A graph polynomial for independent sets of bipartite graphs

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EaGL III
Graph Polynomials

- Algebraic method for analyzing properties of graph
- Some graph polynomials
  - Chromatic polynomial
  - Reliability polynomial
  - Tutte polynomial
Tutte polynomial

- A polynomial in two variables
- Random cluster model

\[ Z(G; q, \mu) = \sum_{S \subseteq E} q^{\kappa(S)} \mu^{|S|} \]

- sum over weighted subgraphs \((V, S)\)
- each edge has weight \(\mu\)
- each connected component has weight \(q\)
Tutte polynomial (cont.)

\[ Z(C_3; q, \mu) = q^3 + 3\mu q^2 + (\mu^3 + 3\mu^2)q \]
A new graph polynomial $R_2$

- The $R_2$ polynomial of a graph $G = (V, E)$
  \[ R_2(G; q, \mu) = \sum_{S \subseteq E} q^{\text{rk}_2(S)} \mu^{|S|} \]

- sum over weighted subgraphs $(V, S)$

- each edge has weight $\mu$

- $\text{rk}_2(S)$ is the rank of the adjacency matrix of $(V, S)$ over $\mathbb{F}_2$
A new graph polynomial $R_2$ (cont.)

$C_3$

$R_2(C_3; q, \mu) = (\mu^3 + 3\mu^2 + 3\mu)q^2 + 1$
A new graph polynomial $R_2$ (cont.)

\[
R_2(G; q, \mu) = \sum_{S \subseteq E} q^{rk_2(S)} \mu^{|S|}
\]

- Properties of the $R_2$ polynomial
  - number of matchings \( P(G; 0) \)
    \[
P(G; \mu) = R_2(G; \mu^{-1/2}, \mu) = \sum_{S \subseteq E} \mu^{|S|-\frac{rk_2(S)}{2}}
    \]
  - number of perfect matchings \( P_2(G; 0) \)
    \[
P_1(G; t, \mu) = t^{|V|} R_2(G; 1/t, \mu)
    \quad P_2(G; \mu) = \mu^{-|V|/2} P_1(G; 0, \mu)
    \]
  - number of independent sets if the graph is bipartite
The $R'_2$ polynomial of a bipartite graph $G = (U \cup W, E)$

$$
R'_2(G; \lambda, \mu) = \sum_{S \subseteq E} \lambda^{|S|} \mu^{r_{k_2}(S)} |S|
$$

- each edge has weight $\mu$
- $r_{k_2}(S)$ is the rank of the bipartite adjacency matrix of $(U \cup W, S)$ over $\mathbb{F}_2$
Polynomial $R'_2$ (cont.)

$K_{2,2}$

\[ R'_2(K_{2,2}; \lambda, \mu) = 1 + \lambda \mu^4 + 2\lambda^2 \mu^2 + 4\lambda \mu^2 + 4\lambda^2 \mu^3 + 4\lambda \mu \]
Polynomial $R'_2$ (cont.)

- For bipartite graphs
  \[ R'_2(G; \lambda, \mu) = R'_2(G; \lambda^2, \mu) \]

- (Main Theorem) Let $G = (U \cup W, E)$ be a bipartite graph. The number of independent sets of $G$ is given by
  \[ 2^{|U| + |W| - |E|} R'_2(G; 1/2, 1) \]
Main Theorem

\[ A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \]

\[ Q = \{ (u, B) \mid u \in \mathbb{F}_2^2, B \in M_{2,2} (\mathbb{F}_2), B \leq A, u^T B \equiv 0 \pmod{2} \} \]

\[ u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \in \ Q \]

\[ u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \notin \ Q \]
Main Theorem

\[ Q = \{ (u, B) \mid u \in \mathbb{F}_2^2, B \in M_{2,2}(\mathbb{F}_2), B \leq A, u^T B \equiv 0 \pmod{2} \} \]

\[ A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \]

\[ B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \]

\[ \mathbb{P}(B) = \sum_{u: u^T B \equiv 0 \pmod{2}} \frac{1}{|Q|} = \frac{2^{n_1 - \text{rk}_2(B)}}{|Q|} = \frac{2^{2 - \text{rk}_2(B)}}{R_2'(G; 1/2, 1)} \]
Main Theorem

\[ A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \]

\[ Q = \{(u, B) \mid u \in \mathbb{F}_2^2, B \in M_{2,2}(\mathbb{F}_2), B \leq A, u^T B \equiv 0 \pmod{2} \} \]

\[ \mathbb{P}(u) = \sum_{B: u^T B \equiv 0 \pmod{2}} \frac{1}{|Q|} = \frac{2\#1(A)-(n_2-k)}{|Q|} = \frac{2^k}{\#BIS(G)} \]
Main Theorem

\[ A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \]

\[ Q = \{ (u, B) \mid u \in \mathbb{F}_2^2, B \in M_{2,2}(\mathbb{F}_2), B \leq A, u^T B \equiv 0 \pmod{2} \} \]

\[ \mathbb{P}(B) = \sum_{u : u^T B \equiv 0 \pmod{2}} \frac{1}{|Q|} = \frac{2^{n_1 - \text{rk}_2(B)}}{|Q|} = \frac{2^{-\text{rk}_2(B)}}{R'_2(G; 1/2, 1)} \]

\[ \mathbb{P}(u) = \sum_{B : u^T B \equiv 0 \pmod{2}} \frac{1}{|Q|} = \frac{2^{\#_1(A) - (n_2 - k)}}{|Q|} = \frac{2^k}{\# \text{BIS}(G)} \]

\[ |Q| = 2^{n_1} R'_2(G; 1/2, 1) = 2^{\#_1(A) - n_2} \# \text{BIS}(G) \]
Complexity of exact evaluation of $R'_2$

- **FP**
  - $\lambda \in \{0, 1\}$
  - $\mu = 0$
  - $(\lambda, \mu) = (1/2, -1)$

- **#P-hard**
  - $\lambda = 1/2$ and $\mu \notin \{0, -1\}$

- **#P-hard assuming GRH**
  - $\lambda \notin \{0, 1, 1/2\}$ and $\mu \neq 0$

$$R'_2(G; \lambda, \mu) = \sum_{S \subseteq E} \lambda^{\text{rk}_2(S)} \mu^{|S|}$$
Approximate sampling problem

- Rank Weighted Subgraphs $\text{RWS}(\lambda, \mu)$
- instance: a bipartite graph $G = (U \cup W, E)$
- output: $S \subseteq E$ with probability

$$\mathbb{P}(S) \propto \lambda^{\text{rk}_2(S)} \mu^{|S|}$$
Sing bond flip chain for \( \text{RWS}(\lambda, \mu) \)

1. pick an edge \( e \in E \) at random
2. let \( S = X_t \oplus \{e\} \)
3. set \( X_{t+1} = S \) with probability
   \[
   (1/2) \min\{1, \lambda^\frac{rk_2(S) - rk_2(X_t)}{\mu |S| - |X_t|} \}
   \]
   and \( X_{t+1} = X_t \) with the remaining probability

- Markov chain Monte Carlo method
- proper Markov chain
- mixing time
Question: for which classes of bipartite graphs does the single bond flip chain mix (in polynomial time)?

- The single bond flip chain for trees mixes in polynomial time
  - $r_{k^2}$ is the size of the maximum matching
  - Mixing time can be bounded by using the canonical paths method
Question: for which classes of bipartite graphs does the single bond flip chain mix (in polynomial time)?

- There exists bipartite graphs for which the single bond flip chain needs exponential time to mix for $\lambda = 1/2$ and $\mu = 1$ [Goldberg & Jerrum 2010]

- How about grids, planar graphs?
Thanks !