1 D-Disjunction Matrix

Definition 1 A d-disjunct matrix is a matrix for which $\forall S \subseteq [n], |S| \leq d, \forall j \in S, \exists i$ s.t. $M_{ij} = 1$ and $\bigcup_{k \in S} M_{ik} = 0$.

Please note that all matrices noted in this lecture are binary.

$$\Omega(d \log n) \leq t(d, n) \leq n$$

(i) Strongly explicit: $t(d, n) \leq O(d_2 \log_2 n)$, (ii) Randomized: $t(d, n) \leq O(d_2 \log_2 n)$

Lemma 1 Where $d \leq d \leq n$, let $M$ be a tnn matrix, (i) $\forall j \in [n], |M_j| \geq w_{\min}$ and (ii) $\forall i \neq j \in [n] |M_i \cap M_j| \leq ?$ for some integers $a_{\max} \leq w_{\min} \leq t$, where $a$ stands for agreement and $w$ stands for weight. Then $M$ is $\left\lfloor \frac{w_{\min} - 1}{a_{\max}} \right\rfloor$-disjunct.

This is stronger than simply having a subset of size $B$, this is saying for a pair of columns. Therefore no matter what column $i$ you choose in the matrix, that column will contain at least $w_{\min}$ 1s, and the total number of 1s shared by two columns is at most $a_{\max}$.

Example 1 Fix an arbitrary $S \subseteq [n], |S| \leq d, j \notin S$, and each column has a different arrangement of 1s in each column. For a value in column $i$ that is equal to 1, there is a match if there exists a 1 in another column $j$. In this case the total number of matches $\leq a_{\max} \cdot d \leq a_{\max} \left( \frac{w_{\min}}{a_{\max}} \right) = w_{\min} - 1 < w_{\min}$. Therefore there must be an all 0 row in $S$.

Step 1: $|C| = n, C \subseteq \{0, 1\}_t$, and $C = \{\bar{c}_1, \cdots, \bar{c}_n\}$

$$M_C = \begin{bmatrix} \uparrow \uparrow \uparrow \\ \bar{c}_1 \bar{c}_2 \cdots \bar{c}_n \\ \downarrow \downarrow \downarrow \end{bmatrix}$$

We need to find a $C_2^*$ s.t. (i) $\forall c \in C^*, |c| \geq w_{\min}$ and (ii) $\forall \bar{c} \neq \bar{c}_1 \in C^*, |\{i \mid \bar{c}_1 = \bar{c}_2 = 1\}| \leq a_{\max}$ Need to show $M_{C^*}$ is $\left\lfloor \frac{w_{\min} - 1}{a_{\max}} \right\rfloor$-disjunct.

Note that the lower bound of the hamming weight for this matrix is greater than $w_{\min}$.
2 Kautz-Singleton Code Concatentation

Pick an Reed-Sullivan code with a block length \( q, c^* = c_{out} \cdot c_{in}, c_{out} \cdot [q, k]_q \)-RS code, \( c_{in} : [q] \rightarrow \{0,1\}^q, j \in [q], c_{in}(j) = (00 \cdots 010 \cdots 00) \), where 1 is in the \( i \)th position.

**Example 2** Let \( k=1, q=3, c_{out} = (0,0,0),(1,1,1),(2,2,2), n = q^k, t = qxq = q^2 \), so \( w_{min}=q \).

\[
M_{c^*} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \quad \rightarrow \quad M_{c^*} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}
\]

Divide the rows into \( q \) sized chunks, and number these rows as \([q] \times [q]\). Each column then corresponds to a codeword, \( M \) defined as \( M_{c^*} \). Where \( \bar{c}_{k_1, k_2} \in c_{out} \), if \( M(i,j), k_1 = M(i,j), k_2 = \bar{c}_{k_1}(i) = \bar{c}_{k_2}(i) = j \). So the corresponding codeword agreed in postion \( i \).

This implies that \( |M_{k_1} \cap M_{k_2}| = q - \Delta(\bar{c}_{k_1}, \bar{c}_{k_2}) \rightarrow k - 1 \) defined as \( a_{max} \). Note that \( \Delta(\bar{c}_{k_1}, \bar{c}_{k_2}) \geq q - k + 1 \) (MDS) and that \( \left\lfloor \frac{w_{min} - 1}{a_{max}} \right\rfloor \) is defined as \( d \).

Now pick \( q \) and \( k \) so the ratio equals \( d \) exactly. (s.t. \( \left\lfloor \frac{q - 1}{k - 1} \right\rfloor = d \)) Now \( \frac{q}{k} \simeq d \Rightarrow q \simeq kd \), and therefore \( q_k = n \Rightarrow k = \frac{\log n}{\log q} \leq \log n \). So one can imply that \( t = q^2 \simeq (kd)^2 \leq (d \log n)^2 \).

Now pick \( d = d^2 \log n \) with a large constant...txn every entry...\( \frac{t}{d} \) expected weight for...and somehow missed the last statement of the lecture...