Techniques for
Private Data Analysis

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Based on joint work with Shiva Kasiviswanathan, Homin Lee, Kobbi Nissim and Adam Smith
Private data analysis

Collections of personal and sensitive data

- census
- medical and public health data
- social networks
- recommendation systems
- trace data: search records, click data
- intrusion-detection
Meta Question

What information can be released?

- Two conflicting goals
  - utility: users can extract ”global” statistics
  - privacy: individual information stays hidden
Related work

Other fields: huge amount of work

• in statistics (statistical disclosure limitation)
• in data mining (privacy-preserving data mining)
• largely: no precise privacy definition
  (only security against specific attacks)

In cryptography (private data analysis)

• [Dinur Nissim 03, Dwork Nissim 04,
  Chawla Dwork McSherry Smith Wee 05,
  Blum Dwork McSherry Nissim 05,
  Chawla Dwork McSherry Talwar 05,
  Dwork McSherry Nissim Smith 06, ...]

• rigorous privacy guarantees
**Differential privacy** [DMNS06]

*Intuition:* Users learn the same thing about me whether or not I participate in the census.

Two databases are *neighbors* if they differ in one row (arbitrarily complex information supplied by one person).

\[
x = \begin{array}{c}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n \\
\end{array}
\]

\[
x' = \begin{array}{c}
  x_1' \\
  x_2' \\
  \vdots \\
  x_n' \\
\end{array}
\]

---

**Privacy definition**

Algorithm $A$ is $\varepsilon$-differentially private if

- for all neighbor databases $x, x'$
- for all sets of answers $S$

\[
\Pr[A(x) \in S] \leq (1 + \varepsilon) \cdot \Pr[A(x') \in S]
\]
Properties of differential privacy

- $\varepsilon$ is non-negligible (at least $\frac{1}{n}$).

- **Composition:** If $A_1$ and $A_2$ are $\varepsilon$-differentially private, then $(A_1, A_2)$ is $2\varepsilon$-differentially private.

- robust in the presence of arbitrary side information.
What can we compute privately?

Research so far:

- **Definitions** [DiNi,DwNi,EGS,DMNS,DwNa,DKMMN,GKS]

- **Function approximation**
  - Protocols [DiNi,DwNi,BDMN,DMNS,NRS,BCDKMT]
  - Impossibility results [DiNi,DMNS,DwNa,DwMT,DwY]
  - Distributed protocols [DKMMN,BNiO]

- **Mechanism design** [McSherry Talwar 07]

- **Learning** [Blum Dwork McSherry Nissim 05, KLNRS08]

- **Releasing classes of functions** [Blum Ligett Roth 08]

- **Synthetic data** [Machanavajjhala Kifer Abowd Gehrke Vilhuber 08]
Road map

I. Function approximation

- Global sensitivity framework [DMNS06]
- Smooth sensitivity framework [NRS07]
- Sample-and-aggregate [NRS07]

II. Learning

- Exponential mechanism [MT07,KLNRS08]
For which functions $f$ can we have:

- **privacy**: differential privacy [DMNS06]
- **utility**: output $A(x)$ is close to $f(x)$
Global sensitivity framework [DMNS06]

**Intuition:** $f$ can be released accurately when it is insensitive to individual entries $x_1, \ldots, x_n$.

Global sensitivity $\text{GS}_f = \max_{\text{neighbors } x, x'} \| f(x) - f(x') \|_1$.

**Example:** $\text{GS}_{\text{average}} = \frac{1}{n}$ if $x \in [0, 1]^n$.

---

**Theorem**

If $A(x) = f(x) + \text{Lap}\left( \frac{\text{GS}_f}{\varepsilon} \right)$ then $A$ is $\varepsilon$-diff. private.
Global sensitivity framework \[\text{[DMNS06]}\]

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**Compare to:** Estimating frequencies (e.g., proportion of people with blue eyes) from $n$ samples: sampling error $\frac{1}{\sqrt{n}}$.

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Functions with low global sensitivity

- Means, variances for data in a bounded interval
- Histograms, contingency tables
- Singular value decomposition
Instance-Based Noise

*Big picture for global sensitivity framework:*

- add enough noise to cover the worst case for $f$
- noise distribution depends only on $f$, not database $x$

*Problem:* for some functions that’s too much noise

*Smooth sensitivity framework* [Nissim Smith Raskhodnikova 07]:

noise tuned to database $x$
Local sensitivity

Local sensitivity $\text{LS}_f(x) = \max_{x': \text{neighbor of } x} \| f(x) - f(x') \|$ 

**Reminder:** $\text{GS}_f = \max_x \text{LS}_f(x)$

**Example:** median for $0 \leq x_1 \leq \cdots \leq x_n \leq 1$, odd $n$

$$\text{median} = \max(x_{m-1}, x_{m+1})$$

**Goal:** Release $f(x)$ with less noise when $\text{LS}_f(x)$ is lower.
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\[ \text{LS}_{\text{median}}(x) = \max(x_m - x_{m-1}, x_{m+1} - x_m) \]

Goal: Release $f(x)$ with less noise when $\text{LS}_f(x)$ is lower.
Instance-based noise: first attempt

Noise magnitude proportional to $LS_f(x)$ instead of $GS_f$?

No! Noise magnitude reveals information.

Lesson: Noise magnitude must be an insensitive function.
Smooth bounds on local sensitivity

Design sensitivity function \( S(x) \)

- \( S(x) \) is an \( \varepsilon \)-smooth upper bound on \( \text{LS}_f(x) \) if:
  - for all \( x \): \( S(x) \geq \text{LS}_f(x) \)
  - for all neighbors \( x, x' \): \( S(x) \leq e^\varepsilon S(x') \)

**Theorem**

If \( A(x) = f(x) + \text{noise} \left( \frac{S(x)}{\varepsilon} \right) \) then \( A \) is \( \varepsilon' \)-differentially private.

*Example:* \( \text{GS}_f \) is always a smooth bound on \( \text{LS}_f(x) \)
**Smooth bounds on local sensitivity**

Design sensitivity function $S(x)$

- $S(x)$ is an $\varepsilon$-smooth upper bound on $\text{LS}_f(x)$ if:
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**Example:** $\text{GS}_f$ is always a smooth bound on $\text{LS}_f(x)$
Smooth Sensitivity

Smooth sensitivity $S^*_f(x) = \max_y (LS_f(y)e^{-\varepsilon \cdot \text{dist}(x,y)})$

Lemma

For every $\varepsilon$-smooth bound $S$: $S^*_f(x) \leq S(x)$ for all $x$.

Intuition: little noise when far from sensitive instances
Smooth Sensitivity

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**Lemma**

For every $\varepsilon$-smooth bound $S$: $S^*_f(x) \leq S(x)$ for all $x$.

**Intuition:** little noise when far from sensitive instances
Computing smooth sensitivity

Example functions with computable smooth sensitivity

- Median & minimum of numbers in a bounded interval
- MST cost when weights are bounded
- Number of triangles in a graph

Approximating smooth sensitivity

- only smooth upper bounds on LS are meaningful
- simple generic methods for smooth approximations
  - work for median and 1-median in $L_1^d$
Road map

I. Function approximation
   • Global sensitivity framework [DMNS06]
   • Smooth sensitivity framework [NRS07]
   • Sample-and-aggregate [NRS07]

II. Learning
   • Exponential mechanism [MT07, KLNRS08]
New goal

• Smooth sensitivity framework requires understanding combinatorial structure of $f$
  – hard in general

• **Goal**: an automatable transformation from an arbitrary $f$ into an $\varepsilon$-diff. private $A$
  – $A(x) \approx f(x)$ for ”good” instances $x$
Example: cluster centers

Database entries: points in a metric space.

- Comparing sets of centers: Earthmover-like metric
- Global sensitivity of cluster centers is roughly the diameter of the space. But intuitively, if clustering is "good", cluster centers should be insensitive.
- No efficient approximation for smooth sensitivity
**Example: cluster centers**

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Sample-and-aggregate framework

Intuition: Replace $f$ with a less sensitive function $\tilde{f}$.

$$\tilde{f}(x) = g(f(sample_1), f(sample_2), \ldots, f(sample_s))$$
Sample-and-aggregate framework

**Intuition:** Replace $f$ with a less sensitive function $\tilde{f}$.

\[
\tilde{f}(x) = g(f(sample_1), f(sample_2), \ldots, f(sample_s))
\]
Good aggregation functions

- **average**
  - works for $L_1$ and $L_2$

- **center of attention**
  - the center of a smallest ball containing a strict majority of input points
  - works for arbitrary metrics
    (in particular, for Earthmover)
  - gives lower noise for $L_1$ and $L_2$
Sample-and-aggregate method

**Theorem**

If $f$ can be approximated *on* $x$

from small samples

then $f$ can be released with little noise
Sample-and-aggregate method

Theorem

If $f$ can be approximated on $x$ within distance $r$

from small samples of size $n^{1-\delta}$

then $f$ can be released with little noise $\approx \frac{r}{\varepsilon} + \text{negl}(n)$
Sample-and-aggregate method

**Theorem**

If $f$ can be approximated on $x$ within distance $r$ from small samples of size $n^{1-\delta}$ then $f$ can be released with little noise $\approx \frac{r}{\varepsilon} + \text{negl}(n)$

- Works in all ”interesting” metric spaces
- Example applications
  - $k$-means cluster centers (if data is separated a.k.a. [Ostrovsky Rabani Schulman Swamy 06])
  - fitting mixtures of Gaussians (if data is i.i.d., using [Achlioptas McSherry 05])
  - PAC concepts (if uniquely learnable, i.e., if learning algorithm always outputs the same hypothesis or something close to it)
Road map

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Learning: the setting

Bank needs to decide which applicants are bad credit risks*

Goal: given sample of labeled data (past customers), produce good prediction rule (hypothesis) for future loan applicants

*Example taken from Blum, FOCS03 tutorial
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Example $y_i$: % down = 10, high debt = No, other accts = Yes, mmp/inc = 0.32, good risk? = Yes

Label $z_i$: % down = 10, high debt = No, other accts = No, mmp/inc = 0.25, good risk? = Yes

Reasonable rules given this data:

- Predict YES iff $100 \times \frac{\text{mmp}}{\text{inc}} - (\% \text{ down}) < 25$
- Predict YES iff (!high debt) AND (\% down > 5)

*Example taken from Blum, FOCS03 tutorial
Learning: the setting
Learning: the setting

- Examples drawn according to distribution $\mathcal{D}$
• Examples drawn according to distribution $\mathcal{D}$
Learning: the setting

- Examples drawn according to distribution $\mathcal{D}$
- A point drawn according to $\mathcal{D}$ has to be classified correctly w.h.p. (over learner randomness and $\mathcal{D}$)
PAC learning [Valiant 84]

Given distribution $\mathcal{D}$ over examples, labeled by function $c$, hypothesis $h$ is good if it mostly agrees with $c$:

$$\Pr_{y \sim \mathcal{D}}[h(y) = c(y)] \text{ is close to } 1.$$
PAC learning [Valiant 84]

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$$\Pr_{y \sim \mathcal{D}}[h(y) = c(y)]$$ is close to 1.

**Definition of PAC learning**

Algorithm $A$ PAC learns a concept class $C$ if

- given polynomially many examples, drawn from $\mathcal{D}$, labeled by some $c \in C$
- $A$ outputs a *good* hypothesis with high probability in polynomial time
**PAC learning** [Valiant 84]

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---

**Definition of PAC* learning**

*Algorithm A PAC* learns a concept class $C$ if

- given polynomially many examples, drawn from $\mathcal{D}$, labeled by some $c \in C$

- $A$ outputs a *good* hypothesis of polynomial length with high probability in polynomial time
Private learning

Input: database \( x = (x_1, \ldots, x_n) \)

\( x_i = (y_i, z_i) \), where \( y_i \sim D \) and \( z_i = c(y_i) \) is the label of example \( y_i \)

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Output: hypothesis

e.g.

“Predict Yes if $100 \times \frac{\text{mmp}}{\text{inc}} - \text{(\% down)} < 25$”
Private learning

Input: database $x = (x_1, ..., x_n)$

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Output: hypothesis

e.g.

“Predict **Yes** if

$$100 \times \frac{\text{mmp}}{\text{inc}} - \frac{\% \text{ down}}{25}$$

Definition

*Algorithm A* privately learns *concept class C* if

- **Utility**: *Algorithm A* PAC learns class *C*
- **Privacy**: *Algorithm A* is differentially private
Private learning

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Output: hypothesis

"Predict Yes if \( 100 \times \frac{\text{mmp}}{\text{inc}} - \text{% down} < 25 \)"

Definition

Algorithm A privately learns concept class \( C \) if

- **Utility:** Algorithm A PAC learns class \( C \) (average-case)
- **Privacy:** Algorithm A is differentially private (worst-case)
Designing private learners: baby steps

View non-private learner as function to be approximated

- **First attempt**: add noise
  - **Problem**: Close hypothesis may mislabel many points
Designing private learners: baby steps

View non-private learner as function to be approximated

- **First attempt**: add noise
  - **Problem**: Close hypothesis may mislabel many points

- **Second attempt**:
  apply sample-and-aggregate to non-private learning algorithm
  - Works when good hypothesis is essentially unique
  - **Problem**: there may be many good hypotheses – different samples may produce different-looking hypotheses
\[ \text{PAC}^* = \text{Private PAC}^* \]

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<th>Theorem (Private analogue of “Occam’s razor”)</th>
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**Theorem (Private analogue of “Occam’s razor”)**

Each \(\text{PAC}^*\) learnable concept class can be learned privately, using polynomially many samples.

**Proof:** Adapt the exponential mechanism of [MT07]:

\[ \text{score}(x, h) = \# \text{ of examples in } x \text{ correctly classified by } h \]
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- Output \( h \) from \( C \) with probability \( \sim e^{\epsilon \cdot \text{score}(x, h)} \)
  - may take exponential time

\[
\text{score} = 4
\]

\[
\text{red dot, blue dot, blue dot}
\]
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**Privacy:** for any hypothesis \( h \):

\[
\frac{\Pr[h \text{ is output on input } x]}{\Pr[h \text{ is output on input } x']} = \frac{e^{\epsilon \cdot \text{score}(x,h)}}{e^{\epsilon \cdot \text{score}(x',h)}} \leq e^{\epsilon}
\]
PAC\* = Private PAC\*

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**Utility (learning):**
Theorem \text{(Private analogue of “Occam’s razor”)}

Each PAC* learnable concept class can be learned privately, using polynomially many samples.

\textbf{Proof:} \quad \text{score}(x, h) = \# \text{ of examples in } x \text{ correctly classified by } h \\
\quad \quad \quad \quad \bullet \text{ Output } h \text{ from } C \text{ with probability } \sim e^{\varepsilon \cdot \text{score}(x, h)}

\textbf{Utility (learning):} \\
\textbf{Good} \quad h \text{ correctly label all examples: } \Pr[h] \sim e^{\varepsilon \cdot n} \\
\textbf{Bad} \quad h \text{ mislabel } \geq 10\% \text{ of examples: } \Pr[h] \sim e^{\varepsilon \cdot 0.9n}
**PAC\(^{*}\) = Private PAC\(^{*}\)**

---

**Theorem (Private analogue of “Occam’s razor”)**

Each PAC\(^{*}\) learnable concept class can be learned privately, using polynomially many samples.

**Proof:**

- \( \text{score}(x, h) = \# \text{ of examples in } x \text{ correctly classified by } h \)
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Sufficient to ensure \( n \gg \log(\# \text{ bad hypotheses}) = \text{ polynomial} \)

Then w.h.p. output \( h \) labels 90\% of examples correctly.
**PAC** = **Private PAC**

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Sufficient to ensure \( n \gg \log(\# \text{ bad hypotheses}) = \text{polynomial} \)

Then w.h.p. output \( h \) labels 90% of examples correctly.

By ”Occam’s razor”, if \( n \gg \log(\# \text{ hypotheses}) \), then \( h \) does well on examples \( \implies h \) does well on distrib. \( \mathcal{D} \)


Road map

I. Function approximation

• Global sensitivity framework [DMNS06]
• Smooth sensitivity framework [NRS07]
• Sample-and-aggregate [NRS07]

II. Learning

• Exponential mechanism [MT07, KLNRS08]
Conclusions

This talk: partial picture of techniques

• current techniques best for
  – function approximation
  – learning

• New ideas needed for
  – combinatorial search problems
  – text processing
  – graph data (definitions?)
  – high-dimensional outputs