# Terrain Reconstruction with an Adaptive Surface Mesh<sup>\*</sup>

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### Abstract

This paper describes a new class of adaptive mesh surface for terrain analysis. The novelty of the contribution resides in the control of the mesh. We use a variance-bias criterion to select the optimal areas for the triangular facets of the mesh. In this way the mesh adapts itself to offer the best tradeoff between increasing the facet area to minimise the noise variance and deccreasing the facet area to minimise the bias of the fitted facet parameters. We provide a illustration of the effectiveness of the new mesh control methodology for the case where the faces of the mesh represent planar patches. The piecewise planar mesh is shown to be effective in the modelling of an area of complex terrain structure in Southern England.

#### 1 Introduction

Automatic terrain analysis has become an important application domain for surface-modelling methodology developed in the fields of both computer graphics and computer vision. The surface model must offer a compromise between economy of representation and fidelity with the underlying terrain structure. It is for this reasons that adaptive mesh surfaces have been widely adopted. The literature is rich with examples. For instance De Floriani et al [3, 4] have developed a multi-scale mesh which has been exploited not only for surface representation, but also for stereoscopic reconstruction. Several authors have reported variable topology meshes. Bulpitt and Efford [1] have a mesh that adapts itself so as to minimise curvature and goodness of fit criteria. However, despite offering ingenious ways of exerting control over the surface model, these all share the feature of being geometrically driven [6, 7]. In particular, they overlook the well known bias-variance dilemma [5, 9] which pervades the statistical fitting of a model to data. Stated succinctly, the problem is one of selecting the model-order, i.e. the number of faces in the mesh. The aim is to strike a compromise between minimising the noise-variance, and minimising the dispersion of the fit-residuals or model-bias. This can be viewed as a tradeoff between increasing the number of sampled data points in each face of the mesh and using a large number of mesh faces.

The aim in this paper is to focus more closely on how this variance-bias tradeoff can be used to exert control over an active terrain model. In particular we consider how to select the optimal local area of each of the faces in the mesh. We present a detailed analysis of variance which shows that the area-dependance has a two-component structure. The first component of variance results from the effects of noise and decreases with increasing facet area. The second variance component results from the model-bias and increases with the area of estimation. As a result of the interplay between these two terms, there is an optimal choice of the facet area that results in a joint minimisation of both the noise variance and the model bias. We develop a semi-empirical model that allows us to compute this optimal area.

### 2 Variance-bias Analysis

Our overall aim is to couple the faces of a triangular mesh to a set of observed height data. Each node of the mesh represents a point at which parameters of surface-patches are sampled. In practice the surface-patches are low-order. For instance, in this paper the surface-model is based on simple planar tangent patches. The mesh density is such that the model fitted to a node has minimum error. This process can be realised using a series of face split and merge operations [10].

In practice however, there is a problem of variancebias tradeoff that hinders the process of estimating the facet model parameters and controlling the mesh in this way.. By increasing the size of the face, the effects of noise variance may be minimised by averaging over a large sample-size. In other words, the temptation is to increase the size of the local surface patches so as to increase the accuracy of the estimated parameters. Unfortunately, as the surface area is increased, then so

<sup>\*</sup>A fuller report of this work is available at  $\rm http://www.cs.york.ac.uk/\simwilson/work.htm$ 

problems of model bias(over-smoothing) emerge. In a nutshell, the problem is that the local model order is insufficient to represent genuine structure in the data.

The basic issue addressed in this paper is how to resolve this dilemma for the specific problem of adaptive mesh fitting.

#### 2.1 Least Squares Tangent Planes

We commence our discussion with a set of 3dimensional data-points  $D = \{p_i | \forall i\}$  derived from range data. In realistic tasks, these points are invariably uncertain in the sense that they deviate from the true surface due to some noise process. In the following, we denote the function of the underlying surface as f(x, y) and the equation of points on this underlying surface is therefore z = f(x, y). The data-point with co-ordinate vector  $p_i = (x_i, y_i, z_i)^T$  is related to the true surface as follows  $z_i = f(x_i, y_i) + n_i$ , where  $n_i$  is the additive noise process. In the case of tangent plane fitting, we must first estimate the average height intercept, i.e.  $f(x_o, y_o)$ . In this new co-ordinate system we can estimate the parameters of the surface patch by performing a least-squares fit of a tangent plane through the origin, i.e. z' = ax' + by'. Suppose that the current face consists of the set of sample datapoints with index-set S. The positions of the sample points are represented by the design matrix

$$X_{p} = \begin{pmatrix} x_{1}' & y_{1}' \\ x_{2}' & y_{2}' \\ \vdots & \vdots \end{pmatrix}$$
(1)

while the corresponding height data is represented by the vector  $Z_p = (z'_1, z'_2, ..)^T$ . With these ingredients, the least-squares fit for the vector of tangent plane parameters  $P = (a, b)^T$  is given by  $\hat{P} = L_p Z_p$ , where  $L_p = (X_p^T X_p)^{-1} X_p^T$  is the pseudo-inverse of the design matrix.

#### 2.2 Analysis of Variance

When the parameter-vector  $\hat{P}$  is estimated in this way, then its covariance structure can be found by propagating the variance in the transformed height data  $Z_p$ . If  $\Sigma_{Z_p}$  is the covariance matrix for the transformed height data, then the covariance matrix for the plane parameters, i.e.  $E[(P - \hat{P})(P - \hat{P})^T]$ , is given by

$$\Sigma_p = L_p \Sigma_{Z_p} L_p^T \tag{2}$$

The total covariance matrix has a two-component structure which reflects the two sources of error in the estimation of the surface normals. The first component is due to the propagation of noise in the surfacedata-point positions, while the second component is a bias term that results from ignoring the higher order terms in the Taylor expansion of the surface function. We make this two-component structure more explicit by writing

$$\Sigma_p = L_p \Sigma_N L_p^T + L_p \Sigma_B L_p^T \tag{3}$$

The noise component of the parameter covariance matrix is modelled under the assumption that the original height data is subject to independent identically distributed Gaussian noise of zero mean and variance  $\sigma^2$ . Under this assumption the noise variance of the least-squares parameter estimates is given by

$$L_p \Sigma_N L_p^T = \sigma^2 \left( \frac{\sum_{i \in S} x_i'^2}{\sum_{i \in S} x_i' y_i'} \frac{\sum_{i \in S} x_i' y_i'}{\sum_{i \in S} y_i'^2} \right)^{-1}$$
(4)

In other words, the noise-component to the total covariance matrix depends on the second-order moments of the points in the surface patch. We assume that the patch parameters are estimated over a square support neighbourhood. As a result, the expectation values of the odd co-ordinate moments are zero. If the density of sampling points is  $\rho$  and the area of the support neighbourhood is A, then it is a straightforward matter to show that the expectation values of the evenmoments are as follows

$$E\left[\sum_{i\in S} x_i^{\prime 2}\right] = E\left[\sum_{i\in S} y_i^{\prime 2}\right] = \frac{\rho A}{12} \tag{5}$$

As a result the noise contribution has a diagonal covariance matrix Specifically,

$$L_p \Sigma_N L_p^T = \frac{12\sigma^2}{\rho A} I \tag{6}$$

where I is the 2x2 identity matrix.

The bias contribution is more complex and depends on the second-order, and higher, derivatives of the local surface. We model the bias term to second-order by computing the covariance matrix for the local deviations from the planar approximation. Accordingly, we write bias-component of the covariance matrix as

$$\Sigma_B = \begin{pmatrix} \epsilon_1 \epsilon_1 & \epsilon_1 \epsilon_2 & \dots & \epsilon_1 \epsilon_n \\ \epsilon_2 \epsilon_1 & \ddots & & \\ \vdots & & & \end{pmatrix}$$
(7)

where  $\epsilon_i = \frac{\partial^2 f}{\partial x^2} (x_i - x_o)^2 + \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} (x_i - x_o) (y_i - y_o) + \frac{\partial^2 f}{\partial y^2} (y_i - y_o)^2 + \dots$  is the non-planar deviation of the point indexed *i*.

Details of the bias model are outside the scope of this paper. Suffice to say that we can compute the expectation values for the elements of the nonplanar bias covariance matrix to second-order, neglecting higher order terms of the Taylor expansion. Under this condition, the bias can be represented as a secondorder polynomial in the patch area A. If  $K_0$ ,  $K_1$  and  $K_2$  represent co-efficient matrices whose elements depend on the second order and higher derivatives of the surface function, then

$$L_p \Sigma_B L_p^T = K_0 + K_1 A + K_2 A^2 + \dots$$
 (8)

As a result the total parameter covariance matrix can be expressed as the following series of area-dependant terms

$$\Sigma_p(A) = \frac{12\sigma^2}{\rho A} I + K_0 + K_1 A + K_2 A^2 + \dots$$
(9)

In other words, the noise propagation term is inversely proportional to the area of the estimating patch. The bias terms, on the other hand, are polynomial in area. As a result the parameter covariance matrix can be minimised with respect to the patch area.

#### 2.3 Optimal Facet Area

The problem of determining the optimal area of estimation is complicated by the fact that we are dealing with a covariance matrix rather than a single scalar quantity. However, since the noise component of  $\Sigma_p$  is diagonal, we confine our attention to minimising the trace of the covariance matrix. To first order in area, the trace is given by

$$Tr[\Sigma_p] = \sigma_a^2 + \sigma_b^2 = 2\left[\left(\frac{12\sigma^2}{\rho A} + k_0 + k_1 A\right)\right]$$
(10)

where  $\sigma_a^2$  and  $\sigma_b^2$  are the measured variances for the plane parameters a and b. This result provides a semiempirical model that we can fit to the observed sum of variances  $\sigma_a^2 + \sigma_b^2$ . In this way we can estimate the semi-empirical model parameters  $k_0$  and  $k_1$ , given knowledge of  $\sigma$ . Once these parameters are to hand, the minimum error surface patch area is given by

$$A^{optimal} = \left[\frac{\rho k_1}{12\sigma^2}\right]^{\frac{1}{2}} \tag{11}$$

We use this result to compute the optimal area for each facet of our triangulated mesh in turn, In the next section, we provide details of how the distribution of mesh facets is controlled.

### 3 Controlling the Mesh

Our overall goal is to use the minimum parametercovariance area to control the split and merge operations. The bias-variance relationship developed in the previous section allows us to fit a semi-empirical model to the computed parameter variances. The strategy that we adopt in determining the optimal local patch area is as follows. For each point on the surface we gradually increase the local patch area and compute the associated parameter variances. This gives a set of data points to which we can fit an appropriate empirical form of the bias-variance curve. The fitted parameters can be used to extract the value of the minimum local patch-area in a stable manner.

Our mesh is based on the Delaunay triangulation of a set of control points or nodes [8, 6, 7, 3]. The basic update process underpinning our surface involves adjusting the mesh-topology by splitting and merging surface-triangles. This process is realised by either inserting or deleting nodes from the mesh. The node insertion and deletion operations take place with the objective of delivering a set of faces whose areas are consistent with the optimal values dictated by the bias-variance criterion outlined in section 2.

The basic aim when reducing mesh density is to merge triangles if the aggregate area is more consistent with the optimal area than the original area. Suppose that the set of triangles  $M_j$  is to be merged to form a new triangle with area  $A_j$ . The average area of the configuration of triangles is

$$A_j^{merge} = \frac{1}{|M_j|} \sum_{i \in M_j} A_i \tag{12}$$

The triangles are merged if the fractional difference between the average area and the optimal area is greater than 10%. In other words, we instantiate the merge if

$$\frac{A_j^{optimal} - A_j^{merge}}{A_j^{optimal}} > 0.1 \tag{13}$$

This tolerancing can be viewed as providing the adaptation of the mesh with a degree of hysteresis.

The split operation proceeds thus: A new node is introduced at the centroid of the original triangle. The new node-set is re-triangulated to update the edge and face sets of the triangulation. The condition for initiating a split operation is that the current fractional difference between the triangle area and it optimal value is greater than 10%. The split condition can therefore be stated as

$$\frac{A_j - A_j^{optimal}}{A_j^{optimal}} > 0.1 \tag{14}$$

#### 4 Experiments

The aim in this section is to illustrate the effectiveness of our adaptive mesh for automatic terrain



Figure 1: Raw terrain height data: The brightness is proportional to the surface height. Note the fine details at the edges of the elevated (i.e. bright) structures.

analysis. The data used for this study is cartographic height data from Salisbury Plain in Wiltshire, England. The raw height data is shown in Figure 1. Here the brightness is proportional to the height of the raw data-points.

The mesh is initialised with its nodes distributed uniformly across the x - y footprint of the height data. Figure 2 shows a perspective view of final mesh configuration overlayed on the rendering of the reconstructed surface. There are several qualitative features which deserve further comment. Firstly, there is a noticeable difference in density of mesh triangles in the basin and upland regions. Secondly, the rilles that encroach into the boundary of the upland are well represented. There is also a localised elevation feature in the basin that is well reconstructed.

# 5 Conclusions

The main contribution in this paper has been to present a simple adaptive surface mesh for terrain modelling. The faces of the mesh represent local tangent planes. We exert control over the mesh using face split and merge operations. These operations are aimed at delivering a mesh in which the faces are consistent with an optimal area criterion. The optimality criterion is based on a tradeoff between noise variance and model-bias.

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Figure 2: The final mesh configuration:Note how the mesh density increases in the vicinity of local surface detail.

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