Mar 27

1 min checkin

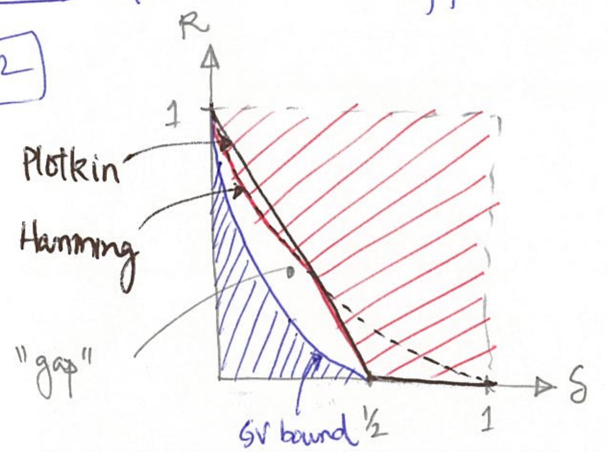
REMINDERS

(•) Mini project report due by 11:59pm on Wed
- ↳ ALL group members INDIVIDUALLY submit the SAME PDF
- ↳ Make sure to follow <u>all</u> instructions for the report

RECAP

<u>Big Q:</u> Optimal tradeoff between R & δ (Recall: worst-case noise model due to Hamming)

$\boxed{q=2}$



$\boxed{\text{Plan for today}}$ (1) Switch gears from worst-case noise model

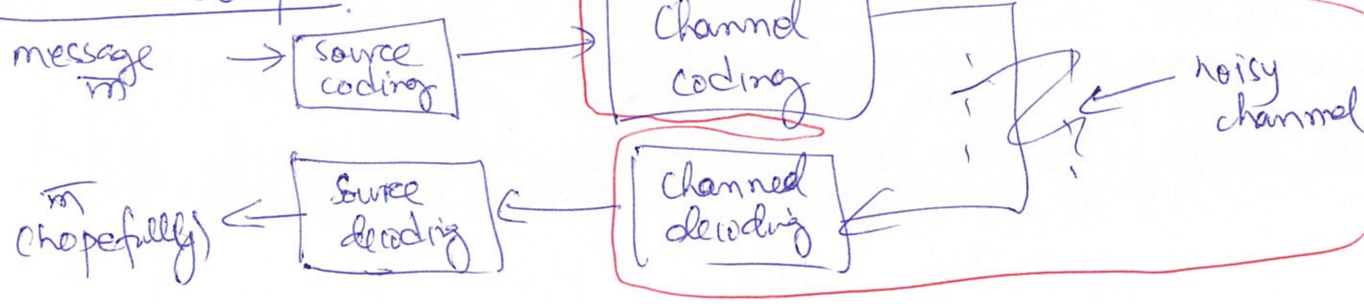
(2) Noise model pioneered by Shannon ← Things are pretty different compared to worst-case noise model
- ↳ Memoryless
- ↳ (Fully specified) random model

<u>Shannon's 1948 paper</u>

(•) Noisy channel (channel coding): Our setup except noise <u>not</u> worst-case (stochastic)

(•) Noisless channel (source coding): No noise during transmission ⇒ do compression

<u>General setup</u>:



In Shannon's setup: One can decouple channel & source coding (ie optimize both separately)
(more on channel coding next.)

Source coding: notion of entropy is the "right" notion of how compressible a message is.

$$\sum_{i=1}^{N} p_i \log \frac{1}{p_i} \qquad (p_1, \dots, p_N)$$

Shannon's noise model

Memoryless: same noise function acts independently on the n transmitted symbols

$$\mathcal{X} \ni x \xrightarrow{\text{Channel}} y \in \mathcal{Y}$$

$\mathcal{X}$: channel i/p alphabet

$\mathcal{Y}$: output alphabet

Transition table

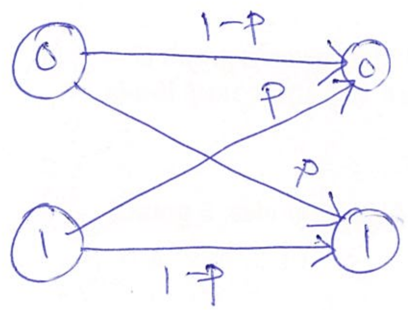
$$x \ni x \longrightarrow \begin{array}{c|c} & (y \in \mathcal{Y}) \\ \hline \end{array} \left[\; M \; \right] \qquad M[x,y] = \Pr(y|x)$$

Each column sums: $= 1$

BSC_p

① Binary symmetric channel $(0 \le p \le 1)$
(p → cross over prob.)

$\mathcal{X} = \mathcal{Y} = \{0,1\}$



$\equiv$ each bit gets flipped w.p. p

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1-p & p \\ 1 & p & 1-p \end{array}$$

Ex: Can assume $0 \le p \le \frac{1}{2}$. Why?

A: If $\frac{1}{2} \le p \le 1$, flip the bits
⇒ reduces to $0 \le p < \frac{1}{2}$ case

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② q-ary symmetric channel qSC_p
$(q = 2 \to BSC_p)$

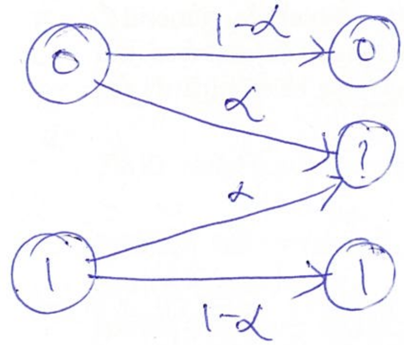
$\mathcal{X} = \mathcal{Y} = \{0, \dots, q-1\}$ $\qquad q \ge 2$ int.

$M[x,y] = \Pr(y \text{ rcvd} \mid x \text{ transmitted})$

$$= \begin{cases} 1-p & x = y \\ \dfrac{p}{q-1} & x \ne y \end{cases}$$

③ Binary Erasure channel:

$X = \{0,1\}$ $\quad Y = \{0,1,?\}$



$\equiv$ each bit gets erased w.p. α

See book for exaple where X/Y is infinite

Error correction

BSC_p: $\rightarrow$ There is $\neq 0$ ~~prob~~ that a codeword ~~gets~~ is received as another valid codeword.

BEC_α: There is non-zero prob (α^n) that all bits get erased

$$\left(\frac{1}{\alpha}\right)^n = 2^{-(\log_2 \frac{1}{\alpha})n}$$

WANT: Every message can be decoded w.p $1 - f(n)$

$\lim_{n \to \infty} f(n) = 0$ $\qquad$ Ideally: $f(n) = 2^{-\Omega(n)}$

Shannon's general result

Big: R vs error $\longrightarrow$ error parameter

$BSC_p \rightarrow p$ $\qquad \alpha$: BEC_α

Shannon's "thm" (informal) For every channel there is a specific number $0 \le C \le 1$ s.t

(•) $\forall R < C \Rightarrow (\exists$ code$)$ reliable communication $\leftarrow$ dec error prob $2^{-\Omega(n)}$

(•) $\forall R > C \Rightarrow (\nexists$ codes$)$ reliable communication is not possible

$C \rightarrow$ capacity of channel.

Next: Capacity thm for BSC_p

Notation: $\bar{e} \sim BSC_p$ $\quad \bar{e} \in \{0,1\}^n$ s.t $\forall i$ $\bar{e}[i] = \begin{cases} 1 & \text{w.p } p \\ 0 & \text{o/w} \end{cases}$

$E(\bar{m}) + \bar{e} \longrightarrow$ valid received word

Shannon's capacity thm for BSC_p

$0 \le p < \frac{1}{2}$, $\quad 0 \le \varepsilon \le 1 - H(p)$.

Following is true for large enough n:

(1) $\exists\, \delta > 0$, an encoding function $E : \{0,1\}^k \rightarrow \{0,1\}^n$

a decoding function $D : \{0,1\}^n \rightarrow \{0,1\}^k$

for $k \le \lfloor (1 - H(p) - \varepsilon) n \rfloor$ s.t

$\forall\, \bar{m} \in \{0,1\}^k$

$$\Pr_{\bar{e} \sim BSC_p} \left[D\left(E(\bar{m}) + \bar{e}\right) \neq \bar{m} \right] \le 2^{-\delta n}$$

(2) If $k \ge \lceil (1 - H(p) + \varepsilon) n \rceil$ then

$\forall\, E : \{0,1\}^k \rightarrow \{0,1\}^n \quad D : \{0,1\}^n \rightarrow \{0,1\}^k$

$\exists\, \bar{m} \in \{0,1\}^k$

$$\Pr_{\bar{e} \sim BSC_p} \left[D\left(E(\bar{m}) + \bar{e}\right) \neq \bar{m} \right] \ge \frac{1}{2} \quad \rightarrow 1 - 2^{-\beta n}$$

$\Rightarrow$ capacity of BSC_p : $1 - H(p) \leftarrow$ prove later

qSC_p : $1 - H_q(p)$

BEC_α : $1 - \alpha$