Mar 2 **REMINDERS:**

(1) Video topic due by 11:59pm tonight

↳ Will email by Th night if any issue with topic choice

(2) HW 2 out by 11:59pm tonight

RECAP: Plotkin bound: Let $C \subseteq [q]^n$ be a code of distance d.

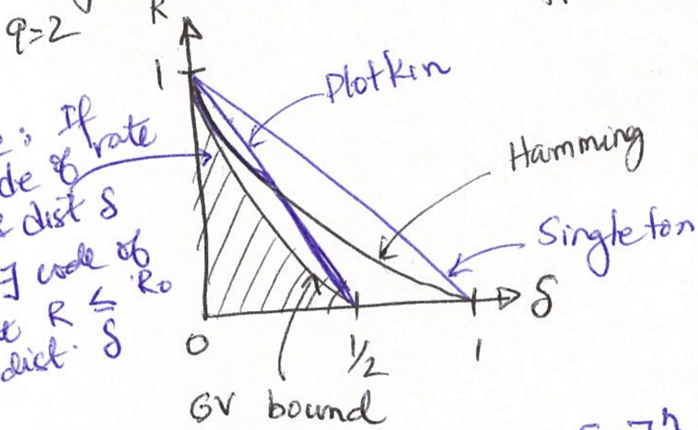
(i) If $d = (1-\frac{1}{q})n \Rightarrow |C| \le 2qn$

(ii) If $d > (1-\frac{1}{q})n \Rightarrow |C| \le \frac{qd}{qd-(q-1)n}$ $(\le qd)$

(•) Corollary: q-ary code with rel. dist δ has $R \le 1 - \left(\frac{q}{q-1}\right)\delta + o(1)$

[eg $q=2$: $R \le 1-2\delta + o(1)$]

Big Q: Optimal tradeoff b/w R and δ?



Note: If ∃ code of rate R_0 & dist δ ⟹ ∃ code of rate $R \le R_0$ + dist. δ

(•) Mapping Lemma: $C \subseteq [q]^n$, $\exists f: C \to \mathbb{R}^{qn}$ s.t.

(i) $\forall \bar{c} \in C$, $\|f(c)\| = 1$

(ii) $\forall \bar{c}_1 \ne \bar{c}_2 \in C$, $\langle f(\bar{c}_1), f(\bar{c}_2)\rangle = 1 - \left(\frac{q}{q-1}\right)\frac{\Delta(\bar{c}_1,\bar{c}_2)}{n}$

(•) Geometric Lemma: Let $\bar{u}_1, \ldots, \bar{u}_m \in \mathbb{R}^N$ be non-zero vectors

(1) If $\langle \bar{u}_i, \bar{u}_j\rangle \le 0$ $\forall i \ne j \Rightarrow m \le 2N$

(2) If $\|u_i\| = 1$ and $\langle \bar{u}_i, \bar{u}_j\rangle \le -\varepsilon < 0$ $\forall i \ne j$

$\Rightarrow m \le 1 + \frac{1}{\varepsilon}$

Proof reader for today: Volunteer? Adam

PLAN for TODAY: ~~(1)~~ Prove part (ii) of Plotkin bound using Mapping + Geometric lemmas

(2) Prove part (2) of Geometric Lemma ←

(3) Prove mapping Lemma

(4) {If there is time} prove part (1) of Geometric lemma

Pf of (ii) Plotkin bound $\qquad C = \{ \bar{c}_1, \dots, \bar{c}_M \}$

$\forall i \neq j \quad \langle f(\bar{c}_i), f(\bar{c}_j) \rangle = 1 - \left(\frac{q}{q-1}\right) \frac{\Delta(\bar{c}_i, \bar{c}_j)}{n}$

$$\leq 1 - \left(\frac{q}{q-1}\right) \frac{d}{n}$$

$d > \left(1 - \frac{1}{q}\right) n = \frac{(q-1)n}{q}$

$\equiv qd > (q-1) n$

$$= \frac{(q-1)n - qd}{(q-1)n}$$

$$= -\underbrace{\left(\frac{qd - (q-1)n}{(q-1)n}\right)}_{= \varepsilon}$$

$M \leq 1 + \frac{1}{\varepsilon} \xLeftarrow{\text{by (2) of GL}}$

$= 1 + \frac{(q-1)n}{qd - (q-1)n}$

$= \frac{qd - (q-1)n + (q-1)n}{qd - (q-1)n} = \frac{qd}{qd - (q-1)n}$ ∎

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part(2) of GL: $\left\{ \begin{array}{l} \|u_i\|_2 = 1 \quad \forall i \neq j \ \langle \bar{u}_i, \bar{u}_j \rangle \leq -\varepsilon < 0 \\ \bar{u}_1, \dots, \bar{u}_m \in \mathbb{R}^N \Rightarrow m \leq 1 + \frac{1}{\varepsilon} \end{array} \right\}$ $m \geq 1$

$\bar{z} = \bar{u}_1 + \dots + \bar{u}_m$

$0 \leq \|z\|^2 = \langle \bar{u}_1 + \dots + \bar{u}_m, \bar{u}_1 + \dots + \bar{u}_m \rangle$

$$= \sum_{i=1}^{m} \underbrace{\|u_i\|^2}_{=1} + 2 \sum_{\underbrace{i<j}_{\binom{m}{2} = \frac{m(m-1)}{2}}} \underbrace{\langle \bar{u}_i, \bar{u}_j \rangle}_{\leq -\varepsilon}$$

$$\leq m \cdot 1 + 2 \cdot \frac{m(m-1)}{2} (-\varepsilon)$$

$$= m - m(m-1)\varepsilon$$

$$= m(1 - (m-1)\varepsilon)$$

$\Rightarrow \cancel{m}(1 - (m-1)\varepsilon) \geq 0 \Rightarrow 1 - (m-1)\varepsilon \geq 0$

$\Rightarrow 1 - \varepsilon m + \varepsilon \geq 0 \Rightarrow 1 + \varepsilon \geq \varepsilon m$

$\Rightarrow 1 + \frac{1}{\varepsilon} \geq m$ ∎

Pf of Mapping Lemma! Only need $\varphi : [q] \to \mathbb{R}^q$

$i \in [q]$

$$\varphi(i) = \Big(\tfrac{1}{q}, \tfrac{1}{q}, \dots, \overset{i}{\tfrac{-(q-1)}{q}}, \dots, \tfrac{1}{q}\Big) \in \mathbb{R}^q$$

$$\begin{array}{rccc} \bar{c} & \boxed{c_1} & \cdots & \boxed{c_n} \\ & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ f(\bar{c}) = & \boxed{\varphi(c_1)} & \cdots & \boxed{\varphi(c_n)} \end{array}$$

Properties of φ: $\forall i \neq j \in [q]$

(1) $\|\varphi(i)\|^2 = (q-1)\cdot\frac{1}{q^2} + 1\cdot\frac{(q-1)^2}{q^2} = \frac{(q-1)+(q-1)^2}{q^2} = \frac{(q-1)(1+q-1)}{q^2} = \frac{(q-1)}{q}$

(2) $\langle \varphi(i), \varphi(j) \rangle = (q-2)\cdot\frac{1}{q^2} - \frac{2(q-1)}{q^2} = \frac{q-2-2q+2}{q^2} = \frac{-q}{q^2} = -\frac{1}{q}$

$f : C \to \mathbb{R}^{qn}$

$\forall\ \bar{c} = (c_1, \dots, c_n)$

$$f(\bar{c}) = \underbrace{\rho}_{\sqrt{\frac{q}{n(q-1)}}} \cdot \big(\varphi(c_1), \varphi(c_2), \dots, \varphi(c_n)\big) \in \mathbb{R}^{qn}$$

(i) $\{\|f(c)\| = 1\}$ $\quad \|f(c)\|^2 = \sum_{i=1}^{n} \|\varphi(c_i)\|^2 \cdot \frac{q}{n(q-1)}$

$$= \sum_{i=1}^{n} \frac{\cancel{q-1}}{\cancel{q}} \cdot \frac{\cancel{q}}{n(\cancel{q-1})} = \sum_{i=1}^{n} \frac{1}{n} = n\cdot\frac{1}{n} = 1.$$

(ii) $\Big\{ \forall\ \bar{c}_1 \neq \bar{c}_2\ \ \langle f(\bar{c}_1), f(\bar{c}_2) \rangle = 1 - \Big(\frac{q}{q-1}\Big)\frac{\Delta(\bar{c}_1, \bar{c}_2)}{n} \Big\}$

$\bar{c}_1 = (x_1, \dots, x_n) \qquad \bar{c}_2 = (y_1, \dots, y_n)$

$$\langle f(\bar{c}_1), f(\bar{c}_2) \rangle = \Big(\sqrt{\frac{q}{n(q-1)}}\Big)^2 \big\langle (\varphi(x_1), \dots, \varphi(x_n)), (\varphi(y_1), \dots, \varphi(y_n)) \big\rangle$$

$$= \frac{q}{n(q-1)} \cdot \sum_{l=1}^{n} \langle \varphi(x_l), \varphi(y_l) \rangle$$

$$= \frac{q}{n(q-1)} \Big(\sum_{l : x_l = y_l} \langle \varphi(x_l), \varphi(x_l) \rangle + \sum_{l : x_l \neq y_l} \langle \varphi(x_l), \varphi(y_l) \rangle \Big)$$

$$= \frac{q}{n(q-1)} \Big((n - \Delta(\bar{c}_1, \bar{c}_2)) \cdot \frac{q-1}{q} + \Delta(\bar{c}_1, \bar{c}_2) \cdot \Big(-\frac{1}{q}\Big) \Big)$$

$$= \frac{q}{n(q-1)} \cdot n \cdot \frac{q-1}{q} - \frac{q}{n(q-1)} \Delta(\bar{c}_1, \bar{c}_2) \cdot \frac{q-1}{q} - \frac{q}{n(q-1)} \Delta(\bar{c}_1, \bar{c}_2) \cdot \frac{1}{q}$$

$$= 1 - \frac{\Delta(\bar{c}_1, \bar{c}_2)}{n} \Big(1 + \frac{1}{q-1}\Big) = 1 - \frac{\Delta(\bar{c}_1, \bar{c}_2)}{n} \cdot \Big(\frac{q}{q-1}\Big)$$