Feb 28

REMINDERS!

(1) Video topic due (via G form) by 11:59pm this Wed
↳ If you miss the deadline, you have a ZERO ON REST of the project

(2) New piazza response time policy

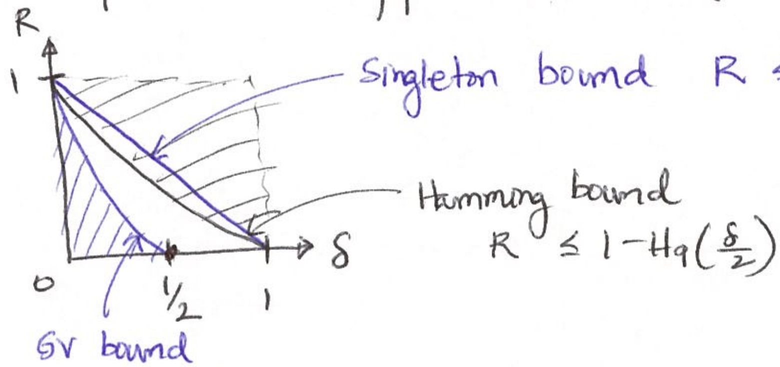
(3) Please ask Q if something doesn't make sense
↳ Things are going to get only more complicated.

(4) Changes to Yunus' office hours coming soon

(5) HW 1 graded
↳ Please read the regrade policy AND FOLLOW IT!

RECAP! Big Q → Optimal tradeoff between R & δ?

$\boxed{q=2}$



Q2) Can we get $R > 0$, $\delta > 1 - \frac{1}{q}$?

Q4) Tighten gap between upper & lower bounds?

Q5) Can we get $R = 1-\delta$ with fixed q?

PROOF READER for today: Thy

PLAN for today: Plotkin bound. (Answers Q2, Q4, Q5)

PLOTKIN BOUND! $C \subseteq [q]^n$ be a code of dist d

(i) If $d = (1-\frac{1}{q})n \Rightarrow |C| \le 2qn$ (for $q=2$, $|C| \le 2n$) ↑ tight.

(ii) If $d > (1-\frac{1}{q})n \Rightarrow |C| \le \frac{qd}{\underbrace{qd-(q-1)n}_{\ge 1}} \le qd$

$\equiv qd > (q-1)n$

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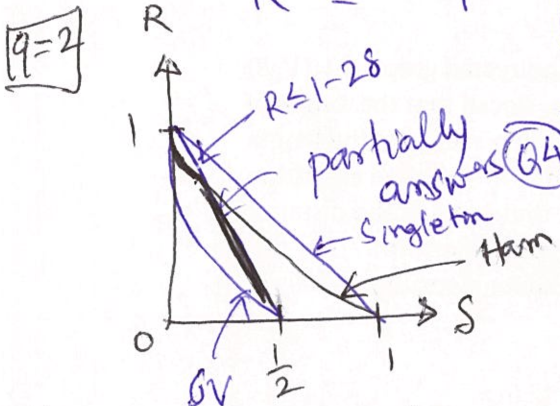
for fixed q, $\dim(C)$ is $O(\log_q 2qn) = O(\log n) \Rightarrow R(C) = 0$

Reminder: Hadamard code has dim $\log_2 n$ & dist $\frac{n}{2}$

⇒ NO to Q2

COR: Any q-ary code of relative dist δ and rate R,

$$R \le 1 - \left(\frac{q}{q-1}\right)\delta + o(1) \quad \left(\text{for } q=2 \quad R \le 1-2\delta\right)$$

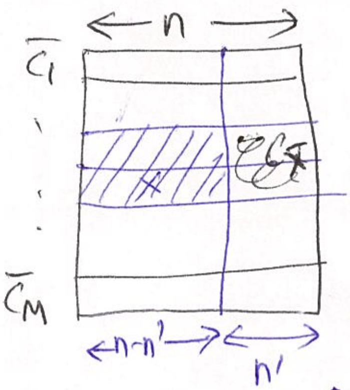


partially answers Q4 $\Longrightarrow$ for fixed q, $R < 1-\delta$

$\Rightarrow$ NO for Q5!

~~Pf~~ of Corollary: (Assume Plotkin bound is true) $C = \{\bar{c}_1, \dots, \bar{c}_M\}$

$d = \delta n$.



$$n' = \left\lfloor \frac{qd}{q-1} \right\rfloor - 1$$

$$\forall\, \bar{x} \in [q]^{n-n'}$$

$$C_{\bar{x}} = \{ (c_{n-n'+1}, \dots, c_n) \mid (c_1, \dots, c_n) \in C, \ (c_1, \dots, c_{n-n'}) = \bar{x} \}$$

$\rightarrow$ $C_{\bar{x}}$ has dist d (as C has dist $\ge d$)

Block length of $C_{\bar{x}}$ is $n' < \frac{qd}{q-1}$

$\equiv qd > n'(q-1)$

$\Rightarrow$ By (ii) of Plotkin bound, $|C_{\bar{x}}| \le qd$.

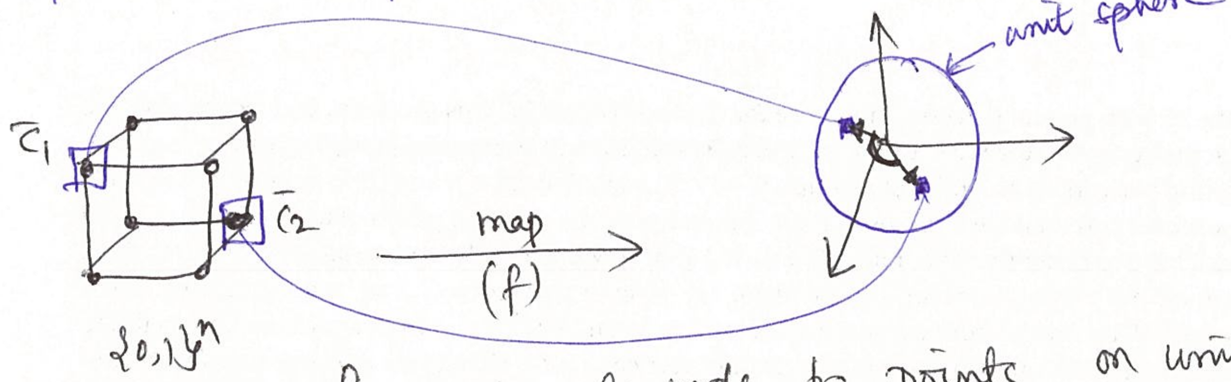
$$q^{Rn} = M = \sum_{\bar{x} \in [q]^{n-n'}} |C_{\bar{x}}| \le \sum_{\bar{x} \in [q]^{n-n'}} qd = q^{n-n'} \cdot \underbrace{qd}_{q^{o(n)}}$$

$$= q^{n-n'+o(n)} \le q^{n - \frac{qd}{q-1} + 2 + o(n)}$$

$$n' \ge \frac{qd}{q-1} - 2 \rightarrow \quad = q^{n - \frac{q\delta n}{q-1} + o(n)}$$

$$\Rightarrow Rn \le n - \frac{q\delta n}{q-1} + o(n) \Rightarrow R \le 1 - \left(\frac{q}{q-1}\right)\delta + o(1) \ \blacksquare$$

(0) Proof overview of Plotkin bound



Mapping lemma: f maps codewords to points on unit sphere s.t if $\Delta(\bar{c}_1, \bar{c}_2) \geq (1-\frac{1}{q})n \Rightarrow$ corresponding points on the sphere are at an obtuse angle.

Geometric lemma: Cannot have too many points on the sphere such that all pairs are at an obtuse angle.

$\bar{u}, \bar{w} \in \mathbb{R}^N$ $\quad \langle \bar{u}, \bar{w} \rangle = \sum_{i=1}^{N} u_i w_i \quad \|u\| = \sqrt{\langle \bar{u}, \bar{u} \rangle}$

$\bar{u}$ is unit if $\|u\| = 1$ $\quad = \sqrt{u_1^2 + \cdots + u_N^2}$

Geometric lemma: Let $\bar{u}_1, \ldots, \bar{u}_m \in \mathbb{R}^N$ be non-zero vectors

(1) If $\langle \bar{u}_i, \bar{u}_j \rangle \leq 0 \Rightarrow m \leq 2N$

(2) Let $\|u_i\| = 1$ for all i. If $\langle \bar{u}_i, \bar{u}_j \rangle \leq -\varepsilon < 0$ $\forall\, i \neq j \Rightarrow m \leq 1 + \frac{1}{\varepsilon}$

Mapping lemma: $C \subseteq [q]^n$, $\exists f: C \longrightarrow \mathbb{R}^{qn}$ s.t.

(i) $\forall\, \bar{c} \in C$, $\|f(c)\| = 1$

(ii) $\forall\, \bar{c}_1 \neq \bar{c}_2 \in C$, $\langle f(\bar{c}_1), f(\bar{c}_2) \rangle = 1 - \left(\frac{q}{q-1}\right) \frac{\Delta(\bar{c}_1, \bar{c}_2)}{n}$

Pf of Plotkin bound: $C = \{\bar{c}_1, \ldots, \bar{c}_M\}$

By mapping lemma $\forall\, \bar{c}_1 \neq \bar{c}_2 \in C$, $\langle f(\bar{c}_1), f(\bar{c}_2) \rangle$
As C has dist d $\leq 1 - \left(\frac{q}{q-1}\right) \frac{d}{n}$

Case 1: $d = (1-\frac{1}{q})n = \left(\frac{q-1}{q}\right)n$

$\Rightarrow \frac{d}{n} = \frac{q-1}{q} \Leftrightarrow 1 - \left(\frac{q}{q-1}\right)\frac{d}{n} = 0 \Rightarrow \forall\, i \neq j \quad \langle f(\bar{c}_i), f(\bar{c}_j) \rangle \leq 0 \xRightarrow{\text{(1) of Geo lemma}} M \leq 2qn$ ∎