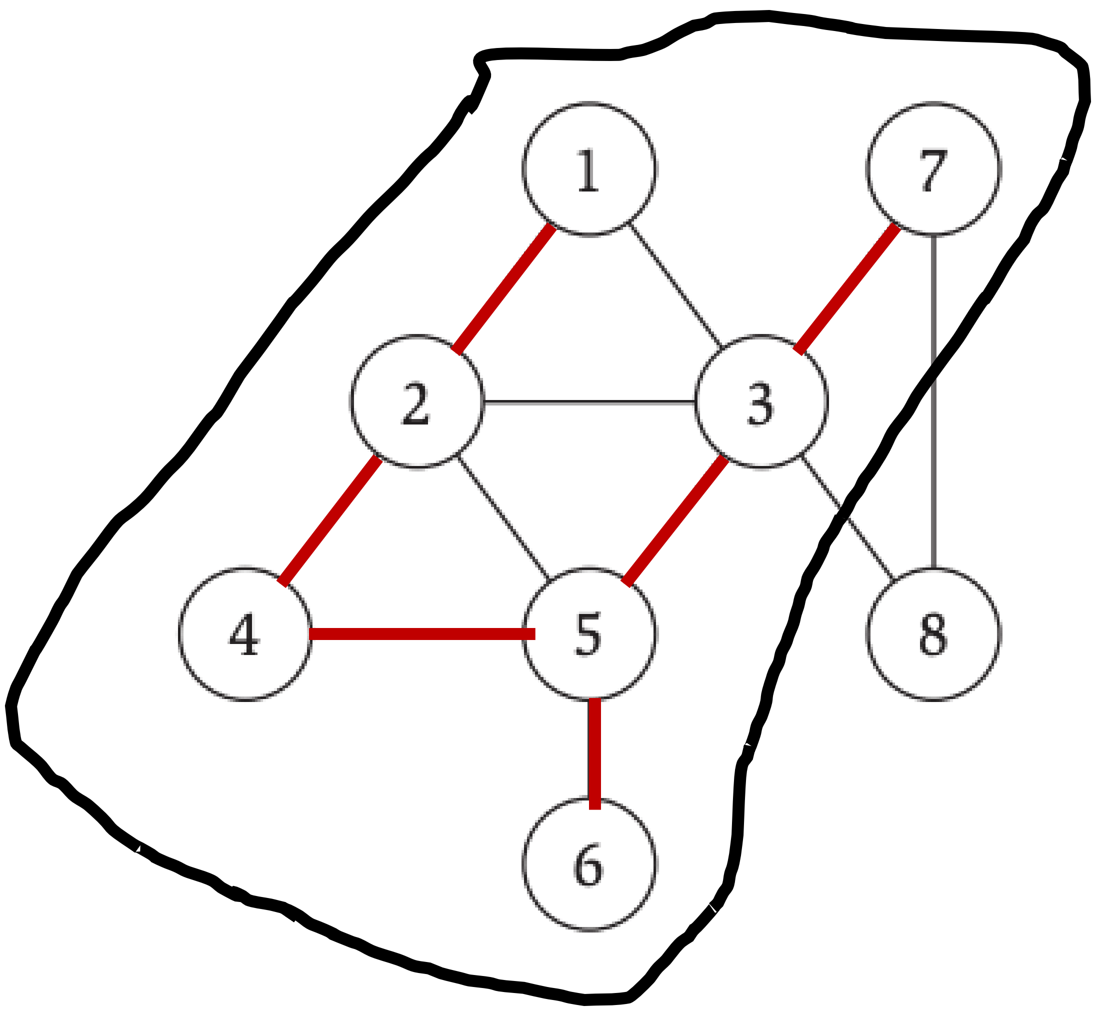
Depth First Search

A = {1,2,4,5,6,3}



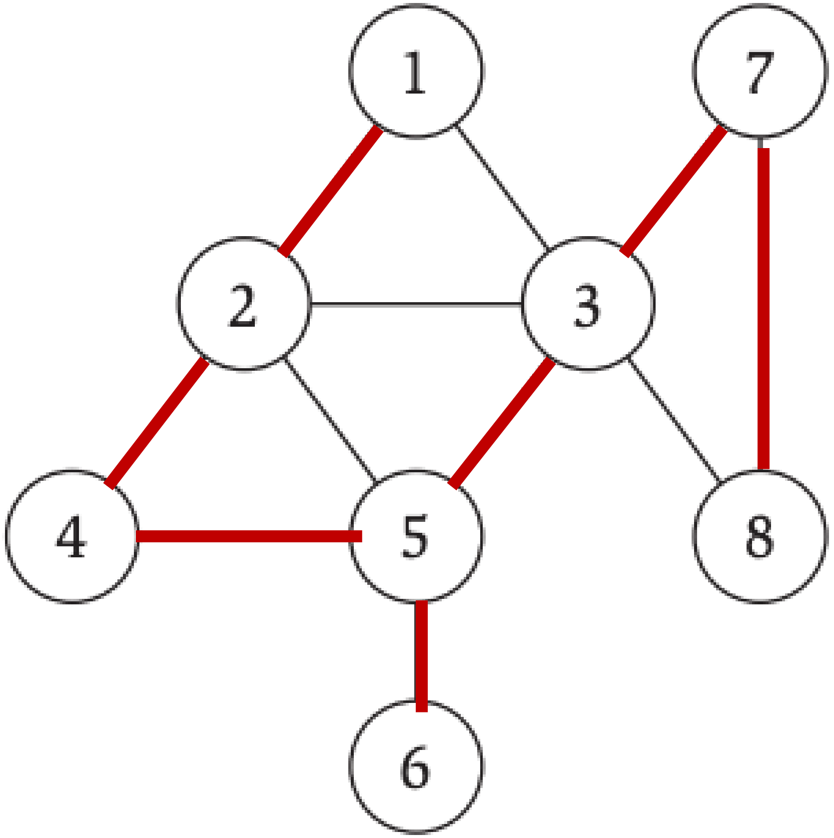
Depth First Search

A = {1,2,4,5,6,3,7}

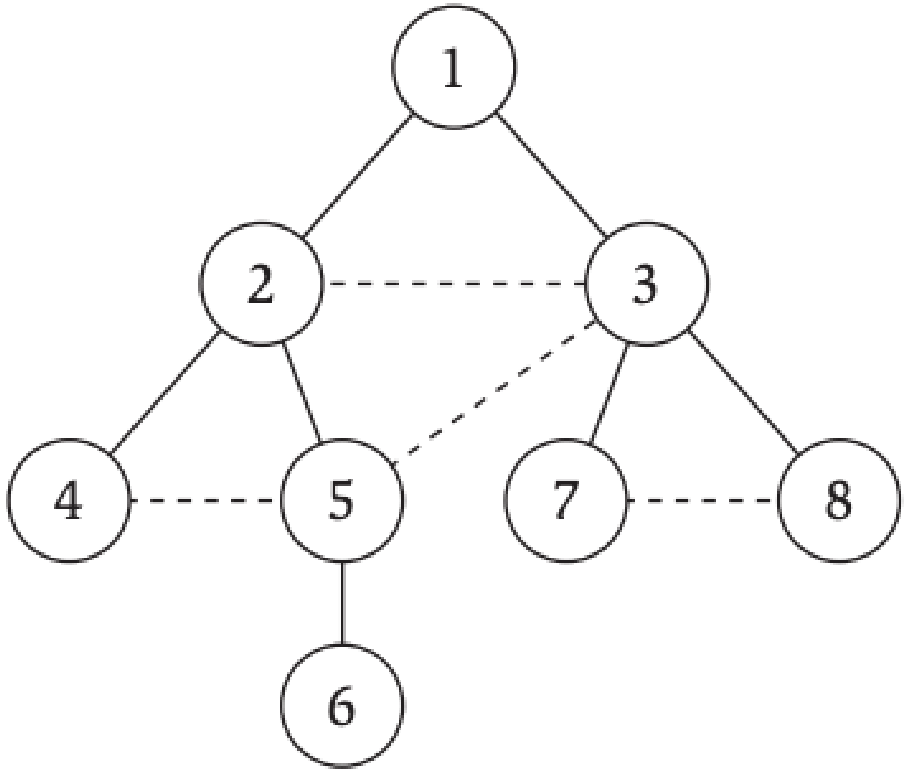


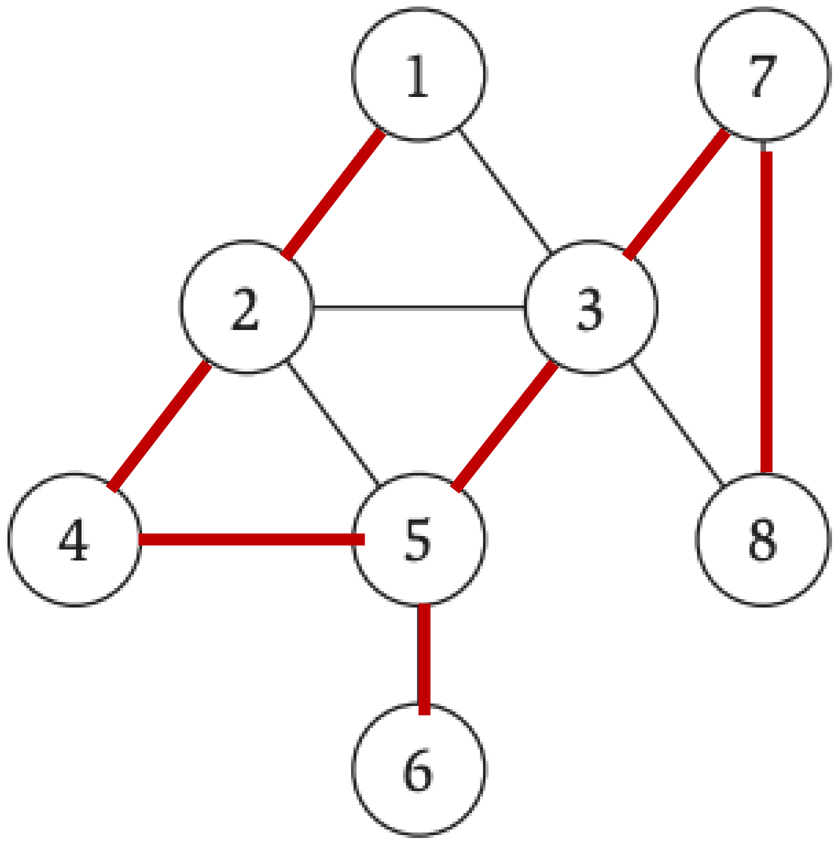
Depth First Search

A = {1,2,4,5,6,3,7,8}



DFS Trees vs BFS Trees





Q: How can you compute all the connected components of a graph?