Mar 1A

REMINDERS

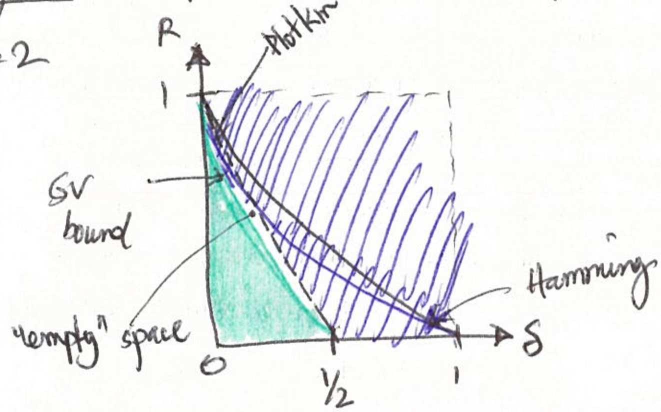
(i) HW 3 due at 11:59pm this Wed ← see post @153 on piazza for more comments

(ii) Mini project 2-pg. report due by 11:59pm Wed, Mar 30
- → Autolab accepting report submission
- → Make sure y'all follow ALL the requirements

RECAP:

Big Q: Optimal tradeoff b/w R & δ (recall: worst-case noise model due to Hamming)

$q=2$



TODAY!

Proof reader volunteer? Kurt Su

→ Switch gears from worst-case noise model

→ Noise model pioneered by Shannon ← things are pretty different compared to worst-case noise model
- → Memoryless
- → (Fully specified) random noise

Shannon's 1948 paper:

(•) Noisy channel (channel coding) Our setup except noise is not worst-case (stochastic)

(•) Noiseless channel (source coding) No noise during transmission ⟹ do compression

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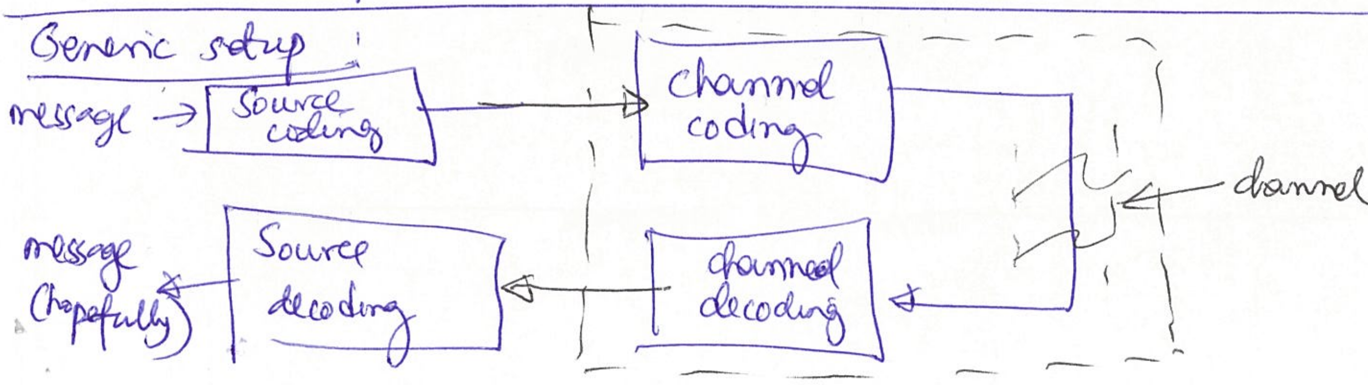
Generic setup:



→ In Shannon's setup can decouple source & channel coding / decoding (optimize separately)
[more on channel coding.]

Source coding: notion of entropy is the "right" notion of compressibility (introduced in Shannon's paper)

Shannon's noise model: memoryless same noise function acts (independently) on all n symbols transmitted.

$$\mathcal{X} \ni x \longrightarrow y \in \mathcal{Y}$$

For us: $\mathcal{X}, \mathcal{Y}$ are finite

Transition matrix

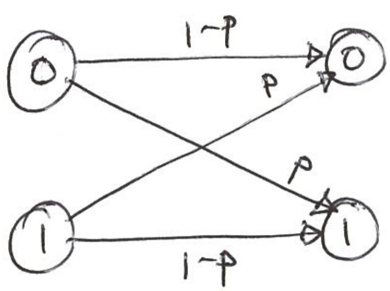
$$\begin{array}{cc} & y \in \mathcal{Y} \\ \mathcal{X} \ni x & \begin{pmatrix} & & \\ & M & \\ & & \end{pmatrix} \end{array}$$

$M[x,y] = \Pr(y \mid x)$

① Binary Symmetric channel $(0 \le p \le 1)$ ⇝ BSC_p

p ← cross-over probability

$\mathcal{X} = \mathcal{Y} = \{0,1\}$



≡ each bit gets flipped w.p. p

$$\equiv \begin{array}{ccc} & 0 & 1 \\ 0 & 1-p & p \\ 1 & p & 1-p \end{array}$$

Q: Can assume $0 \le p \le \frac{1}{2}$. Why?

A: If $1 \ge p > \frac{1}{2}$, flip all received bits

$\Rightarrow$ $0 \le p \le \frac{1}{2}$ range.

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② q-ary Symmetric channel (qSC_p) $q \ge 2$ as int

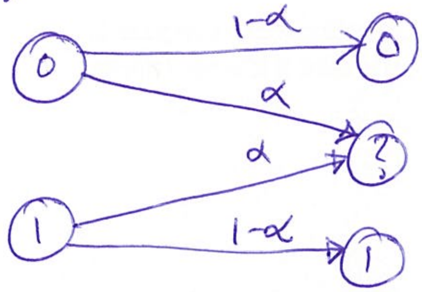
($q=2 \Rightarrow BSC_p$)

$\mathcal{X} = \mathcal{Y} = [q]$ $\qquad 0 \le p \le 1 - \frac{1}{q}$

$$M[x,y] = \begin{cases} 1-p & \text{if } y = x \\ \frac{p}{q-1} & \text{if } y \ne x \end{cases}$$

③ Binary erasure channel (BEC_α) $0 \le \alpha \le 1$

$\mathcal{X} = \{0,1\}$ $\mathcal{Y} = \{0,1,?\}$



i.e. each bit gets erased w.p. α

Error correction

$BSC_p \rightarrow$ There is non-zero prob that the transmitted codeword is received as another valid codeword.

$BEC_\alpha \longrightarrow$ There is a non-zero prob $= \alpha^n = 2^{-\Omega(n)}$ that all locations get erased.

$\alpha^n = \left(\frac{1}{\alpha}\right)^{-n} = 2^{-\log_2\left(\frac{1}{\alpha}\right) \cdot n}$

So cannot decode w.p. 1

WANT! every message is decoded correctly w.p $1 - f(n)$

$\lim_{n \to \infty} f(n) = 0$ ideally, $f(n) = 2^{-\Omega(n)}$

Shannon's general result

Big Q! R vs error $\longrightarrow$ error parameter

p : BSC_p

α : BEC_α

Shannon's thm (informal) For every channel there is a specific number $0 \le C \le 1$ s.t

$\forall\ R < C \Rightarrow (\exists$ a code$)$ reliable comm happens ← dec prob $1 - 2^{-\Omega(n)}$

$\forall\ R > C \Rightarrow (\forall$ codes$)$ reliable comm cannot happen.

$C \rightarrow$ capacity of channel

<u>Next:</u> capacity theorem for BSC_p

$\bar{e} \sim BSC_p$ $\quad \bar{e} \in \{0,1\}^n$ $\quad \forall i \; \bar{e}[i] = \begin{cases} 1 & \text{wp } p \\ 0 & \text{o/w} \end{cases}$

<u>Shannon's capacity Thm for BSC_p</u>

$0 \le p < \frac{1}{2}$, $0 \le \varepsilon \le 1 - H(p)$. following are true for large enough n:

① $\exists\, \delta > 0$, an encoding function $E : \{0,1\}^k \to \{0,1\}^n$
a decoding —— $D : \{0,1\}^n \to \{0,1\}^k$
for $k \le \lfloor (1 - H(p) - \varepsilon) n \rfloor$ s.t

$\forall\, \bar{m} \in \{0,1\}^k$

$$\Pr_{\bar{e} \sim BSC_p} \left[D(E(\bar{m}) + \bar{e}) \neq \bar{m} \right] \le 2^{-\delta n}.$$

② If $k \ge \lceil (1 - H(p) + \varepsilon) n \rceil$ then $\forall\, E : \{0,1\}^k \to \{0,1\}^n$
$D : \{0,1\}^n \to \{0,1\}^k$

$\exists\, \bar{m} \in \{0,1\}^k$ s.t

$$\Pr_{\bar{e} \sim BSC_p} \left[D(E(\bar{m}) + \bar{e}) \neq \bar{m} \right] \ge \frac{1}{2} \leftarrow 1 - 2^{-\beta n}$$

$\Rightarrow$ capacity for BSC_p: $\quad 1 - H(p) \leftarrow$ do this in class.

qSC_p: $\quad 1 - H_q(p)$

BEC_α: $\quad 1 - \alpha$