Hannan-Kalai-V

$c_1 \ldots c_{t-1},\ c_0$ (FPL)

$x_t = M(c_0 \ldots c_{t-1}) = \arg\min_{x} \sum_{i=0}^{t-1} c_i^T x$

$FPL_{n}(1\ T) \leq BPL_{n}(1\ T) + \delta RT$

$BPL_{n}(1, T) \leq OPT(1, T) + E[c_0 \cdot M(c_{1..T}) - M(c_0)]$

$FPL_{n}(1..T) \leq [1+O(\varepsilon n)]\, OPT(1..T) + O\left(\frac{\ln(n)}{\varepsilon}\right)$

Hedge: $[1 + O(\varepsilon)]$

$\|C\|_\infty \le 1 \quad \|C\|_1 \le n$

$C_1, C_2, \ldots, C_T$

$C_i \Longrightarrow C_i^1 \cdot \vec{e}_1, \; C_i^2 \vec{e}_2, \ldots \; C_i^n \vec{e}_n \qquad \vec{e}_i = [0, 0, \ldots \overset{i}{1} \ldots 0]^T$

$\Longrightarrow C'_1, C'_2 \cdots \; C'_{nT} \quad |C'_i|_1 \le 1$

$\underline{FPL_u(1 \ldots T)} \le \underline{FPL_u(1, \ldots, nT)} \le (1 + O(\varepsilon))\, OPT(1, \ldots, nT) + O\left(\frac{\ln n}{\varepsilon}\right)$

$OPT(1, \ldots, nT) = OPT(1 \ldots T)$

$E[FPL_u(1 \ldots T)] \le \underline{E[FPL_u(1 \ldots nT)]}$

e.g. $C_i^k \cdot \vec{e}_k$, $\underline{(i-1)n+k}$

$\cdot \Pr[\text{cost} = c_i^k \mid N_{(i-1)\cdot n+k}] \ge \text{Prob}[\text{cost} = c_i^k \mid O_i]$

e.g. $k = ?$

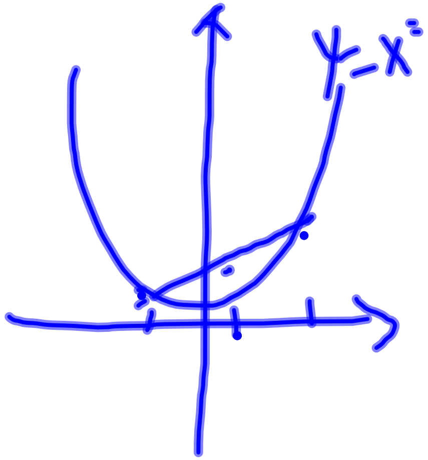
2. online convex optimization

① $L \subset \mathbb{R}^n$, convex set

② $f_1, f_2, \cdots, f_T$, convex cost function

③ $\|x-y\|_2 \le D$, $\forall x, y \in L$

④ $\|\nabla f\|_2 \le A$ for $\forall f_i$ $1 \le i \le T$



$$P(z) = \underset{x \in L}{\arg\min} \|x - z\|_2$$

Given strategies $x_1, x_2, \cdots, x_T$, $\eta_1 \geq \cdots \geq \eta_T$

$$x_{t+1} = P(x_t - \eta_t \nabla f_t(x_t))$$ (Zinkevich)

Thm: For every $x \in L$, the strategies $x_1, x_2, \cdots, x_T$ selected by

$$\sum_{t=1}^{T} f_t(x_t) \leq \sum_{t=1}^{T} f_t(x) + \frac{D^2}{\eta_T} + \frac{1}{2} A^2 \sum_{t=1}^{T} \eta_t$$

$$\|x_t - x\|_2 \to 0 \quad t \to \infty$$

$$\Phi(y) = \|y - x\|_2$$

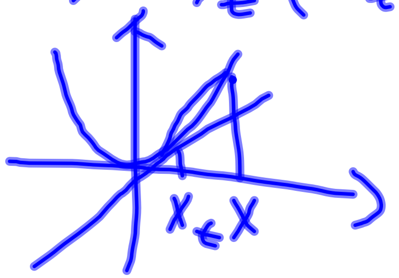
$$\Phi(P(z)) \leq \Phi(z)$$

$$\Phi(x_{t+1}) - \Phi(x_t) \leq \Phi(x_t - \eta_t \nabla f_t(x_t)) - \Phi(x_t)$$

$$= \|(x_t - x) - \eta_t \nabla f_t\|_2^2 - \|x_t - x\|_2^2$$

$$= -2\eta_t \nabla f_t \cdot (x_t - x) + \eta_t^2 \|\nabla f_t\|^2$$

f_t is convex $\Rightarrow \nabla f_t \cdot (x_t - x) \geq f_t(x_t) - f_t(x)$



$$f(x) \geq f(x_t) + (x - x_t) \cdot \nabla f_t$$

$$\Phi(x_{t+1}) - \Phi(x_t) \leq -2\eta_t [f_t(x_t) - f_t(x)] + A^2 \eta_t^2$$

$$f_t(x_t) - f_t(x) \leq \frac{\Phi(x_t) - \Phi(x_{t+1})}{2\eta_t} + \frac{A^2 \eta_t}{2}$$

$$\sum_{t=1}^{T} \left(f_t(x_t) - f_t(x)\right) \le \frac{\Phi(x_1) - \Phi(x_{T+1})}{2\eta_T} + \frac{1}{2}A^2 \sum_{t=1}^{T} \eta_t$$

$$\le \frac{D^2}{\eta_T} + \frac{1}{2}A^2 \sum_{t=1}^{T} \eta_t$$

$$\|x_1 - x\|_2^2 - \|x_{T+1} - x\|_2^2$$

$$=$$

$$\eta_t = \frac{D}{A}\sqrt{1/t}$$

$$\therefore \sum_{t=1}^{T} f_t(x_t) \le \sum_{t=1}^{T} f_t(x) + O\left(DA\sqrt{T}\right)$$

For sequence $z_1, z_2, \ldots z_T \in \mathcal{L}$

$$\mathcal{L}(z_1, \ldots, z_T) = \sum_{t=1}^{T-1} \|z_{t+1} - z_t\|_2^2$$

Thm: For any $z_1 \ldots z_T$,

$$\sum_{t=1}^{T} f_t(x_t) \le \sum_{t=1}^{T} f_t(z_t) + \frac{D^2}{\eta_T} + \frac{2D\mathcal{L}(z_1, \ldots z_T)}{\eta_T} + \frac{A^2}{2}\sum_{t=1}^{T}\eta_t$$

Proof: $\Phi(x, z) = \|x - z\|_2^2$

$$\Phi(x_{t+1}, z_{t+1}) - \Phi(x_t, z_t) = [\Phi(x_{t+1}, z_{t+1}) - \Phi(x_{t+1}, z_t)] + [\Phi(x_{t+1}, z_t) - \Phi(x_t, z_t)]$$

$$\leq [\Phi(x_{t+1}, z_{t+1}) - \Phi(x_{t+1}, z_t)] - 2\eta_t [f_t(x_t) - f_t(z_t)] + A^2 \eta_t^2$$

$$\Phi(x_{t+1}, z_{t+1}) - \Phi(x_{t+1}, z_t)$$

$$= \|\underbrace{x_{t+1} - z_{t+1}}_{v}\|_2^2 - \|\underbrace{x_{t+1} - z_t}_{w}\|_2^2$$

$$= (v+w)\cdot(v-w)$$

$$\leq \|v+w\|_2 \, \|v-w\|_2$$

$$\leq 2D \, \|z_t - z_{t+1}\|_2$$

$$\Phi(x_{t+1}, z_{t+1}) - \Phi(x_t, z_t) \leq 2D\|z_t - z_{t+1}\|_2 - 2\eta_t [f_t(x_t) - f_t(z_t)] + A^2\eta_t^2$$

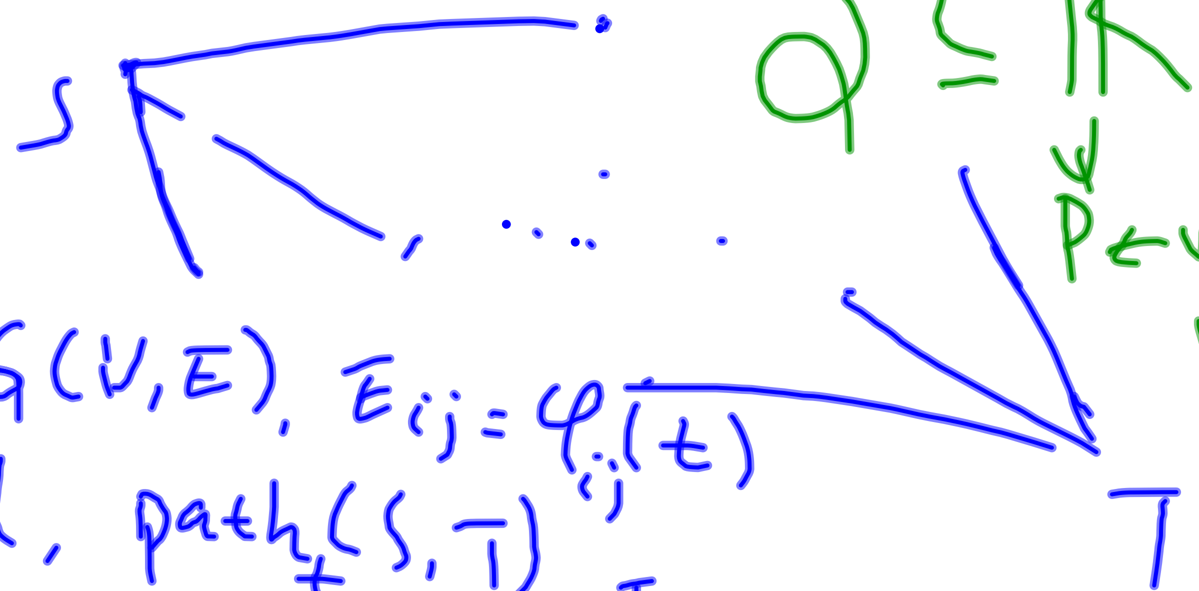
$$f_t(x_t) - f_t(z_t) \leq \frac{2D}{\eta_t}\|z_{t+1} - z_t\|_2 + \frac{\Phi(x_{t+1}, z_{t+1}) - \Phi(x_t, z_t)}{2\eta_t} + \frac{A^2}{2}\eta_t$$

$$\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(z_t) \leq \frac{2D}{\eta_T} L(z_1, \ldots z_T) + \frac{D^2}{\eta_T} + \frac{A^2}{2}\sum_{t=1}^{T}\eta_t$$

online shortest path.

$S \subseteq \mathbb{R}^{|E|}$

$P \leftarrow$ valid path



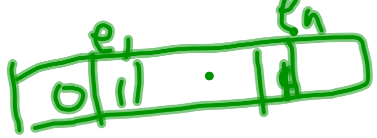
given $G(V,E)$, $E_{ij} = \varphi_{ij}(t)$

goal, $path_t(S,T)$

$$\sum_{t=1}^{T} cost(P_t(S,T)) \le \sum_{t=1}^{T} cost\, P_t(X) + \delta$$

$c_t = (c_1^t, \dots, c_1^{|E|})$

$e_1, \dots, e_n$



Tree update problem.

Give $D = \{x_1, \dots x_n\}$

goal: $\sum_{t=1}^{T} \text{cost}(\text{tree}_t) + \sum_{j=1}^{M} \text{cost}_{\text{u}}(j) \leq \sum_{t=1}^{T} \text{cost}(\text{tree}) + \Delta$

Ran select S_a, V_a

After each query, query a.

$S_a = S_a + 1$

if $S_a > V_a$

$V_a = V_a + N.$

update tree.

$N = \sqrt{T/n}$ $\quad E[\text{cost of } (\alpha = \gamma)] \leq E(\text{cost of best} + 2n\sqrt{nT})$