February 26, 2019

REMINDERS:

(1) HW 1 due 11:59 pm <u>this Thursday</u>.

(2) Group composition + video topic due by 11:59pm next Tue (DUE in 1 WEEK!)

(3) HW 2 out this Th and due Th next week.

RECAP:

- <u>Plotkin Bound</u>: Let $C \subseteq [q]^n$ be a code of dist d:

(i) If $d = \left(1-\frac{1}{q}\right)n \Rightarrow |C| \le 2qn$

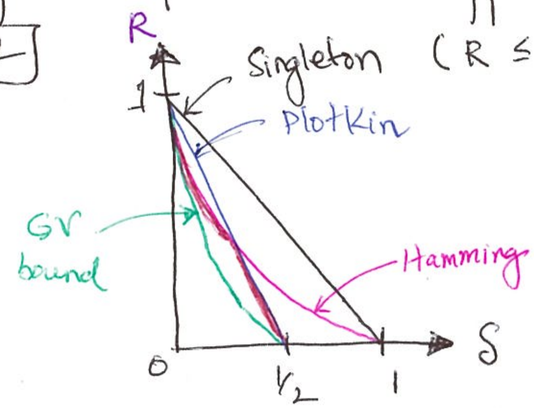
(ii) If $d > \left(1-\frac{1}{q}\right)n \Rightarrow |C| \le \dfrac{qd}{qd-(q-1)n}$ $\quad (\le qd)$

- <u>Corollary</u>: q-ary code with rel. dist δ has rate $R \le 1 - \left(\frac{q}{q-1}\right)\delta + o(1)$

(eg. for $q=2 \Rightarrow R \le 1-2\delta+o(1)$)

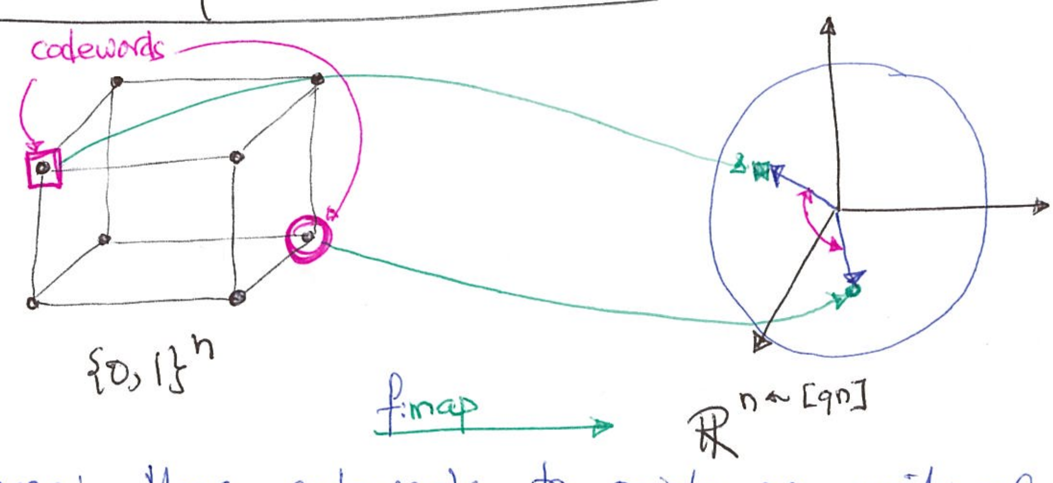
<u>Big Q</u>: Optimal tradeoff between R & δ?

$\boxed{q=2}$



Proof reader: Sai

- <u>Proof overview of Plotkin bound</u>:



<u>Mapping lemma</u>: Maps codewords to points on unit sphere s.t.
If $\Delta(\bar{c}_1, c_2) \ge \left(1-\frac{1}{q}\right)n \Rightarrow$ points on sphere are at an obtuse $\angle$.

<u>Geometric lemma</u>: Cannot have too many pts on sphere w/ obtuse $\angle$.

$\bar{u}, \bar{w} \in \mathbb{R}^N$ $\qquad \|u\| = \sqrt{u_1^2 + \cdots + u_N^2}$

u is unit iff $\|u\| = 1$ $\qquad \langle u, w \rangle = \sum_{i=1}^{N} u_i \cdot w_i$

Geometric Lemma: Let $\bar{u}_1, \ldots, \bar{u}_m \in \mathbb{R}^N$ be non-zero vectors

(1) If $\langle \bar{u}_i, \bar{u}_j \rangle \leq 0 \;\; \forall i \neq j \Rightarrow m \leq 2N$.

(2) Let $\bar{u}_i$ be all unit vector.

If $\langle \bar{u}_i, u_j \rangle \leq -\varepsilon < 0 \Rightarrow m \leq 1 + \frac{1}{\varepsilon}$

Mapping Lemma: $C \subseteq [q]^n \quad \exists f : C \to \mathbb{R}^{qn}$ s.t

(i) $\forall \bar{c} \in C, \quad \|f(c)\| = 1$

(ii) $\forall \bar{c}_1 \neq \bar{c}_2 \in C \qquad \langle f(\bar{c}_1), f(\bar{c}_2) \rangle = 1 - \left(\frac{q}{q-1}\right) \frac{\Delta(\bar{c}_1, \bar{c}_2)}{n}$

Pf. of Plotkin bound: $\quad C = \{\bar{c}_1, \ldots, \bar{c}_m\}$

$$\forall i \neq j \quad \langle f(\bar{c}_i), f(\bar{c}_j) \rangle \underset{\text{(ii) of ML}}{=} 1 - \left(\frac{q}{q-1}\right) \frac{\Delta(\bar{c}_1, \bar{c}_2)}{n}$$

$$\leq 1 - \left(\frac{q}{q-1}\right) \frac{d}{n}$$

Case 1: $d = \left(1 - \frac{1}{q}\right) n$

$$\Leftrightarrow \frac{d}{n} = \left(\frac{q-1}{q}\right) \Leftrightarrow 1 - \left(\frac{q}{q-1}\right) \frac{d}{n} = 0$$

$$\Rightarrow \forall i \neq j \quad \langle f(\bar{c}_i), f(\bar{c}_j) \rangle \leq 0 \underset{\text{(1) of GT}}{\Longrightarrow} m \leq 2qn$$

Case 2: $\langle f(\bar{c}_i), f(\bar{c}_j) \rangle \leq 1 - \frac{qd}{n(q-1)} = \frac{n(q-1) - qd}{n(q-1)}$

$$= -\underbrace{\left(\frac{qd - n(q-1)}{n(q-1)}\right)}_{=\varepsilon} < 0$$

$d > (1 - \frac{1}{q})$

$$\Rightarrow m \leq 1 + \frac{1}{\varepsilon}$$

$$= 1 + \frac{n(q-1)}{qd - n(q-1)}$$

$$= \frac{qd}{qd - n(q-1)}$$

part(2) of Geometric lemma: $\|u_i\| = 1$, $\forall i$ $\langle \bar{u}_i, \bar{u}_j \rangle \le -\varepsilon$

$\Rightarrow m \le 1 + \frac{1}{\varepsilon}$

$$\bar{Z} = \bar{u}_1 + \cdots + \bar{u}_m$$

$$0 \le \|Z\|^2 = \sum_{i=1}^{m} \underbrace{\|u_i\|^2}_{=1} + 2 \underbrace{\sum_{i<j}}_{\binom{m}{2}} \underbrace{\langle \bar{u}_i, \bar{u}_j \rangle}_{\le -\varepsilon}$$

$$\le m \cdot 1 + 2 \cdot \binom{m}{2}(-\varepsilon) = m - m(m-1)\varepsilon$$

$$= m(1 - \varepsilon m + \varepsilon)$$

$$\Rightarrow 1 - \varepsilon m + \varepsilon \ge 0 \Rightarrow \varepsilon m \le 1 + \varepsilon \Rightarrow m \le 1 + \frac{1}{\varepsilon}.$$

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part (1) of GL: $\forall i \ne j$ $\langle \bar{u}_i, \bar{u}_j \rangle \le 0$ Goal: $m \le 2N$

$\bar{u}_i \in \mathbb{R}^N$

$\langle \bar{u}_i, \bar{u}_j \rangle = \|\bar{u}_i\| \|u_j\| \cos(\theta_{ij})$

By induction on N:

Base case: $N = 0 \Rightarrow m = 0 \Rightarrow 0 \le 2 \cdot 0$.

As rotation (all vector rotated by same $\angle$) & any scaling of vectors doesn't change the sign of $\langle \bar{u}_i, u_j \rangle$

WLOG: $\bar{u}_m = (1, 0 \cdots 0)$

$\forall i \ne m$ let $\bar{u}_i = (\alpha_i, \bar{y}_i)$ $\bar{y}_i \in \mathbb{R}^{N-1}$

$\forall i \ne m$: $\langle \bar{u}_i, \bar{u}_1 \rangle = \alpha_i \cdot 1 + \langle \bar{0}, \bar{y}_i \rangle = \alpha_i \Rightarrow \alpha_i \le 0$.

for any $1 \le i < j \le m-1$

$$\langle \bar{y}_i, \bar{y}_j \rangle = \langle \bar{u}_i, \bar{u}_j \rangle - \underbrace{\alpha_i \alpha_j}_{\ge 0} \le \langle \bar{u}_i, \bar{u}_j \rangle \le 0$$

$N \to N-1$, $m \to m-1$ - - - - - - - - - - $m \le N$.

not correct as it need not be the case that $y_i \ne 0$ $\forall 1 \le i \le m$.

Claim: At most one of $\bar{y}_1, \ldots, \bar{y}_{m-1}$ is 0.

(WLOG: $\bar{y}_{m-1} = \bar{0}$) Now we can apply induction

$N \to N-1$, $(\bar{u}_1, \ldots, \bar{u}_m) \to (\bar{y}_1, \ldots, \bar{y}_{m-2})$ i.e. $m \to m-2$

- - N tm $m \le 2N$.

Pf of claim: If not $\exists\ \bar{y}_i = \bar{y}_j = \bar{0}$

$$\Rightarrow \langle \bar{u}_i, \bar{u}_j \rangle = \alpha_i \alpha_j + \langle \bar{y}_i, \bar{y}_j \rangle = \alpha_i \cdot \alpha_j > 0$$

as $\bar{u}_i \neq 0$, $\bar{u}_j \neq 0 \Rightarrow \alpha_i \neq 0$, $\alpha_j \neq 0 \Rightarrow \alpha_i, \alpha_j < 0$ $\rightarrow$ contradiction

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Pf of Mapping Lemma: Only need $\phi : [q] \rightarrow \mathbb{R}^q$ (final map f will apply ϕ to each co-ordinate)

$i \in [q]$

$$\phi(i) = \left(\frac{1}{q}, \frac{1}{q}, \ldots, \underset{\substack{\uparrow \\ i}}{\frac{-(q-1)}{q}}, \frac{1}{q} \cdots \frac{1}{q}\right)$$

$f : C \rightarrow \mathbb{R}^{qn}$

$$\forall \bar{c} \in C, \quad f(\bar{c}) = \sqrt{\frac{q}{n(q-1)}} \cdot \left(\phi(c_1), \phi(c_2) \ldots, \phi(c_n)\right)$$

Properties of ϕ:

(1) $$\|\phi(i)\|^2 = \langle \phi(i), \phi(i) \rangle = (q-1) \cdot \frac{1}{q^2} + \frac{(q-1)^2}{q^2} = \frac{(q-1)(1+q-1)}{q^2} = \frac{q-1}{q}$$

(2) $\forall\ i \neq j$

$$\langle \phi(i), \phi(j) \rangle = (q-2) \cdot \frac{1}{q^2} - \frac{2(q-1)}{q^2} = \frac{q-2-2q+2}{q^2} = \frac{q-2q}{q^2} = -\frac{1}{q}$$

(i) $[\|f(c)\| = 1]$

$$\|f(c)\|^2 = \left(\sqrt{\frac{q}{n(q-1)}}\right)^2 \sum_{i=1}^{n} \underbrace{\|\phi(c_i)\|^2}_{= \frac{q-1}{q}}$$

$$= \frac{q}{n(q-1)} \cdot \frac{n(q-1)}{q} = 1$$

(ii) $\forall\ i \neq j$ $$\langle f(\bar{c}_i), f(\bar{c}_j) \rangle = 1 - \left(\frac{q}{q-1}\right) \frac{\Delta(\bar{c}_1, \bar{c}_2)}{n}$$

$\bar{c}_i = (x_1, \ldots, x_n)$, $\bar{c}_j = (y_1, \ldots, y_n)$

$$\langle f(\bar{c}_i), f(\bar{c}_j) \rangle = \sum_{l=1}^{n} \langle \phi(x_l), \phi(y_l) \rangle \cdot \frac{q}{n(q-1)}$$

$$= \frac{q}{n(q-1)} \left(\sum_{l : x_l = y_l} \underbrace{\langle \phi(x_l), \phi(y_l) \rangle}_{= \frac{q-1}{q}} + \sum_{l : x_l \neq y_l} \underbrace{\langle \phi(x_l), \phi(y_l) \rangle}_{= -\frac{1}{q}} \right)$$

$$= \frac{q}{n(q-1)} \left\{ \left(n - \Delta(\bar{c}_i, \bar{c}_j) \right) \left(\frac{q-1}{q} \right) \cancel{+} - \frac{\Delta(\bar{c}_i, \bar{c}_j)}{q} \right\}$$

$$= 1 - \frac{q}{n(\cancel{q-1})} \Delta(\bar{c}_i, \bar{c}_j) \cdot \frac{\cancel{q-1}}{\cancel{q}} - \frac{q}{n(q-1)} \frac{\Delta(\bar{c}_i, \bar{c}_j)}{\cancel{q}}$$

$$= 1 - \frac{\Delta(\bar{c}_i, \bar{c}_j)}{n} \left(1 + \frac{1}{q-1} \right) = 1 - \frac{q}{q-1} \frac{\Delta(\bar{c}_i, \bar{c}_j)}{n}.$$