A graph polynomial for independent sets of bipartite graphs

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joint work with Daniel Štefankovič

EaGL III

Graph Polynomials

- Algebraic method for analyzing properties of graph
- Some graph polynomials
  - Chromatic polynomial
  - Reliability polynomial
  - Tutte polynomial

Tutte polynomial

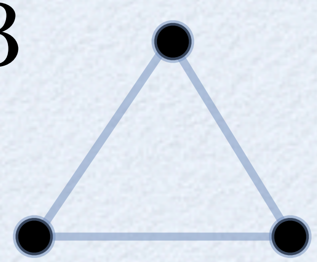
- A polynomial in two variables
- Random cluster model

$$Z(G; q, \mu) = \sum_{S \subseteq E} q^{\kappa(S)} \mu^{|S|}$$

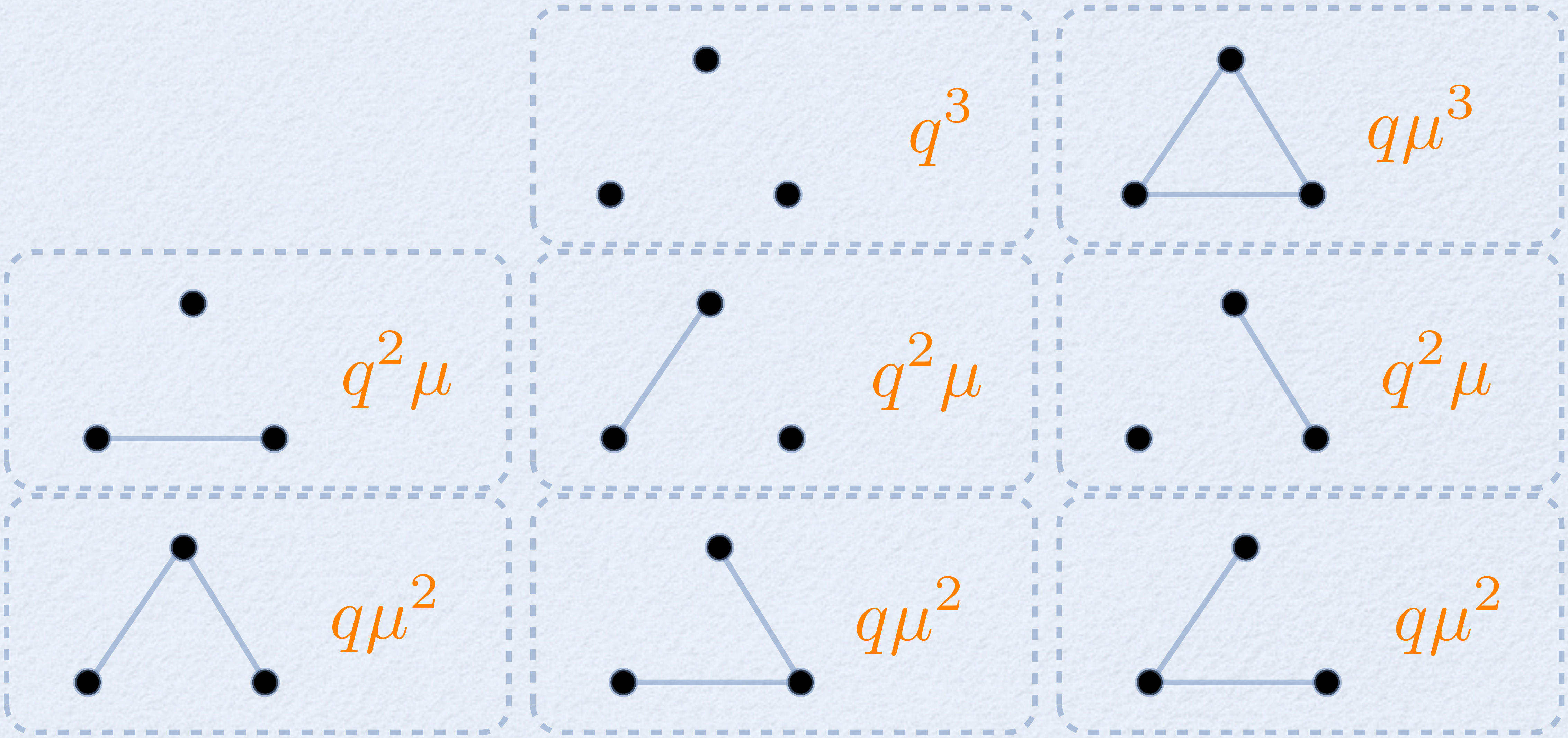
  - sum over weighted subgraphs (V, S)
  - each edge has weight μ
  - each connected component has weight q

Tutte polynomial (cont.)

C_3



$$Z(C_3; q, \mu) = q^3 + 3\mu q^2 + (\mu^3 + 3\mu^2)q$$



A new graph polynomial R_2

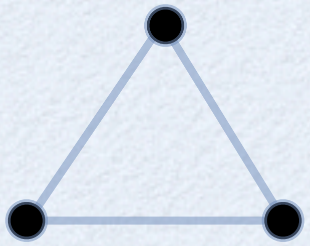
- The R_2 polynomial of a graph $G = (V, E)$

$$R_2(G; q, \mu) = \sum_{S \subseteq E} q^{\mathrm{rk}_2(S)} \mu^{|S|}$$

- sum over weighted subgraphs (V, S)
- each edge has weight μ
- $\mathrm{rk}_2(S)$ is the rank of the adjacency matrix of (V, S) over $\mathbb{F}_2$

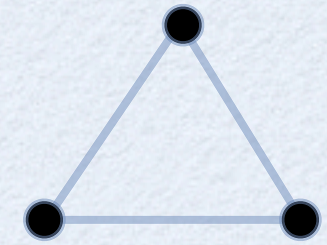
A new graph polynomial R_2 (cont.)

C_3

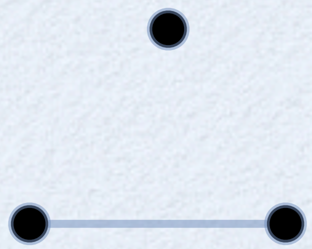


$$R_2(C_3; q, \mu) = (\mu^3 + 3\mu^2 + 3\mu)q^2 + 1$$

$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ 1



$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ $q^2\mu^3$



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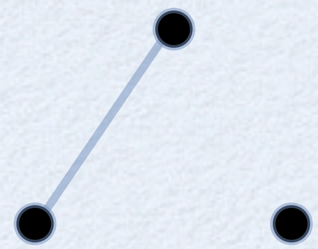
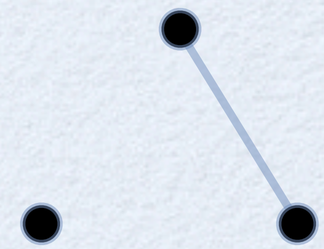
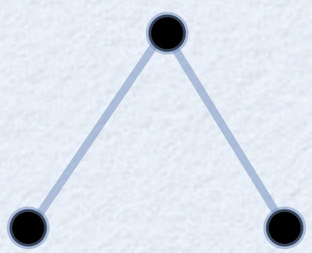


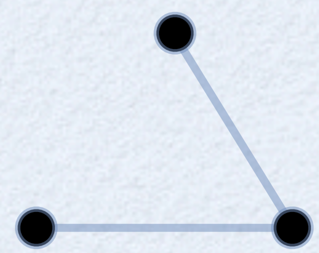
$q^2\mu$



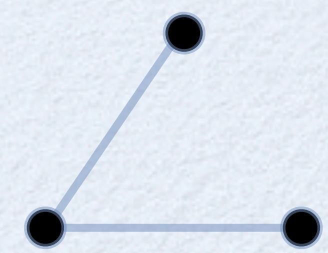
$q^2\mu$



$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $q^2\mu^2$



$q^2\mu^2$



$q^2\mu^2$

A new graph polynomial R_2 (cont.)

$$R_2(G;q,\mu) = \sum_{S\subseteq E} q^{\mathrm{rk}_2(S)} \mu^{|S|}$$

- Properties of the R_2 polynomial
  - number of matchings $P(G;0)$

$$P(G;\mu) = R_2(G;\mu^{-1/2},\mu) = \sum_{S\subseteq E} \mu^{|S|-\mathrm{rk}_2(S)/2}$$

  - number of perfect matchings $P_2(G;0)$

$$P_1(G;t,\mu) = t^{|V|} R_2(G;1/t,\mu) \qquad P_2(G;\mu) = \mu^{-|V|/2} P_1(G;0,\mu)$$

  - number of independent sets if the graph is bipartite

Polynomial R_2'

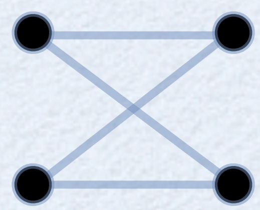
- The R_2' polynomial of a bipartite graph $G = (U \cup W, E)$

$$R_2'(G; \lambda, \mu) = \sum_{S \subseteq E} \lambda^{\mathrm{rk}_2(S)} \mu^{|S|}$$

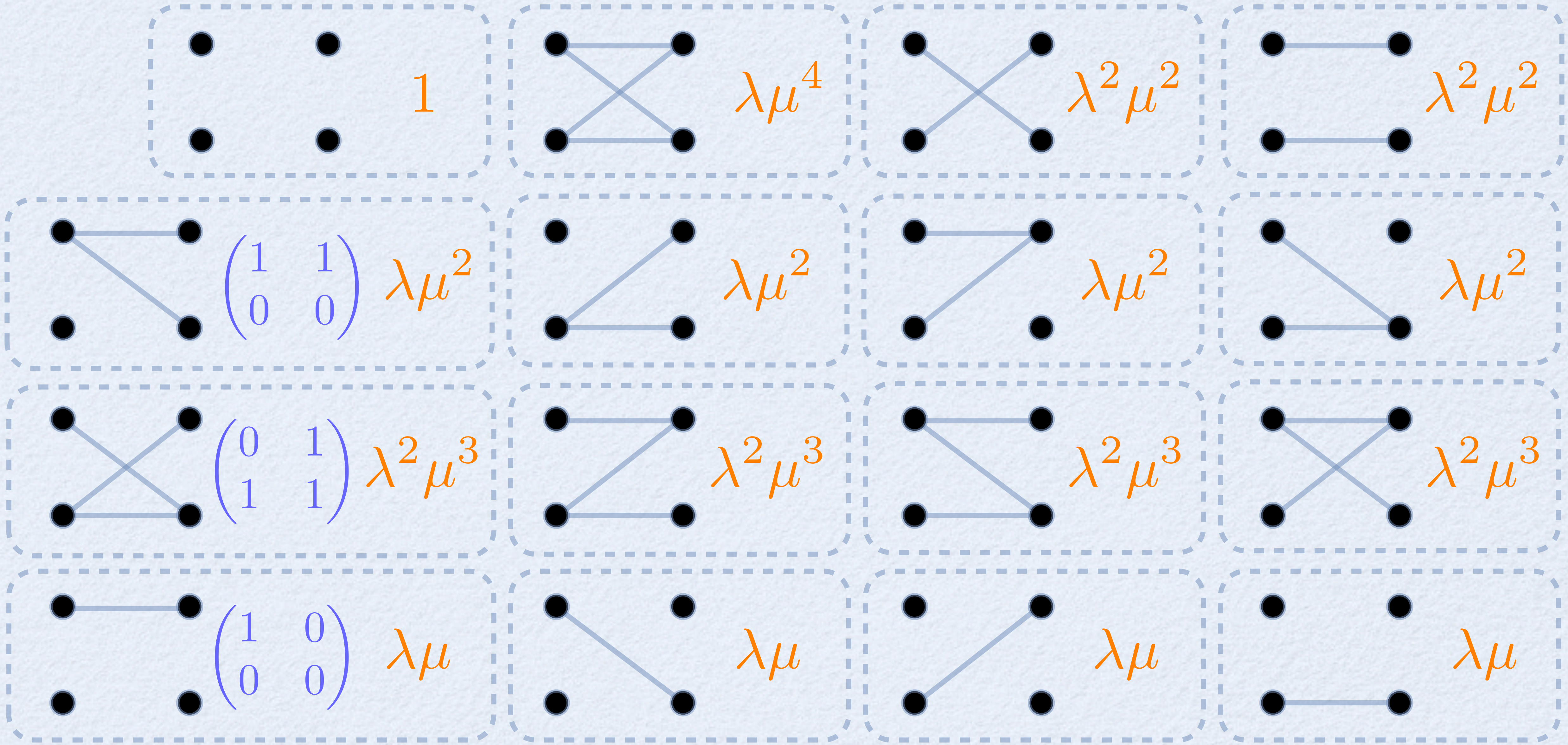
  - each edge has weight μ
  - $\mathrm{rk}_2(S)$ is the rank of the bipartite adjacency matrix of $(U \cup W, S)$ over $\mathbb{F}_2$

Polynomial R'_2 (cont.)

$K_{2,2}$



$$R'_2(K_{2,2};\lambda,\mu) = 1 + \lambda\mu^4 + 2\lambda^2\mu^2 + 4\lambda\mu^2 + 4\lambda^2\mu^3 + 4\lambda\mu$$



Polynomial R'_2 (cont.)

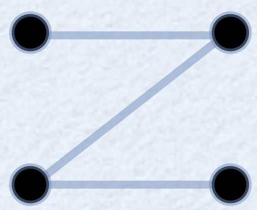
- For bipartite graphs

$$R_2(G;\lambda,\mu) = R'_2(G;\lambda^2,\mu)$$

- (Main Theorem) Let $G = (U \cup W, E)$ be a bipartite graph. The number of independent sets of G is given by

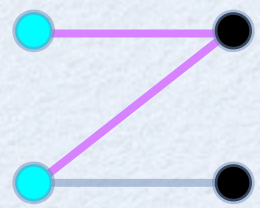
$$2^{|U|+|W|-|E|} R'_2(G;1/2,1)$$

Main Theorem

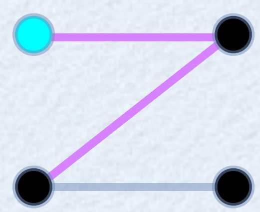


$$Q = \{(u, B) \mid u \in \mathbb{F}_2^2, B \in M_{2,2}(\mathbb{F}_2), B \leq A, u^{\mathrm{T}} B \equiv 0 \pmod 2\}$$

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

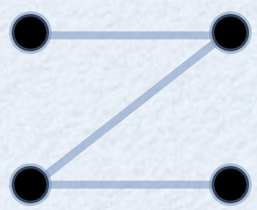


$$u = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \qquad \in Q$$



$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \qquad \notin Q$$

Main Theorem



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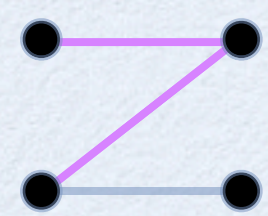
$\mathbb{P}(B)$?

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$



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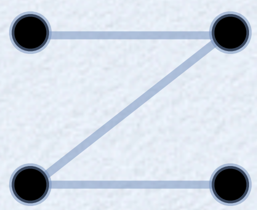
$$\mathbb{P}(B) = \sum_{u : u^{\mathrm{T}} B \equiv 0 \pmod 2} \frac{1}{|Q|} = \frac{2^{n_1 - \mathrm{rk}_2(B)}}{|Q|} = \frac{2^{-\mathrm{rk}_2(B)}}{R_2'(G; 1/2, 1)}$$

Main Theorem



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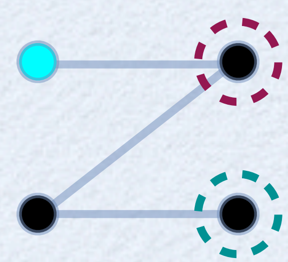
$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$\mathbb{P}(u)$?

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



$$B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

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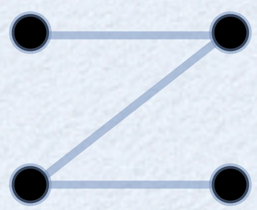
$$\mathbb{P}(u) = \sum_{B : u^{\mathrm{T}} B \equiv 0 \pmod 2} \frac{1}{|Q|} = \frac{2^{\#_1(A) - (n_2 - k)}}{|Q|} = \frac{2^k}{\#BIS(G)}$$

Main Theorem



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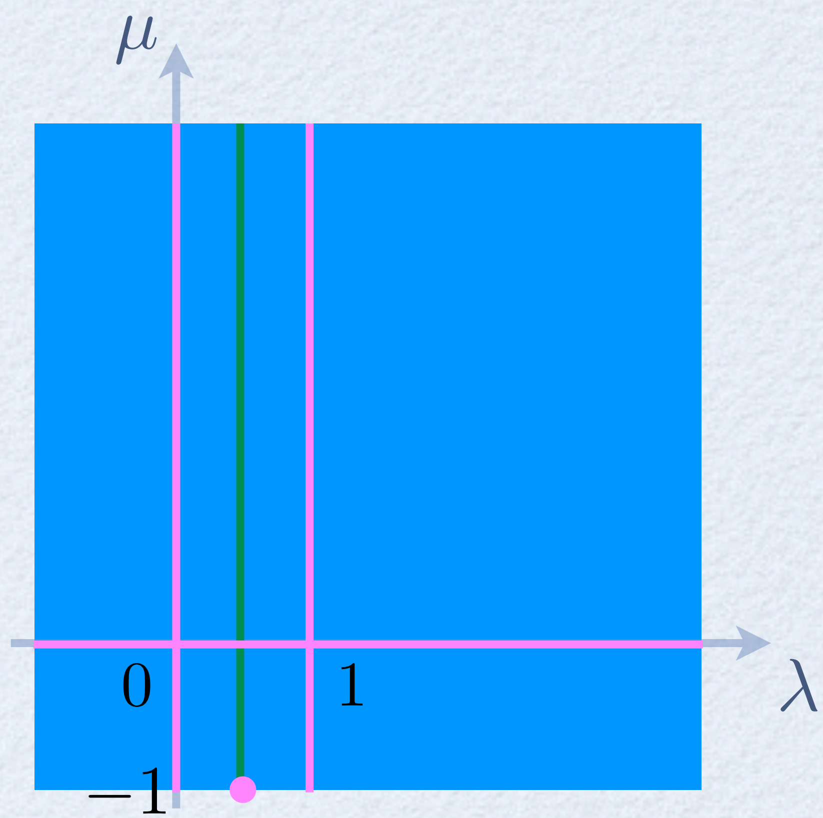
$$|Q| = 2^{n_1} R_2'(G; 1/2, 1) = 2^{\#_1(A) - n_2} \#BIS(G)$$

Complexity of exact evaluation of R'_2

$$R'_2(G;\lambda,\mu) = \sum_{S \subseteq E} \lambda^{\mathrm{rk}_2(S)} \mu^{|S|}$$

- FP
  - $\lambda \in \{0,1\}$
  - $\mu = 0$
  - $(\lambda,\mu) = (1/2,-1)$
- #P-hard
  - $\lambda = 1/2$ and $\mu \notin \{0,-1\}$
- #P-hard assuming GRH
  - $\lambda \notin \{0,1,1/2\}$ and $\mu \neq 0$



Approximate sampling problem

- Rank Weighted Subgraphs $\mathrm{RWS}(\lambda, \mu)$
  - instance: a bipartite graph $G = (U \cup W, E)$
  - output: $S \subseteq E$ with probability

$$\mathbb{P}(S) \propto \lambda^{\mathrm{rk}_2(S)} \mu^{|S|}$$

$\mathrm{RWS}(\lambda, \mu)$

- Markov chain Monte Carlo method
  - proper Markov chain
  - mixing time

Sing bond flip chain for $\mathrm{RWS}(\lambda, \mu)$

1. pick an edge $e \in E$ at random
2. let $S = X_t \oplus \{e\}$
3. set $X_{t+1} = S$ with probability

$$(1/2)\min\{1, \lambda^{\mathrm{rk}_2(S)-\mathrm{rk}_2(X_t)}\mu^{|S|-|X_t|}\}$$

and $X_{t+1} = X_t$ with the remaining probability

$\mathrm{RWS}(\lambda, \mu)$ (cont.)

Question: for which classes of bipartite graphs does the single bond flip chain mix (in polynomial time)?

- The single bond flip chain for trees mixes in polynomial time
  - rk_2 is the size of the maximum matching
  - mixing time can be bounded by using the canonical paths method

$\mathrm{RWS}(\lambda, \mu)$ (cont.)

Question: for which classes of bipartite graphs does the single bond flip chain mix (in polynomial time)?

- There exists bipartite graphs for which the single bond flip chain needs exponential time to mix for $\lambda = 1/2$ and $\mu = 1$ [Goldberg & Jerrum 2010]
- How about grids, planar graphs?

Thanks !