Mar 11

REMINDERS

(•) HW 3 due by 11:59pm Wed ← Atri needs to create Autolab submission

(•) Project 2-pg report due by 11:59 pm Wed, Mar 30

RECAP

Group testing problem:

(•) Unknown vector $\bar{x} \in \{0,1\}^N$, $wt(\bar{x}) \le d$

(•) Query / test: $S \subseteq [N]$

↳ Answer $A(S) = \bigvee_{i \in S} x_i$

(•) Goal: Figure out queries $S_1, \ldots, S_t$ s.t $A(S_1), \ldots, A(S_t)$ gives $\bar{x}$

(→) Adaptive vs. non-adaptive queries

$t^a(d,N)$ $\qquad$ $t(d,N)$

(→) $t^a(d,N) = \Theta\left(d \log \frac{N}{d}\right)$

(→) $t^a(d,N) \le t(d,N) \le N$

PROOF READER for today: Any volunteer? Prakshal

PLAN for today: (i) $t(1,N) = \Theta(\log N)$

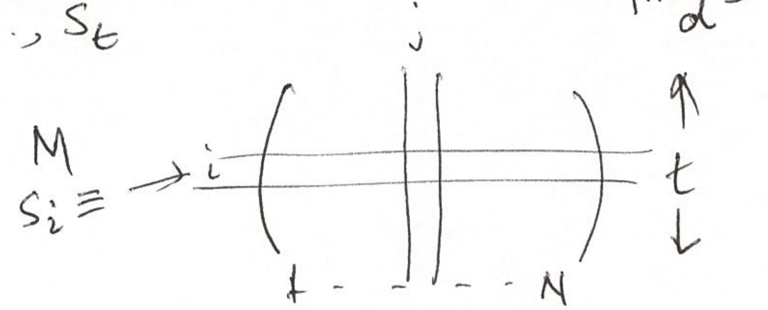
(ii) Disjunct matrices (iii) Use RS codes to show $t(d,N)$ is $O(d^2(\log_d N)^2)$

Non-Adaptive Group Testing (NAGT)

(fix N, d)

Define a "testing" matrix

$S_1, \ldots, S_t$



$M_{ij} = \begin{cases} 1 & \text{if } j \in S_i \\ 0 & \text{o/w} \end{cases}$

$N = 3$

$S = \{1, 3\}$

1	0	1
1	2	3

$$\begin{pmatrix} & & \\ & M & \\ & & \end{pmatrix} \odot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \to \begin{pmatrix} r_1 \\ \vdots \\ r_t \end{pmatrix} \to r_i = \bigvee_{j \in S_i} x_j$$

$= \bigvee_{n} M_{ij} \wedge x_j$

matrix vect mult if add is ∨ mult is ∧

$t(1,N) \qquad t(1,N) \geq t^a(1,N) \geq \log_2 N$

<u>Claim:</u> $t(1,N) \leq \lceil \log_2(N+1) \rceil + 1$

Assume $wt(\bar{x}) = 1$

$$\begin{array}{c} \xleftarrow{\quad N \quad} \\ t \left(\begin{matrix} \uparrow & \uparrow & & \uparrow \\ M^1 & M^2 & \cdots & M^N \\ \downarrow & \downarrow & & \downarrow \end{matrix} \right) \odot \left(\bar{e}_i \right) = \left(\begin{matrix} \uparrow \\ M^i \\ \downarrow \end{matrix} \right) \longrightarrow i \end{array}$$

$\log_2 N \approx t$

$M^i = Bin(i)$

parity check matrix of a Hamming code

$[2^t - 1, \ 2^t - t - 1, 3]_2$ Hamming code

$N = 2^t - 1 \Rightarrow t = \log_2(N+1)$

$\rightarrow$ To handle general N, you need $t \leq \lceil \log_2(N+1) \rceil + 1$

$t = 3$

$$\begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & \cdots \\ 1 & 0 & 1 & \cdots \end{pmatrix} \odot \begin{pmatrix} 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\odot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$\rightarrow$ d-seperable

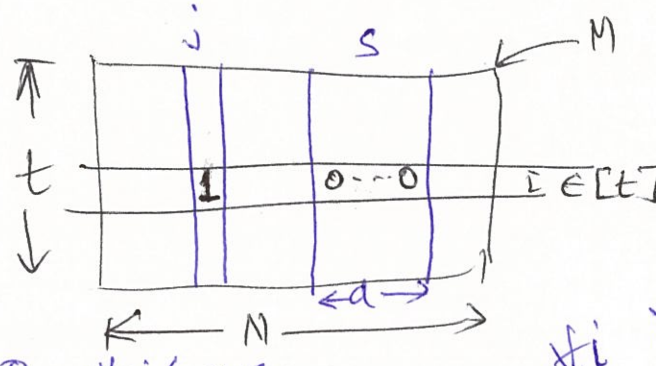
<u>Def</u>: d-Disjunct matrices

$M \in \{0,1\}^{t \times N}$ is d-disjunct

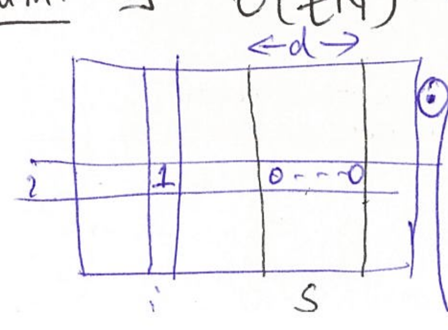
if $\forall S \subseteq [N]$ s.t. $|S| = d$

$\forall j \notin S \Rightarrow$

$\exists i$ s.t. $M_{ij} = 1$ but $M_{ij'} = 0 \ \forall j' \in S$



<u>Claim:</u> $\exists \ O(tN)$ recovery algo



$\odot \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} r_1 \\ \vdots \\ r_t \end{pmatrix} \quad r_i = 0$

$S = \{i \mid x_i = 1\}$

$\forall j \in [N] \setminus S$ s.t. $M_{ij} = 1$ $\quad \equiv \forall j \ x_j = 0$

$\Rightarrow r_i = 0$

If $r_i = 0 \Leftrightarrow \forall j$ s.t. $M_{ij} = 1, \ x_j = 0$

Algo: (i) $X_j \leftarrow 1 \quad \forall j \in [N]$
(ii) for $i = 1 \dots t$
  if $r_i = 0$
    for $j = 1 \dots N$
      If $M_{ij} = 1$
        $X_j \leftarrow 0$

Pf: This works i.e. $\bar{X}$ is recovered exactly.

Book: general connection between codes & d-disjunct matrices.

Today: Kautz-Singleton (uses RS codes) $[q, k, q-k+1]_q$

Step 1: $M' = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ c_1 & c_2 & \cdots & c_N \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$

$N = q^k$

$\bar{c}_1, \dots, \bar{c}_N$

$\phi : \mathbb{F}_q \to \{0,1\}^q \qquad \phi(i) = \bar{e}_i^{\,T} = \begin{array}{c} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \end{array} \begin{array}{c} \\ \\ i \\ \\ \\ \end{array}$

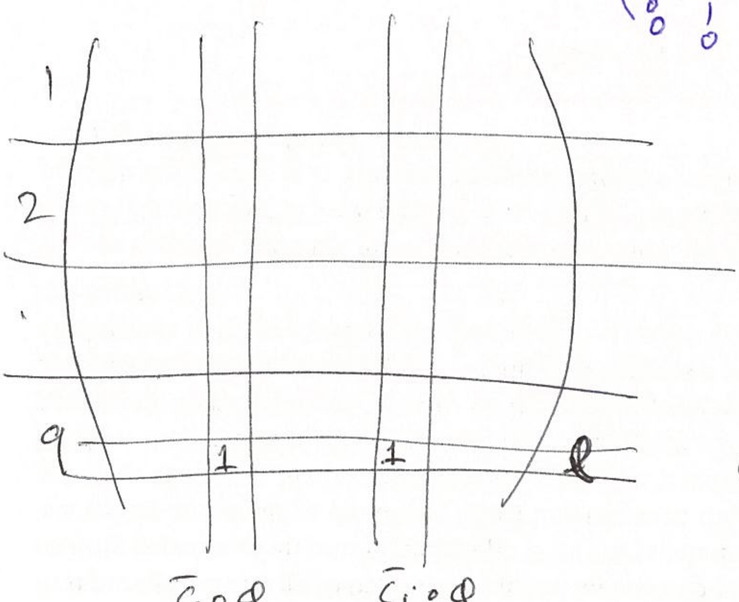
Apply ϕ to each entry of M'

$$[3,1]_3 \text{ RS} \quad \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow[\phi]{} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{array}{c} \\ \\ \\ \uparrow \\ q \\ \downarrow \\ \\ \\ \\ \end{array}$$

$t = q^2$

$d = \left\lfloor \frac{q-1}{k-1} \right\rfloor$

M is d-disjunct



$c_i[q] = c_j[q] = \ell$