CSE 331: Algorithms & Complexity

"Greedy Algorithms"

Prof. Charlie Anne Carlson (She/Her)

Lecture 15

Wednesday October 1st, 2025

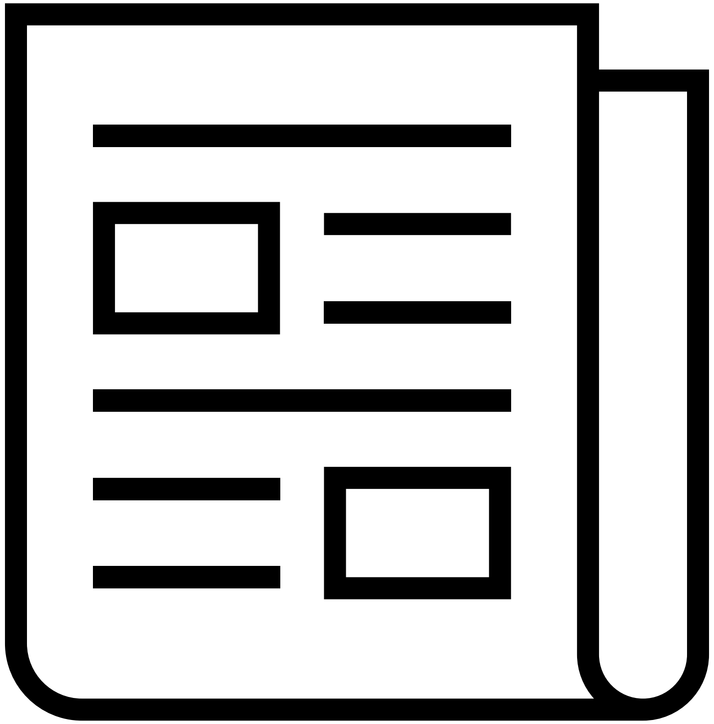
University at Buffalo

Schedule

1. Course Updates
2. Strong Connectivity
3. Greedy Algorithms
4. Change Making
5. Interval Scheduling

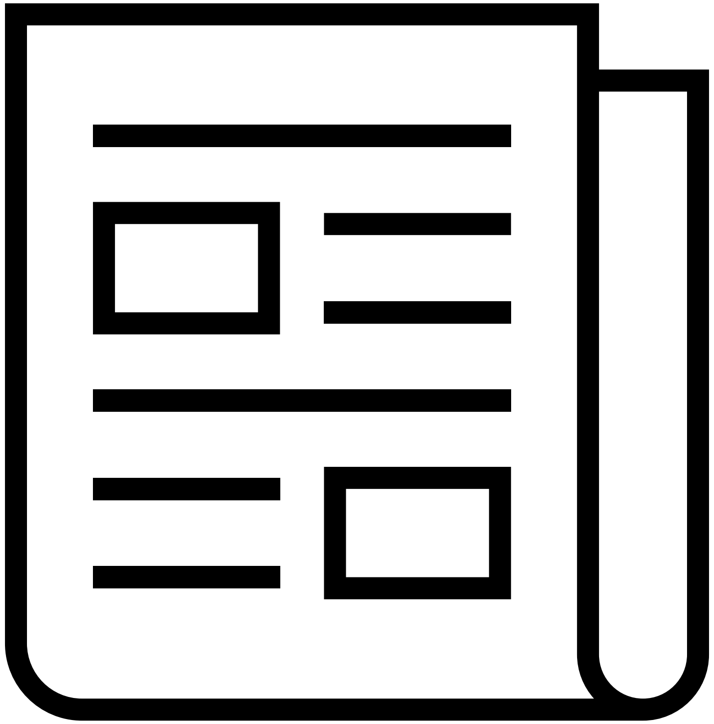
Course Updates

- HW 2 Grading Out
- HW 3 Solutions Out
- HW 4 Out Soon
    - Not Due Next Week!
- Group Project
    - First Problems Oct 31st
- Sample Midterms Out
- Midterms Oct 6 and Oct 8



Midterms

- Advice:
  - Start Studying
  - Go to Recitations this Week
  - Review Book Chapters
  - Review Solutions to HW/Quiz
  - Try Sample Midterm
  - Make Good Use of Time



Strong Connectivity Problem

Input: Directed graph $G = (V, E)$
Output: True if strongly connected and False otherwise.

Proof Idea:

- Pick a vertex s in V
- Use BFS to find all vertices I can reach from s.
- Use ______________ to find all vertices that can reach s.
- If both sets are equal return true and otherwise return false.

Strong Connectivity Algorithm

- **Definition**: We say that a directed graph is strongly connected if for any two vertices in the graph, there exists a directed path from one to the other.
- **Observation**: If u and v are mutually reachable, and v and w are mutually reachable, then u and w are mutually reachable.
- **Strong Connectivity Problem:**
  - **`Input`**`:` `Directed graph` $G = (V, E)$
  - **`Output`**`:` `True if strongly connected and False otherwise.`

Strong Connectivity Algorithm

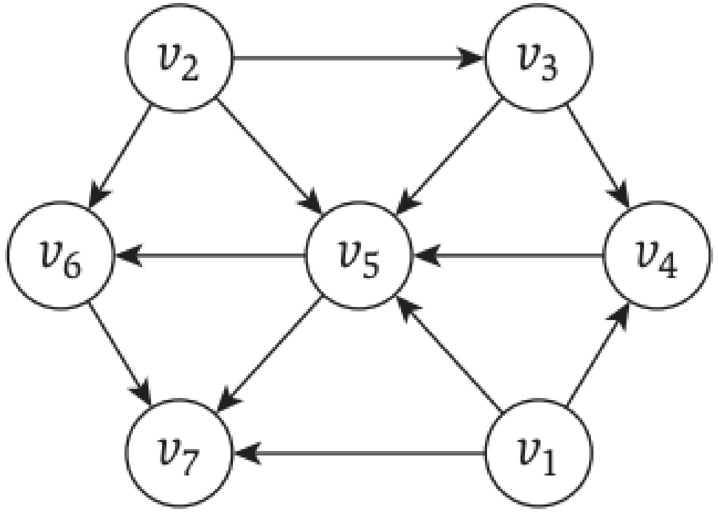
Input: Directed graph $G = (V, E)$
Output: True if strongly connected and False otherwise.

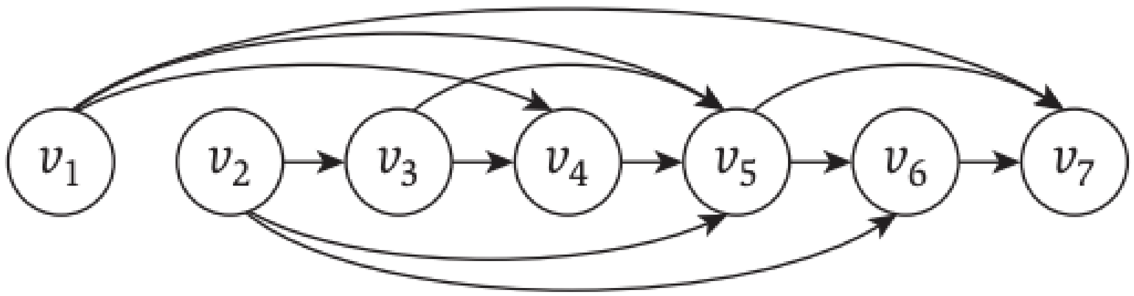
Proof Idea:

- Pick a vertex s in V
- Use BFS to find all vertices I can reach from s.
- Use BFS on "Reversed Graph" to find all vertices that can reach s. For each edge (u,v) replace with edge (v,u)
- If both sets are equal return true and otherwise return false.

Directed Acyclic Graphs (DAGs)

- **Definition:** A directed graph is a DAG if it has no directed cycles.
- **Definition:** A topological ordering of a directed graph $G = (V, E)$ is an ordering of its nodes as $v_1, v_2, \ldots, v_n$ so that for every edge (v_i, v_j), we have i < j.





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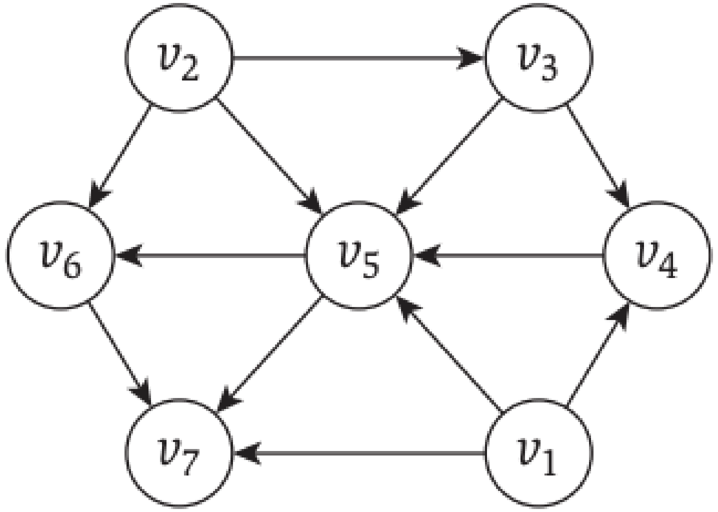

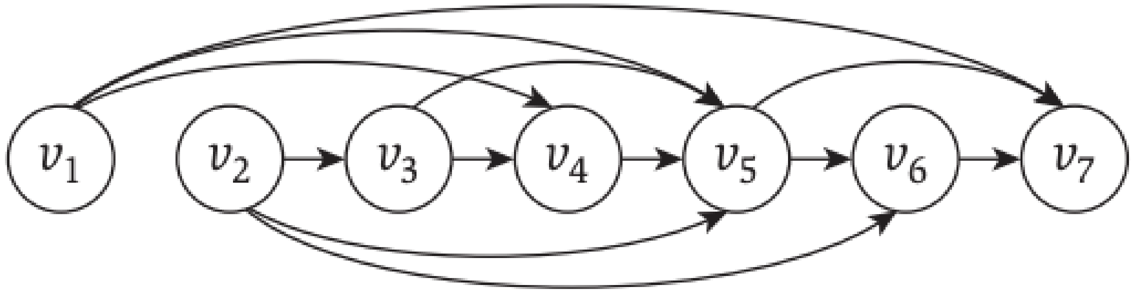

All edges are going “forward”

Directed Acyclic Graphs (DAGs)

Definition: A directed graph is a DAG if it has no directed cycles.

Definition: A topological ordering of a directed graph $G =$

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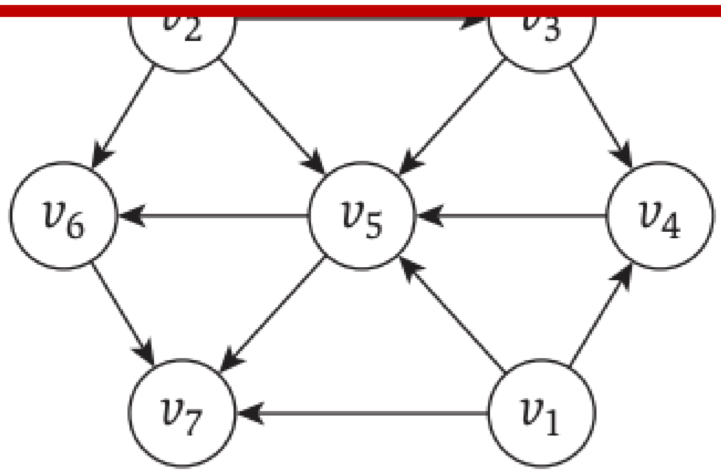
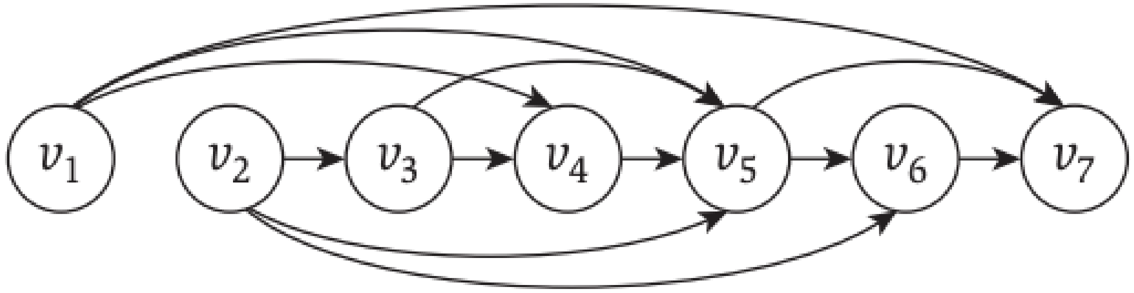
Read KT Section 3.6 and Review Care Packaged on Topological Ordering





All edges are going “forward”

Midterm Check Point

What is next?

- Greedy Algorithms
- Divide and Conquer
- Dynamic Programming
- Network Flows (maybe)
- Computation Complexity

“How do we design new algorithms?”

- **Greedy Algorithms**
- **Divide and Conquer**
- **Dynamic Programming**
- **Network Flows (maybe)**
- Computation Complexity

“How do we use reduce another problem?”

- Greedy Algorithms
- Divide and Conquer
- Dynamic Programming
- **Network Flows (maybe)**
- **Computation Complexity**

"How do we know when to give up?"

- Greedy Algorithms
- Divide and Conquer
- Dynamic Programming
- Network Flows (maybe)
- **Computation Complexity**

What are Greedy Algorithm?

What are Greedy Algorithm?

- Build solution one piece at a time.
- Only look at immediate information to make choices.
- Never go back on a decision.
- NOT ALWAYS THE BEST CHOICE!

Coin Change Problem

- **Problem**: Given U.S. currency denominations {1.00, 0.25, 0.10, 0.05, 0.01} find an algorithm to pay an amount to a customer using the fewest coins possible.

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- **Algorithm**: At each iteration, add a coin of the largest value that is less than the amount needed to be paid.

Q: Is this algorithm always optimal?

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Proof Ideas for Optimality

- **Proof Idea:**
  - Suppose there it wasn't optimal.
    - Then there exists a budget B such that the algorithm returns S and the answer is S' (S != S').

Proof Ideas for Optimality

- **Proof Ideas:**
  - Suppose there it wasn’t optimal.
    - Then there exists a budget B such that the algorithm returns a set S(B) and the answer is a different set S’(B).
  - If S’(B) has the largest coin that S(B) has, then you can remove it and you get a smaller bad budget B’ = B - <large coin value>

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Proof Ideas for Optimality

- **Proof Ideas:**
  - If S’(B) has the largest coin that S(B) has, then you can remove it and you get a smaller bad budget B’ = B - <large coin value>
  - We now show that S’(B) has to have the largest coin that S(B) has.
    - That is, the greedy choice was good!

Proof Ideas for Optimality

- **Proof Ideas:**
  - We now show that S'(B) has to have the largest coin that S(B) has.
  - If S'(B) doesn't have the largest coin that can fit in the budget, then it must be replaced with smaller coins.
    - We can check optimal for restricted settings of coins. ← Sounds like a few base cases

Proof Ideas for Optimality

- **Proof Ideas:**
  - We can show that in an optimal solution we have:
    - At most 4 pennies
    - At most 1 nickel
    - At most 2 nickels + dimes
    - At most 3 quarters
  - We show these by contradicting the optimality!

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Big Ideas

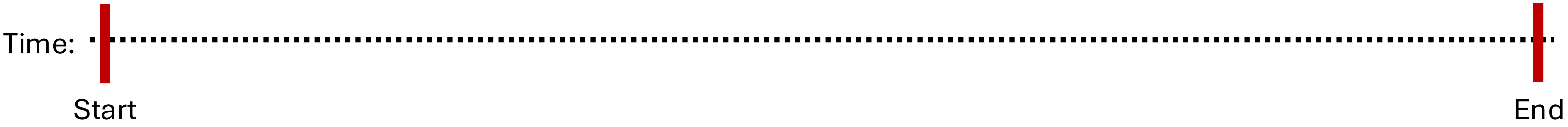
- Sometimes Greedy Works but sometimes it doesn’t…
  - If you use a different set of coins, you may not be able to use the cashier algorithm.
  - Consider {1,10,21,34,70,100,350,1225} and the budget 140.

Big Ideas

- Sometimes Greedy Works but sometimes it doesn't…
  - If you use a different set of coins, you may not be able to use the cashier algorithm.
  - Consider {1,10,21,34,70,100,350,1225} and the budget 140.
    - Algorithm Output: 100, 34, 1x6
    - Answer: 70 x 2

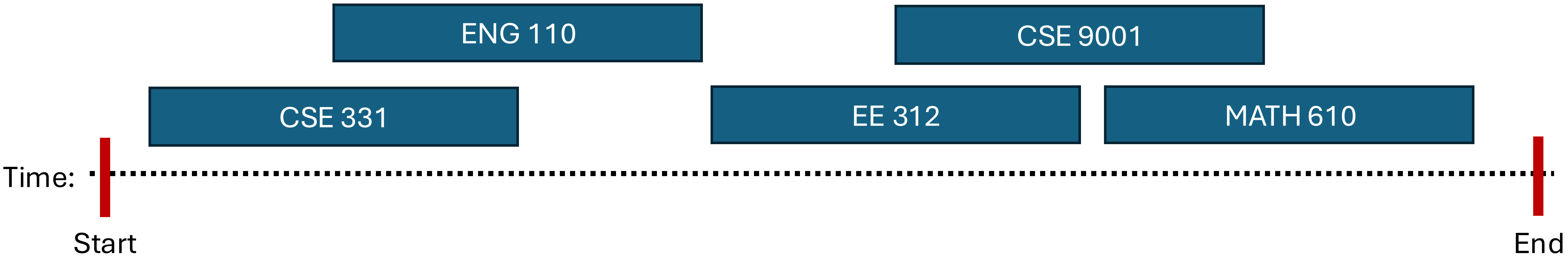
Interval Scheduling

- Consider an interval of time (e.g. Wednesday).



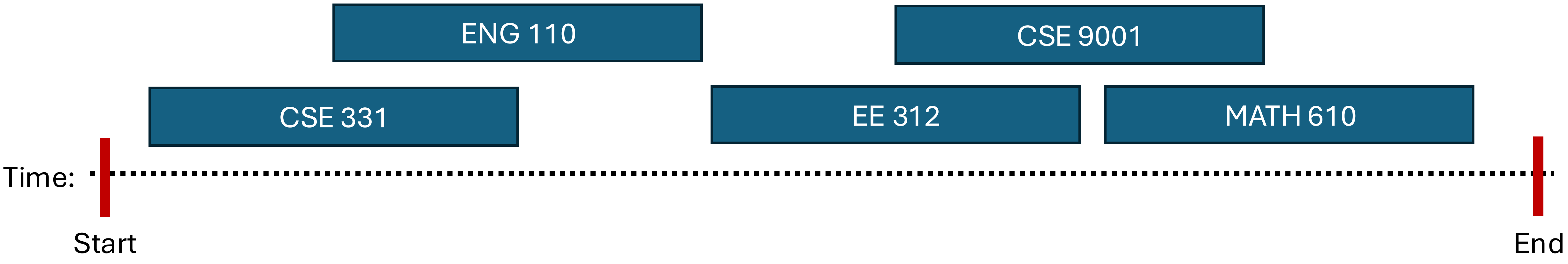
Interval Scheduling

- Consider an interval of time (e.g. Wednesday)
- Consider tasks that need to be completed during specific times (e.g. classes)



Interval Scheduling

- Consider an interval of time (e.g. Wednesday).
- Consider tasks that need to be completed during specific times (e.g. classes).
- We want to fit as many tasks as possible into the day such that no two overlap.



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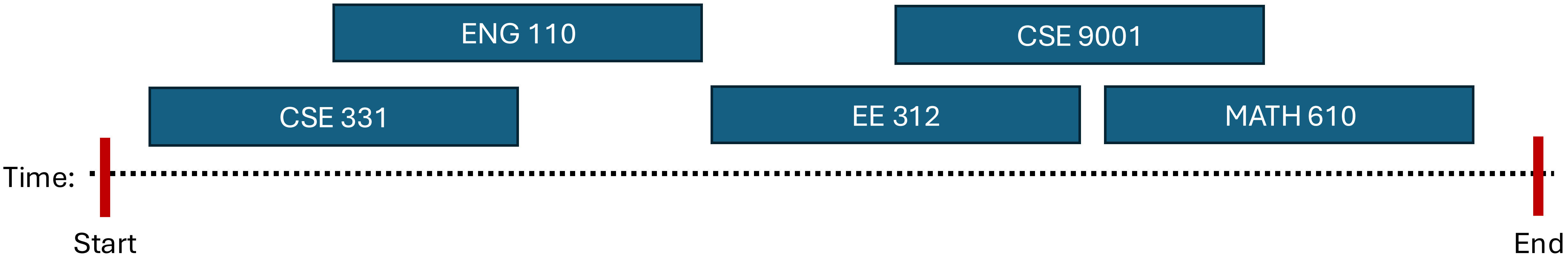

Interval Scheduling

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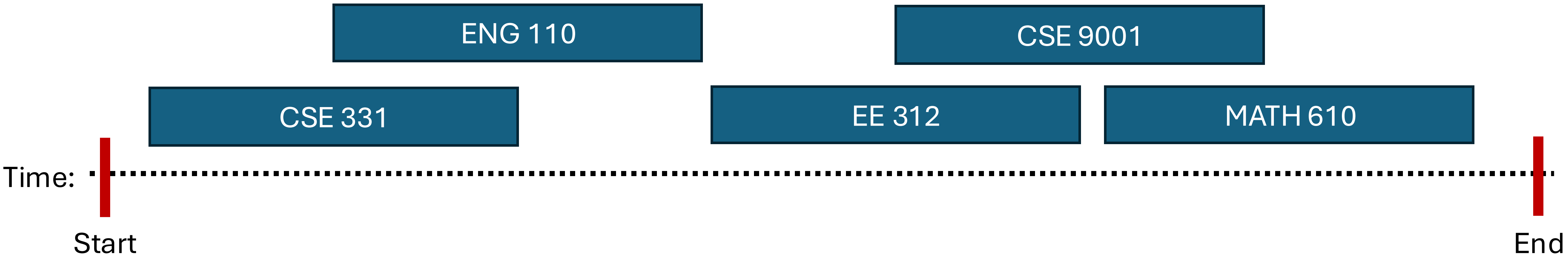
Q: Do we ever pick CSE 9001?



Interval Scheduling

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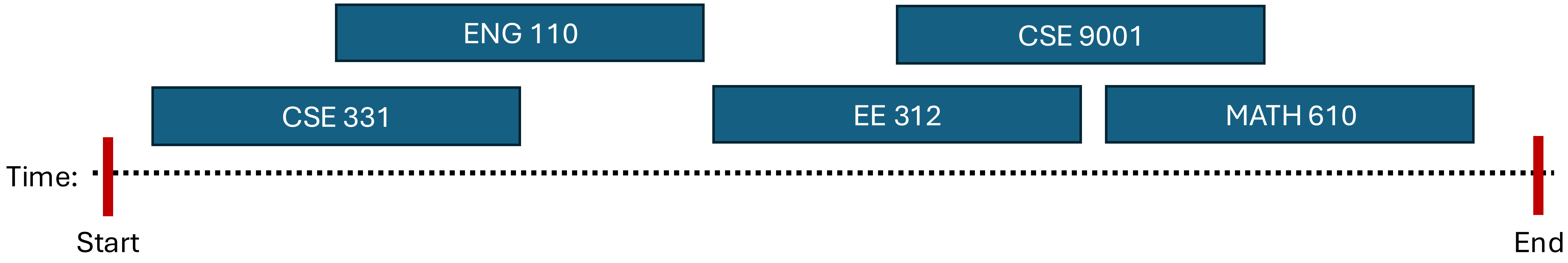
A: No, because it blocks two classes!



Optimal Solution #1

- Consider an interval of time (e.g. Wednesday).
- Consider tasks that need to be completed during specific times (e.g. classes).
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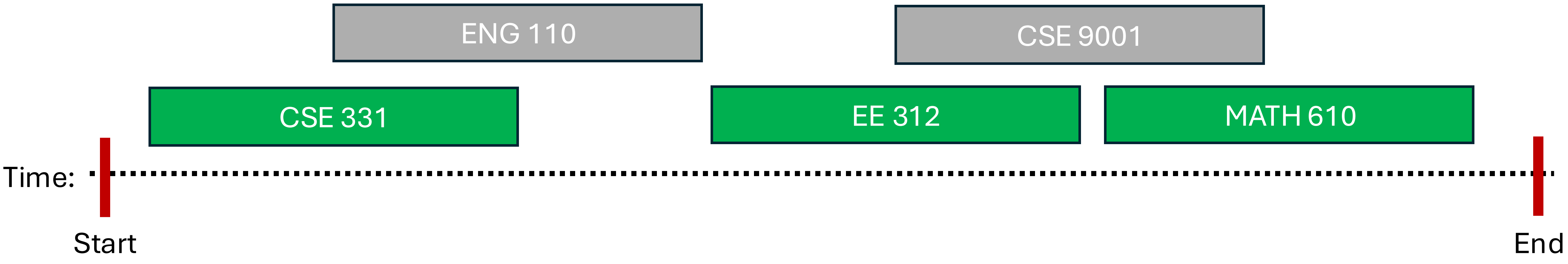

Optimal Solution #2

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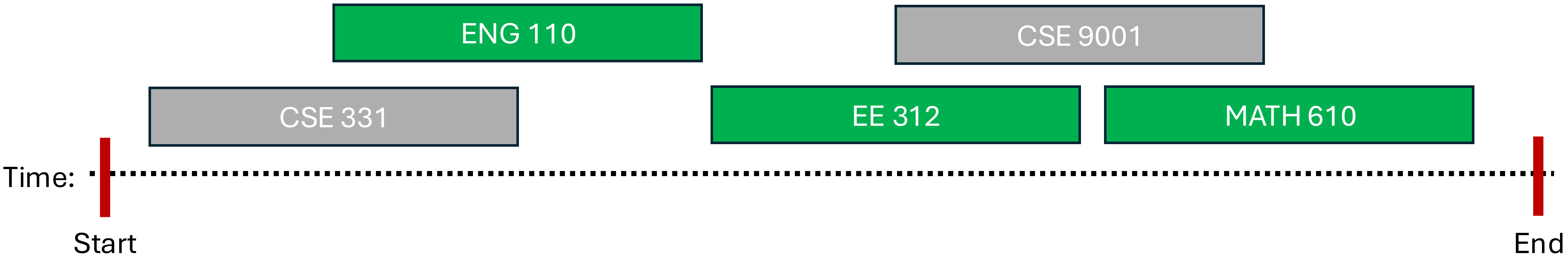


Interval Scheduling Problem (Support Page)

Interval Scheduling via examples

In which we derive an algorithm that solves the Interval Scheduling problem via a sequence of examples.

The problem

In these notes we will solve the following problem:

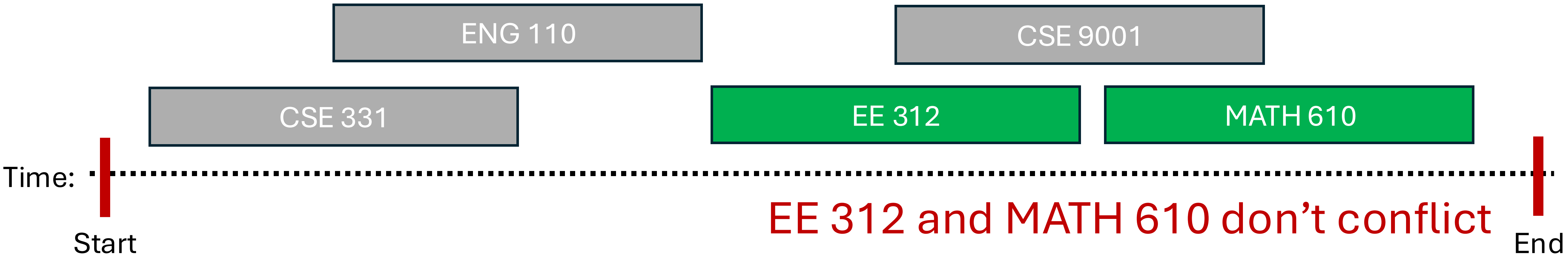
Interval Scheduling Problem

`Input:` An input of n intervals $[s(i), f(i))$, or in other words, $\{s(i), \ldots, f(i) - 1\}$ for $1 \leq i \leq n$ where i represents the intervals, $s(i)$ represents the start time, and $f(i)$ represents the finish time.

`Output:` A schedule S of n intervals where no two intervals in S conflict, and the total number of intervals in S is maximized.

Interval Scheduling Problem

- **Input**: A set of n intervals with start and finish times.
  - For $1 \le \mathrm{i} \le \mathrm{n}$, $[s(\mathrm{i}), \mathrm{f}(\mathrm{i}))$ where $s(\mathrm{i})$ and $\mathrm{f}(\mathrm{i})$ are start and finish times of task i respectively.
- **Output**: A schedule (subset of intervals) S such that no two intervals in S conflict and the total number of intervals is maximized.



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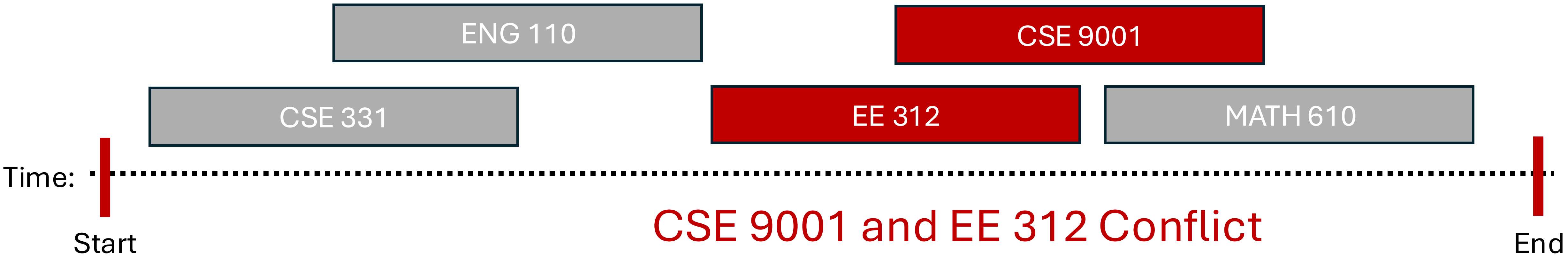
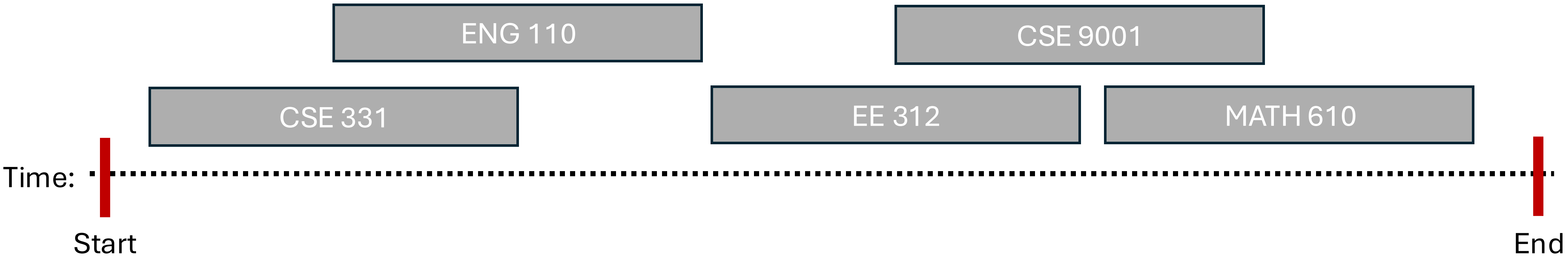


Q: How should we try to solve this?

- **Input**: A set of n intervals R with start and finish times.
  - For $1 \leq \mathrm{i} \leq \mathrm{n}$, $[s(\mathrm{i}), \mathrm{f}(\mathrm{i}))$ where $s(\mathrm{i})$ and $\mathrm{f}(\mathrm{i})$ are start and finish times of task i respectively.
- **Output**: A schedule (subset of intervals) S such that no two intervals in S conflict and the total number of intervals is maximized.



Q: How should we try to solve this?

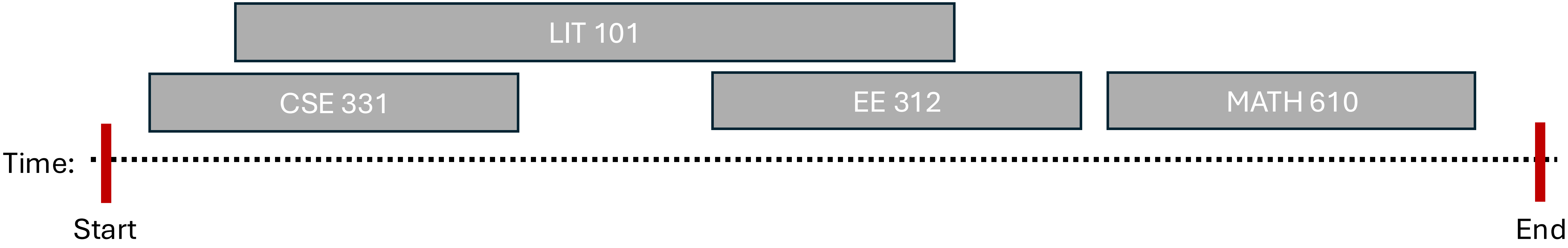
A: Let’s try to generate some examples and see what works and what doesn’t.

Build an Algorithm

- Basic Algorithm Outline:
  - S is empty
  - While R is not empty:
    - Pick i in R
    - Add i to S
    - Remove i from R

Build an Algorithm

- Basic Algorithm Outline:
  - S is empty
  - While R is not empty:
    - Pick i in R
    - Add i to S You might add conflicts!
    - Remove i from R



Build an Algorithm

- Basic Algorithm Outline:
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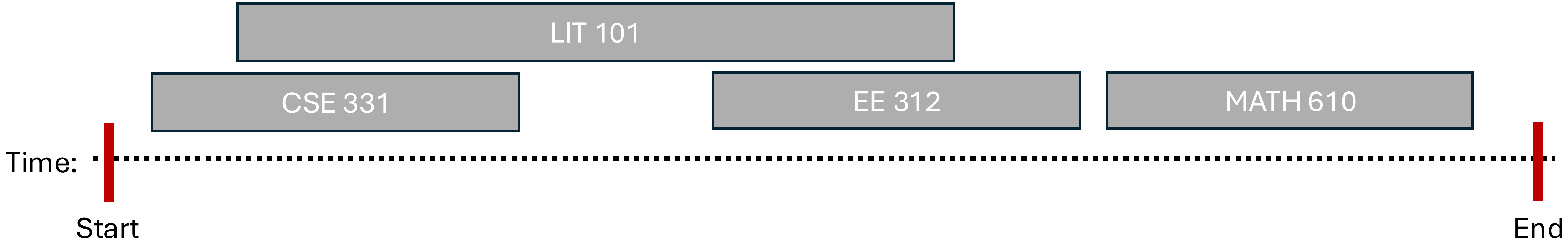

Build an Algorithm

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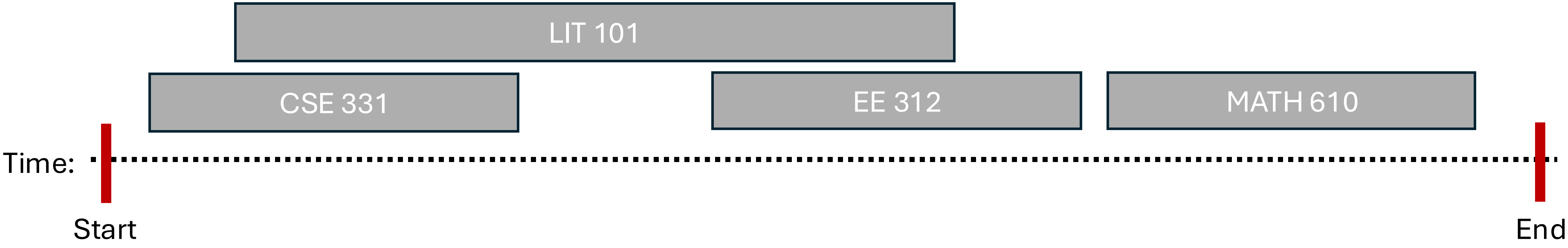

Build a Greedy Algorithm

- Basic Algorithm Outline:
  - S is empty
  - While R is not empty:
    - Pick i in R that minimizes v(i)
    - Add i to S
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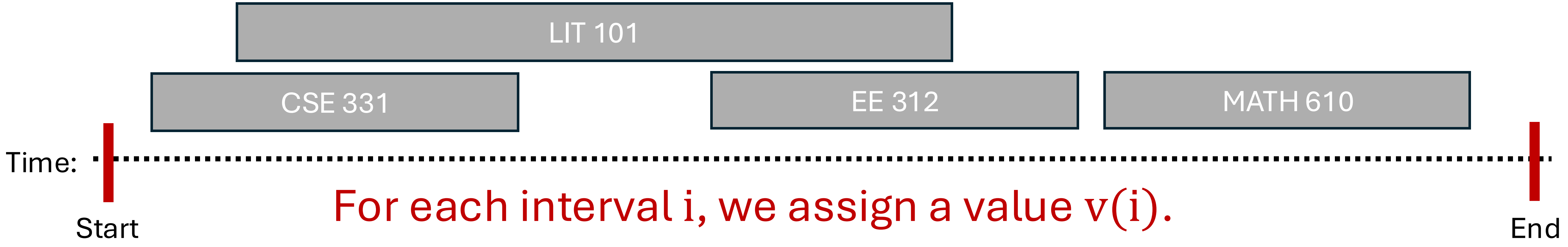

For each interval i, we assign a value v(i).

Q: What should we pick for $v(i)$?

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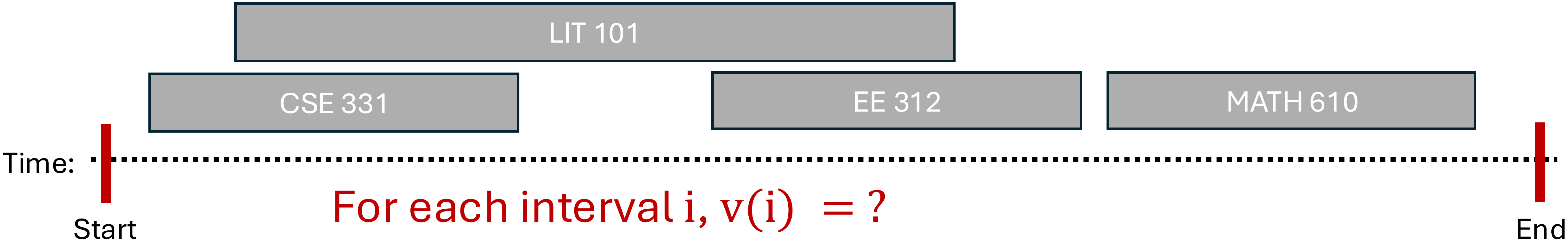

Attempt I: Interval Length (Okay)

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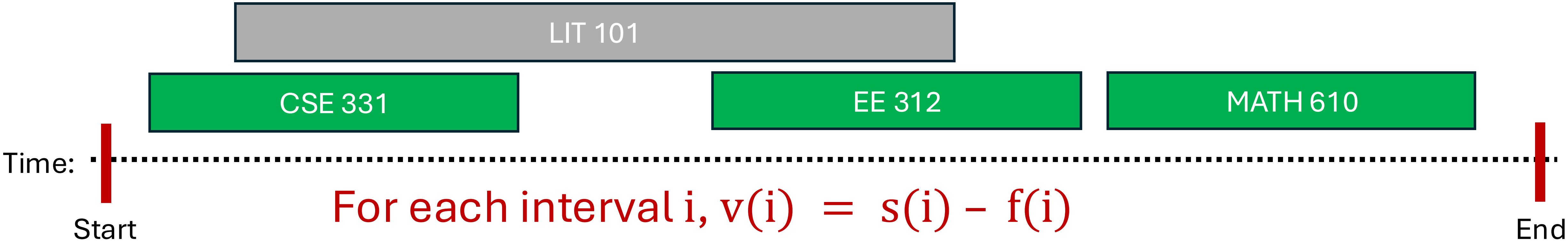

For each interval i, $v(i) = s(i) - f(i)$

Attempt I: Interval Length (Oh no!)

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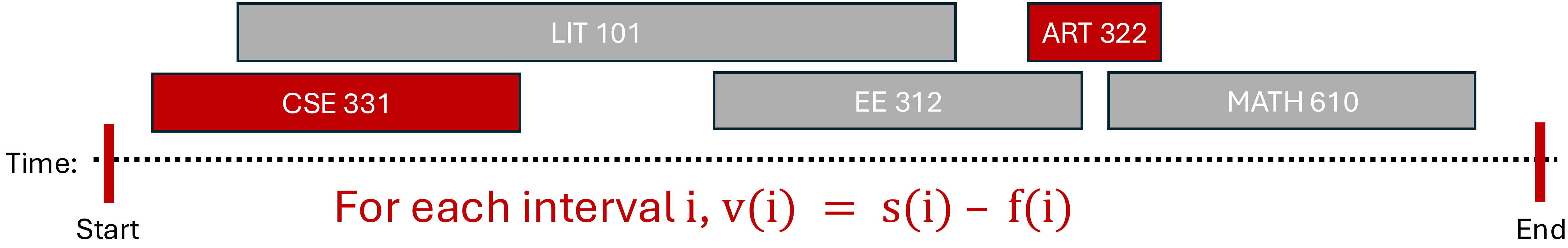

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Attempt II: Start Time (Okay)

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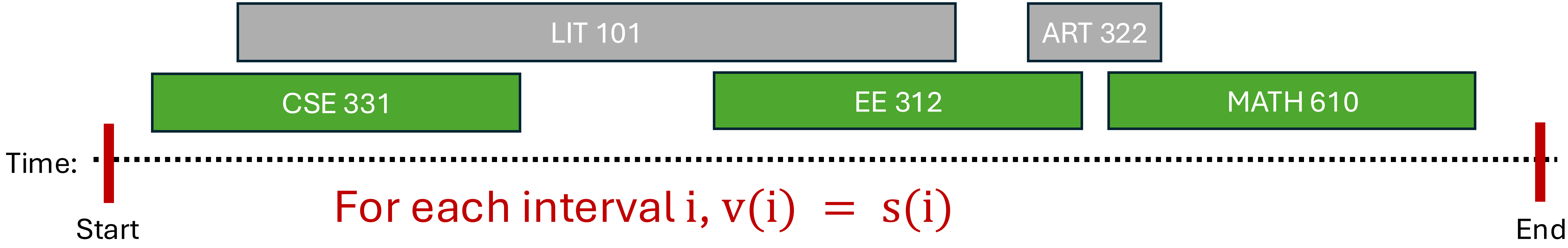

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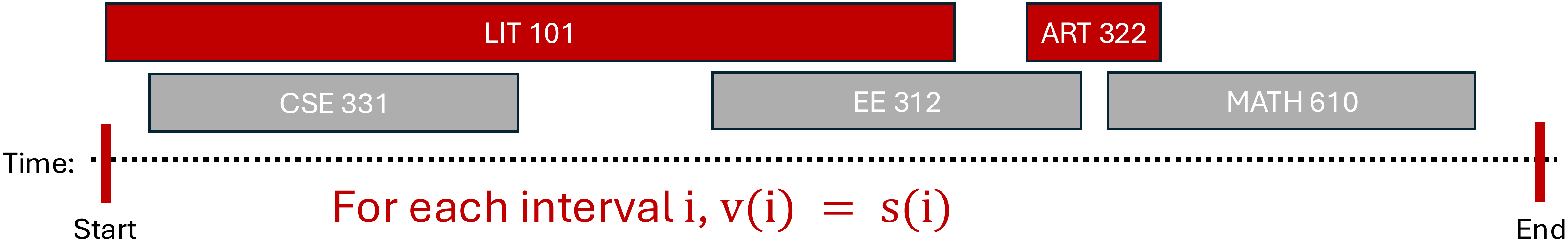

For each interval i, $v(i) = s(i)$

Attempt III: Min Conflicts (Okay)

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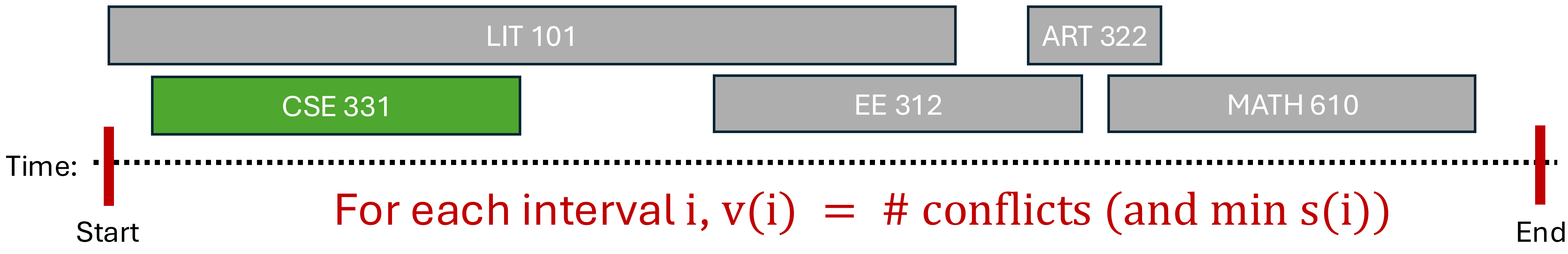

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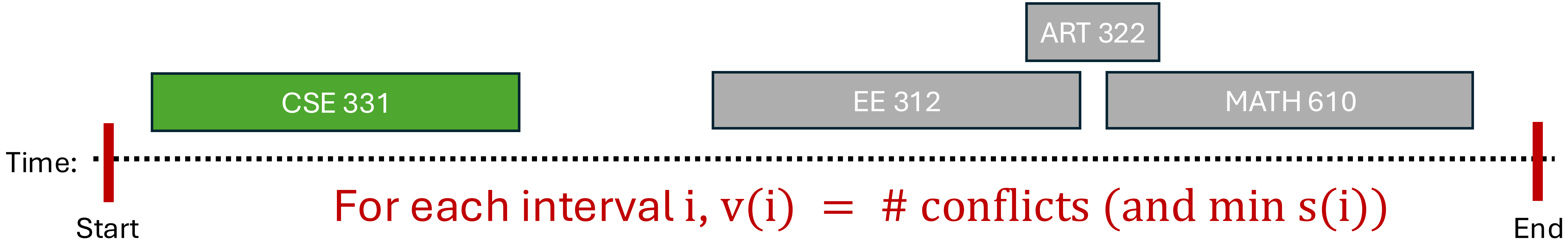

For each interval i, v(i) = # conflicts (and min s(i))

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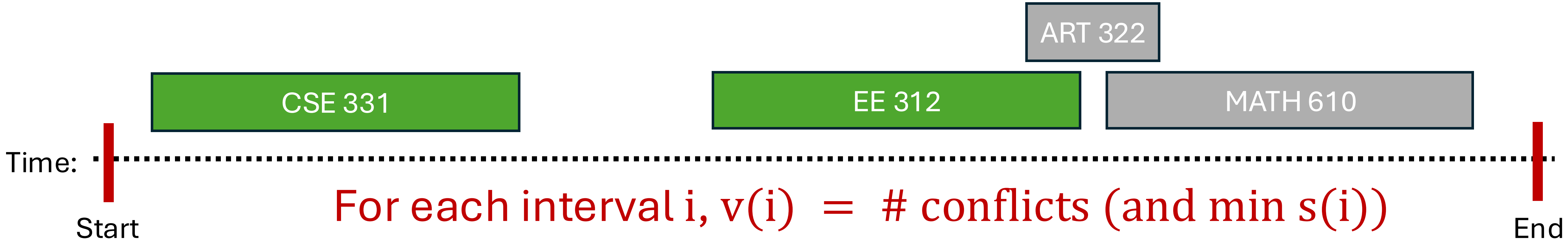

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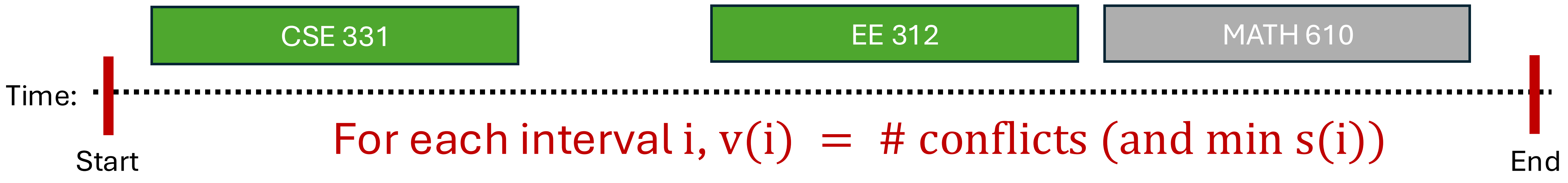

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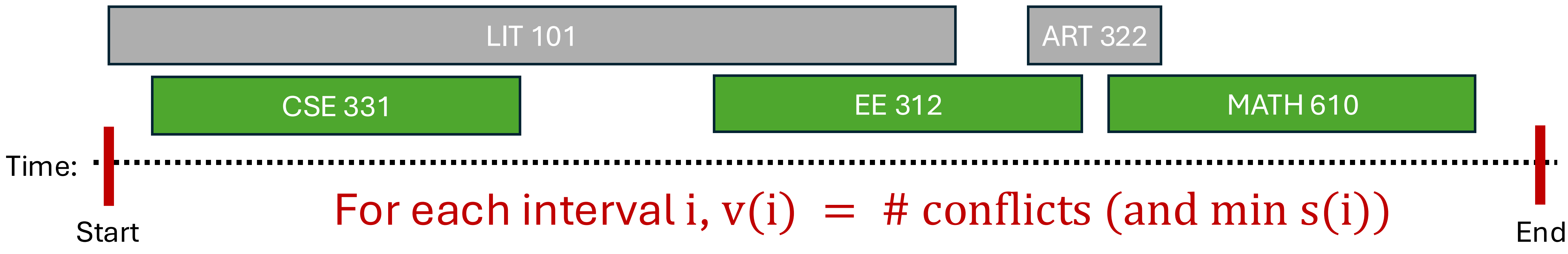

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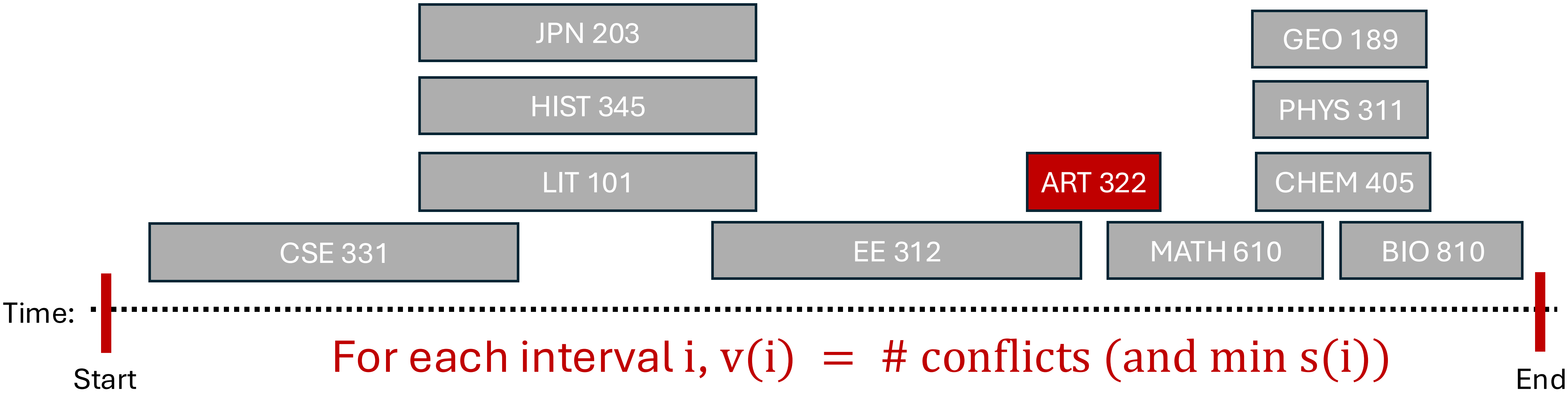

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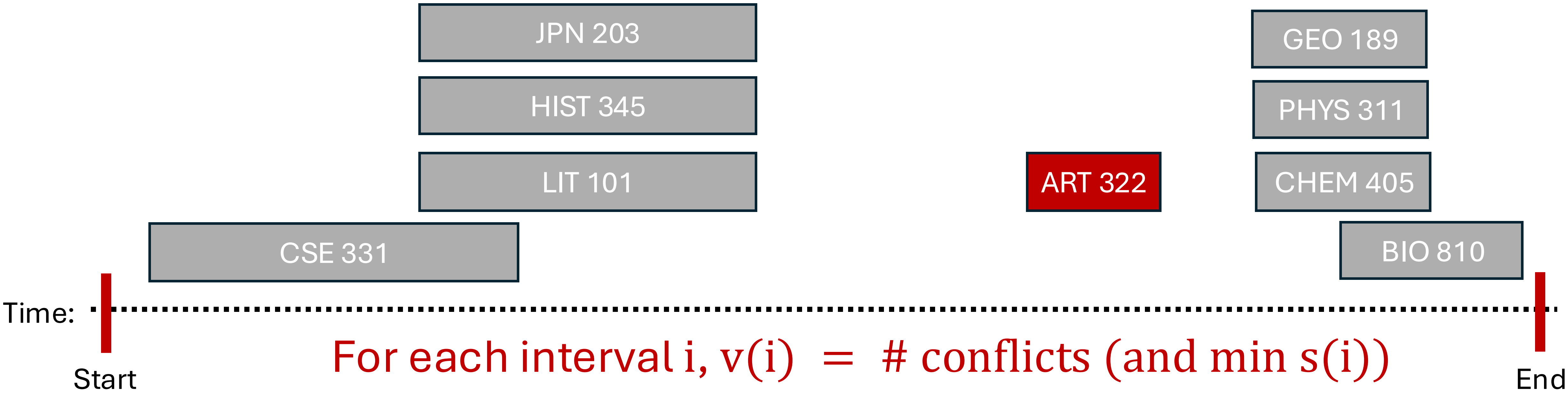

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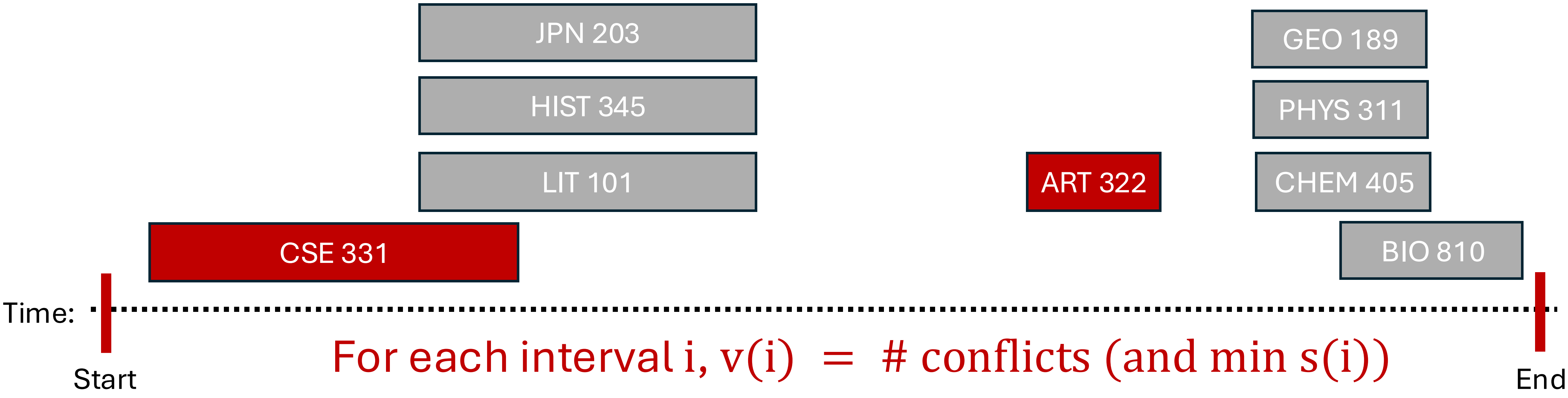
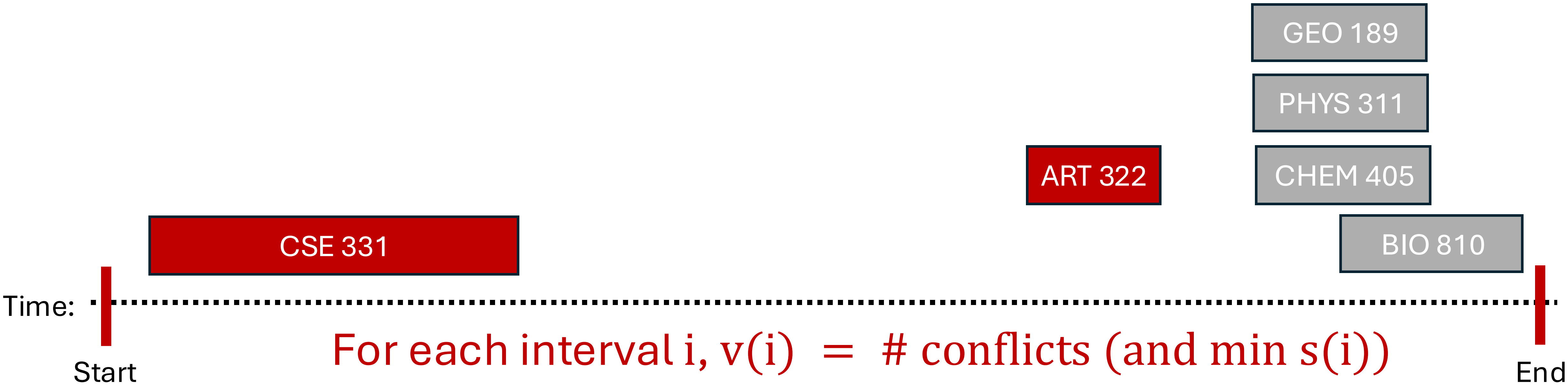
Attempt III: Min Conflicts (Oh no!)



For each interval i, $v(i) = \#$ conflicts (and min $s(i)$)

Attempt III: Min Conflicts (Oh no!)



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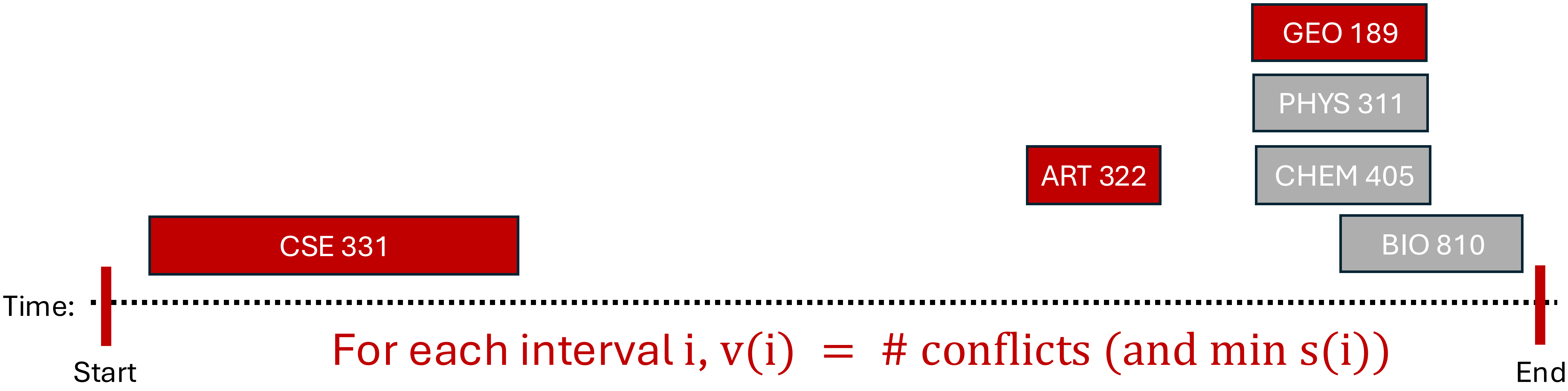

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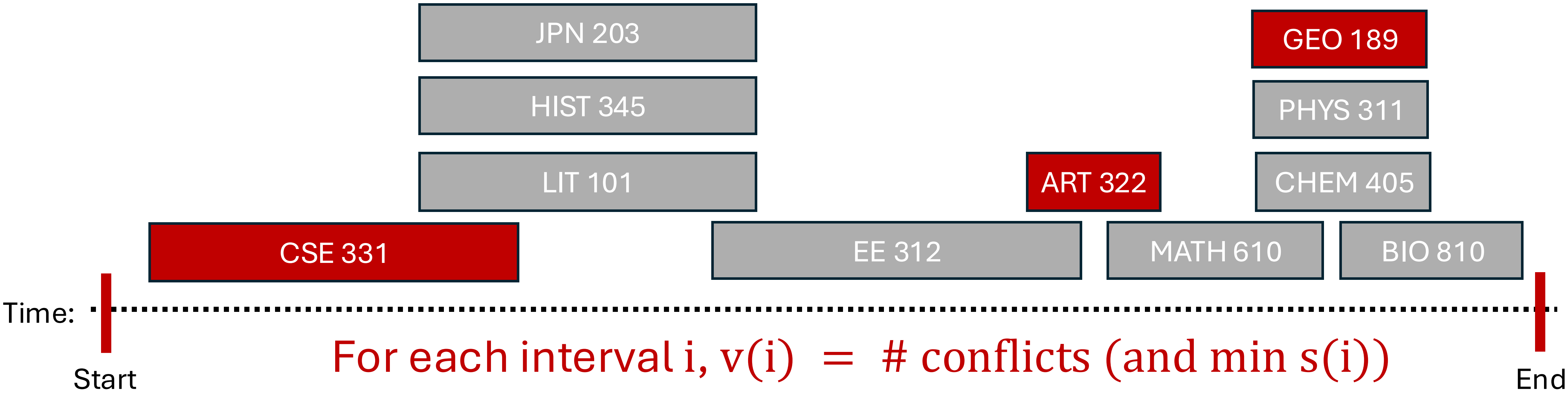
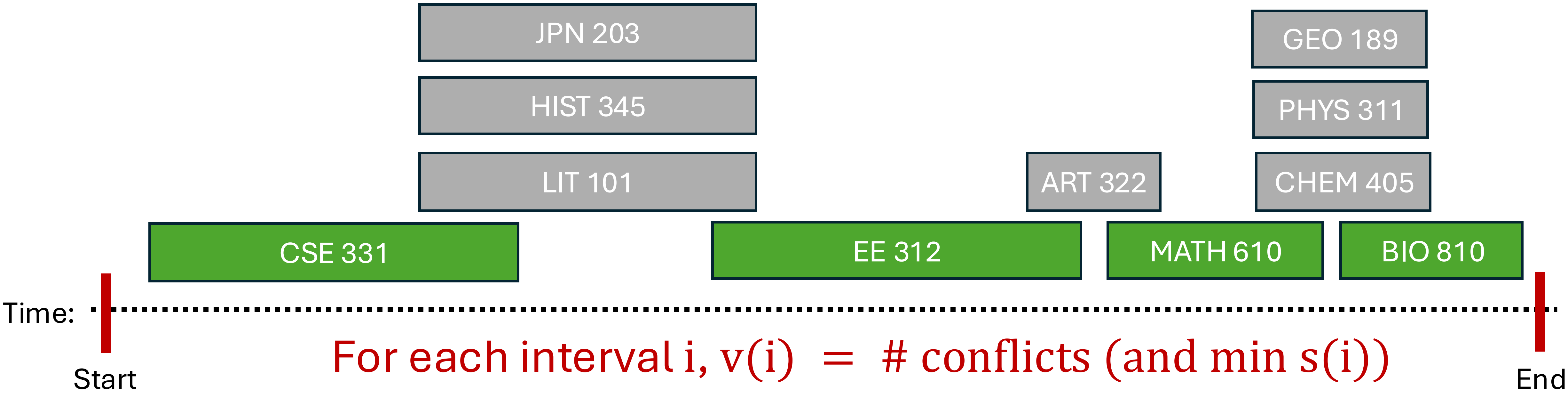


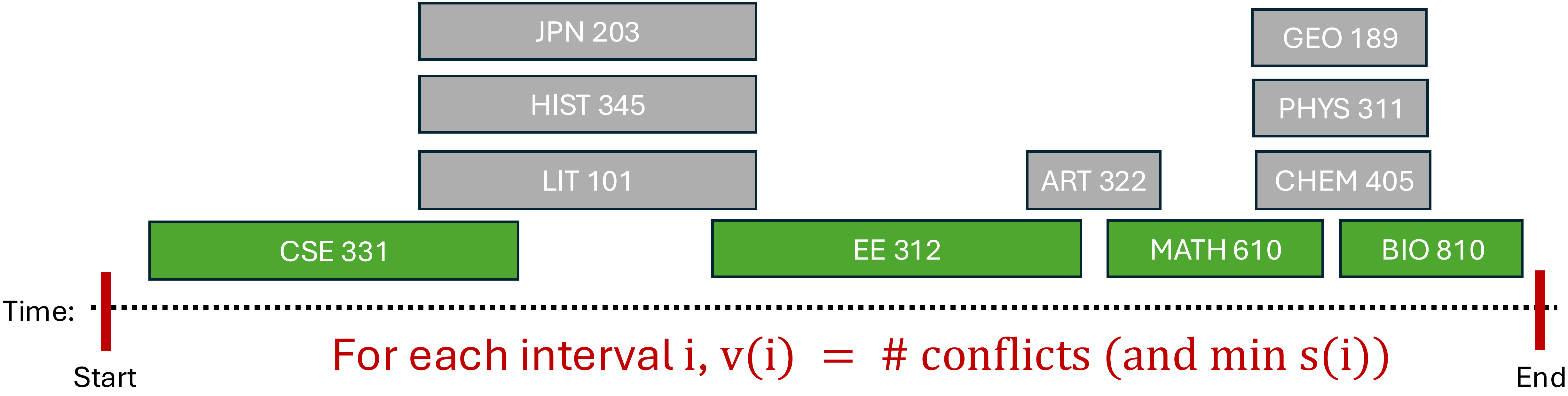
For each interval i, v(i) = # conflicts (and min s(i))

Attempt III: Min Conflicts (Oh no!)



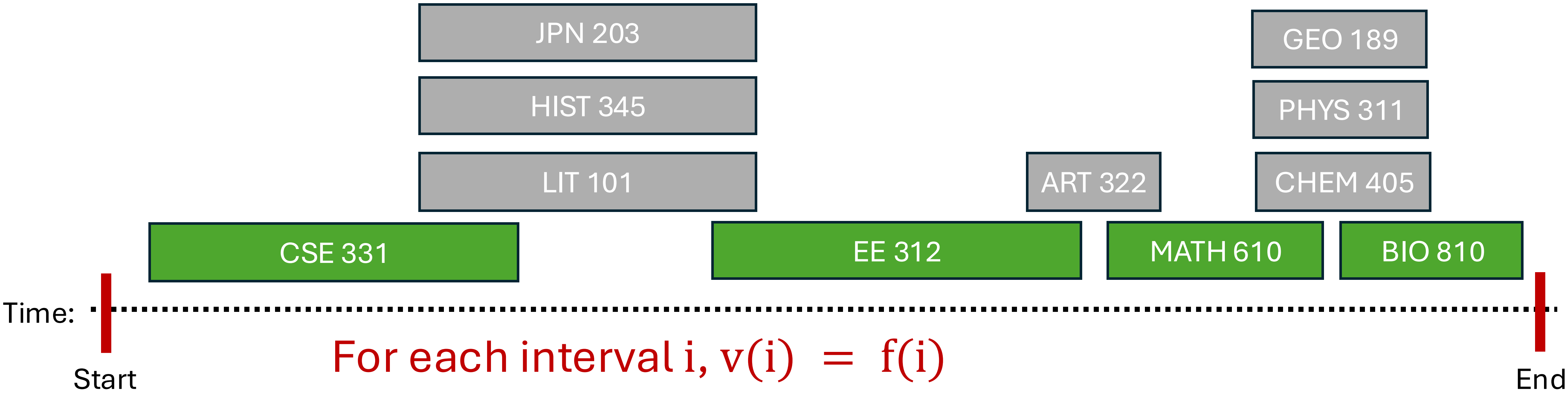
For each interval i, v(i) = # conflicts (and min s(i))

I GIVE UP!!!!



For each interval i, v(i) = # conflicts (and min s(i))

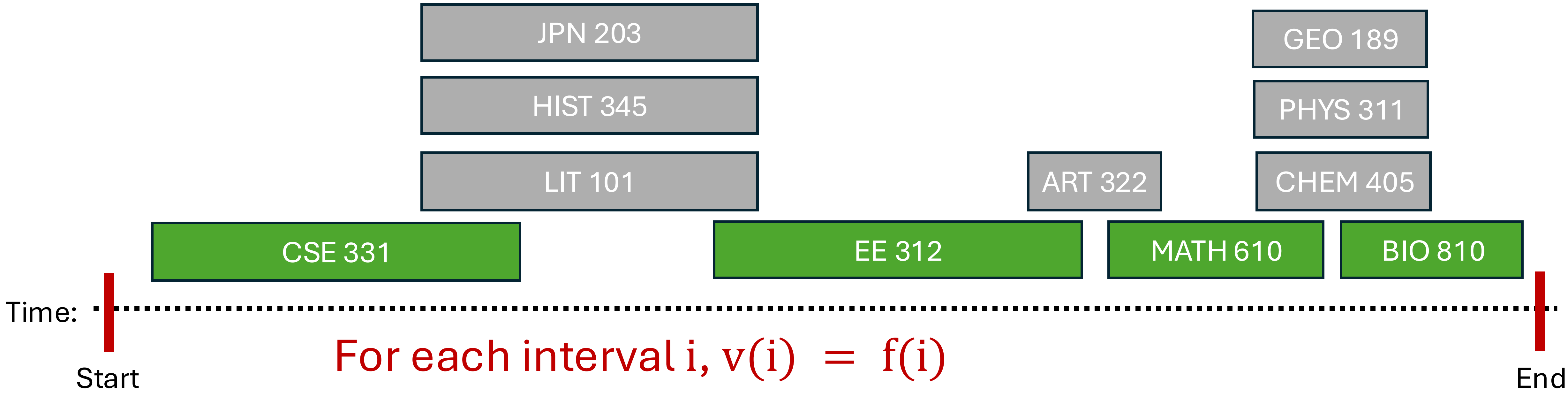
Attempt IV: Finish Time (Okay)



For each interval i, $v(i) = f(i)$

Attempt IV: Finish Time (....)

Wait... does that actually work?

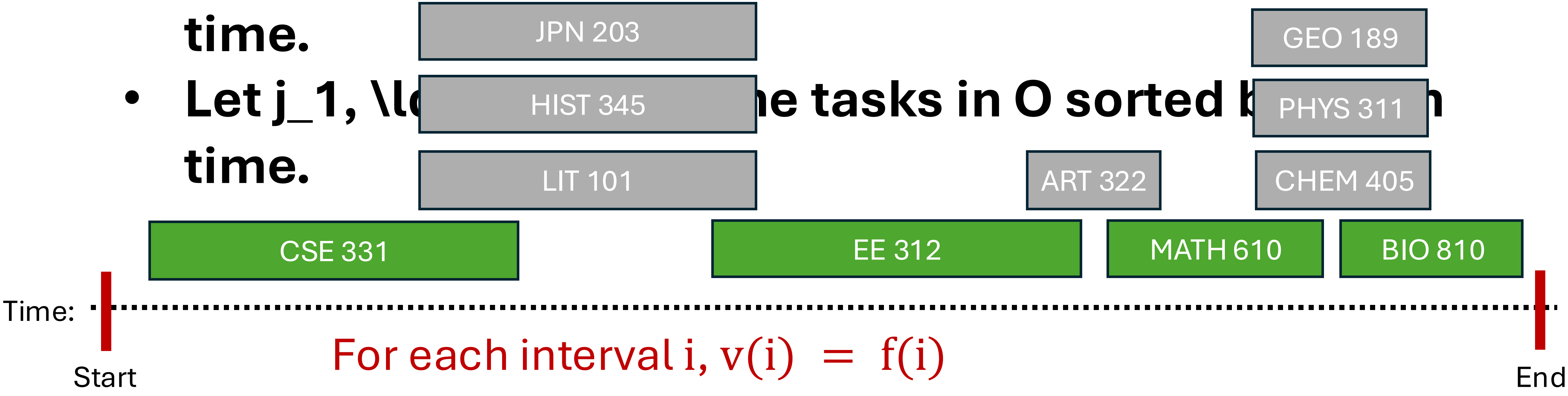


For each interval i, $v(i) = f(i)$

Claim: The Finish First Algorithm is Optimal

Proof Ideas:

- **Let A be the set returned by the algorithm and O be the optimal list.**
- **Let i_1, \ldots, i_k be the tasks in A sorted by add time.**
- **Let j_1, \l[...]e tasks in O sorted b[...]n time.**



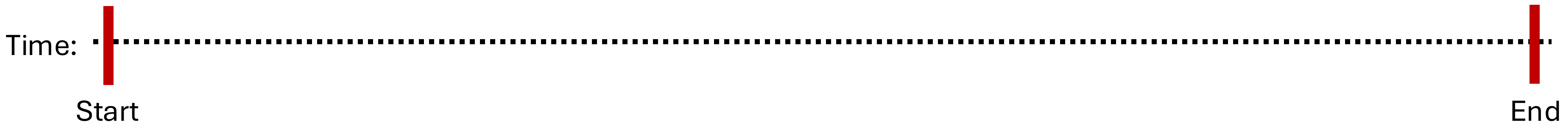
For each interval i, $v(i) = f(i)$

Claim: The Finish First Algorithm is Optimal

Proof Ideas:

- Let $i_1, \ldots, i_k$ be the tasks returned by algorithm (sorted by finish/add time).
- Let $j_1, \ldots, j_m$ be the tasks in optimal solution (sorted by finish time)
- We want to show k = m

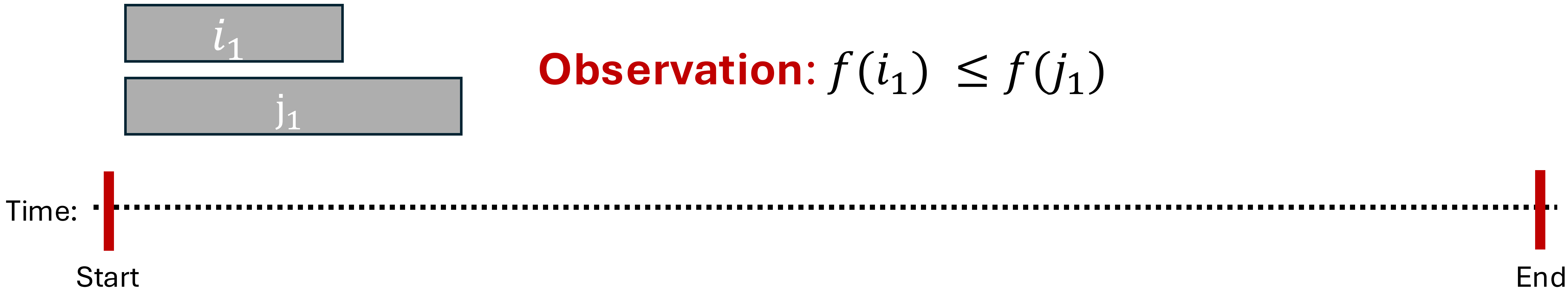
Q: What can we say about the first job in each list?



Claim: The Finish First Algorithm is Optimal

Proof Ideas:

- Let $i_1, \ldots, i_k$ be the tasks returned by algorithm (sorted by finish/add time).
- Let $\mathrm{j}_1, \ldots, \mathrm{j}_\mathrm{m}$ be the tasks in optimal solution (sorted by finish time)



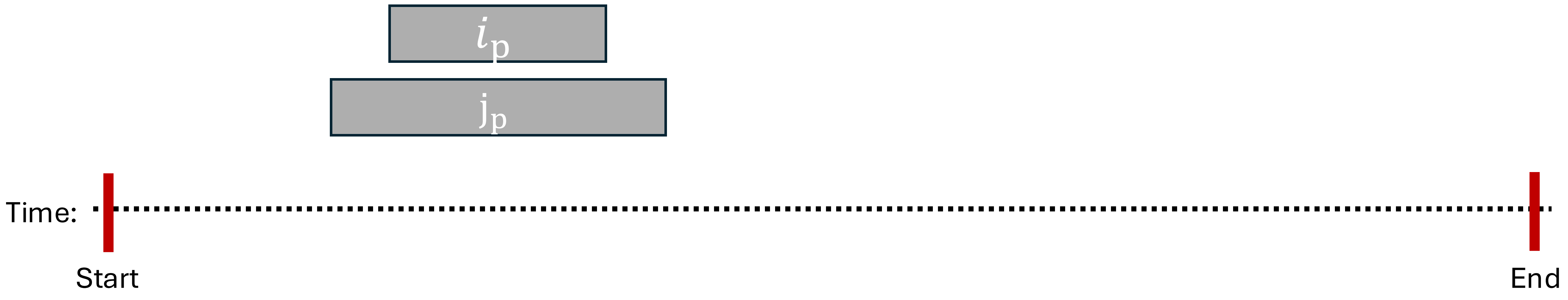
Observation: $f(i_1) \leq f(j_1)$

Claim: The Finish First Algorithm is Optimal

Proof Ideas:

- Assume now that $f(i_p) \leq f(j_p)$ for some p.
  - That is, assume the pth in the algorithms list ends before the pth job in the optimal list.

Q: What can we say about the $\mathrm{p}+1$ job in each list?

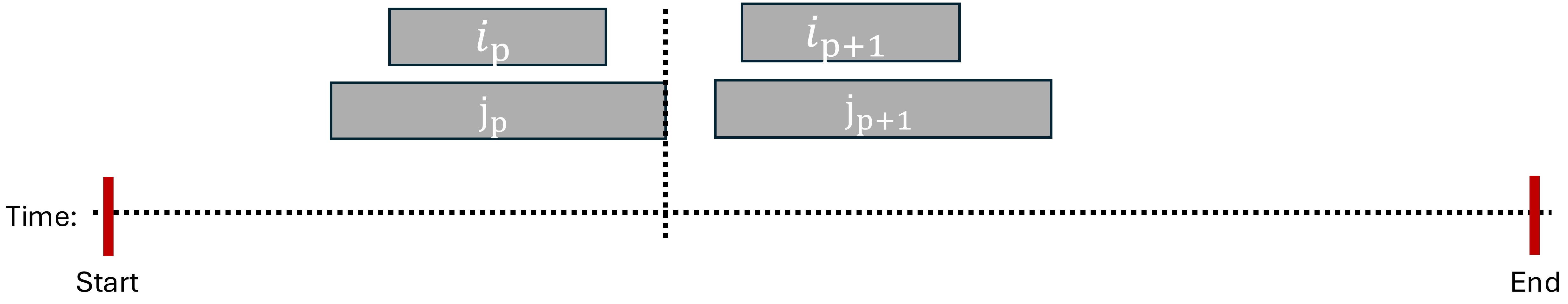


Claim: The Finish First Algorithm is Optimal

Proof Ideas:

- Assume now that $f(i_p) \leq f(j_p)$ for some p.
  - That is, assume the pth in the algorithms list ends before the pth job in the optimal list.

Observation: The algorithm could have added j_{p+1}!

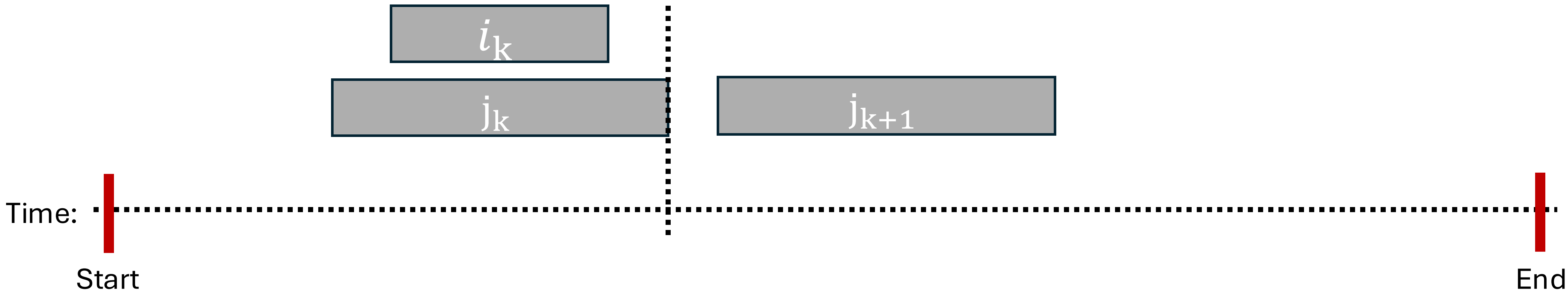


Claim: The Finish First Algorithm is Optimal

Proof Ideas:

- Assume now that $f(i_k) \leq f(j_k)$ and m > k.

Q: Why is this a problem?



Claim: The Finish First Algorithm is Optimal

Proof Ideas:

- Assume now that $f(i_k) \leq f(j_k)$ and m > k.

Observation: The algorithm could have added j_{k+1}!