$G = k \times n$ matrix

$$\begin{bmatrix} g_1 & g_2 & \cdots & g_k & g_{k+1} & \cdots & g_n \end{bmatrix} \quad g_i \in \mathbb{F}_q^{k^T}$$

$$GT = G' = \left[I_k \,/\, A \right], \quad T \text{ invertible.}$$

RECAP:

Regret(A) = Cost(A) − Cost(OPT)

(for $\varepsilon < \frac{1}{2}$) $\mathrm{Cost(Hedge)} < (1+2\varepsilon)\,\mathrm{Cost(OPT)} + \frac{1}{\varepsilon}\ln n$

$$\Rightarrow R(\mathrm{Hedge}) < 2\varepsilon \underbrace{\boxed{\mathrm{Cost(OPT)}}}_{\leq T} + \ln n/\varepsilon$$

$$\leq 2\sqrt{2T\ln n} \quad \left(\varepsilon = \sqrt{\frac{\ln n}{2T}}\right)$$

TODAY:

(i) Remove the need to know T in advance

(ii) Definition of normal games & Nash Equilibrium

(iii) von Neumann's Minimax theorem on 'zero-sum' games.

↳ Proof using Hedge (& it's low regret property)

$\text{Hedge}^* \to$ At every time $t = 2^j$ $\quad j \geq 0, 1, \ldots$ (x_t is chosen expert)

Run $\text{Hedge}\left(\varepsilon = \sqrt{\frac{\ln n}{2^{j+1}}}\right)$ (restart at each 2^j)

$\underline{T = 2^{k+1} - 1}$ $\quad (2^k \leq T < 2^{k+1})$

$$R(\text{Hedge}^*) = \mathbb{E}\left[\max_{x \in [n]} \sum_{t=1}^{T} c_t(x_t) - c_t(x)\right]$$

$$= \mathbb{E}\left[\max_{x \in [n]} \sum_{j=0}^{k} \sum_{t=2^j}^{2^{j+1}-1} c_t(x_t) - c_t(x)\right]$$

$$\leq \sum_{j=0}^{k} \mathbb{E}\left[\max_{x \in [n]} \sum_{t=2^j}^{2^{j+1}-1} c_t(x_t) - c_t(x)\right]$$

$$= \sum_{j=0}^{k} R\left(\text{Hedge}\left(\varepsilon = \sqrt{\frac{\ln n}{2^{j+1}}}\right)\right)$$

$$\leq \sum_{j=0}^{k} 2\sqrt{2^{j+1} \cdot \ln n} < 2\sqrt{2^{k+1} \ln n} \sum_{i=0}^{\infty} \left(\sqrt{2}\right)^{-i} \quad (**)$$

$(**)$

$$< 2\sqrt{2T \ln n} \cdot \left(\frac{1}{1 - 1/\sqrt{2}}\right)$$

$$< 7\sqrt{T \ln n}.$$

Next: For any algo (even rand.) A

$$R(A) \geq \Omega\left(\sqrt{T \ln n}\right) \quad \{\text{for large enough } T\}$$

Bad instance: $c_t(x) = \begin{cases} 1 & \text{w.p. } \frac{1}{2} \quad 1 \leq t \leq T,\ x \in [n] \\ 0 & \text{w.p. } \frac{1}{2} \end{cases}$

$\forall t, x, \quad \mathbb{E}[c_t(x)] = \frac{1}{2}$

$\Rightarrow \mathbb{E}\left[\sum_{t=1}^{T} c_t(x_t)\right] = \frac{T}{2} \cdot \mathcal{N}(0,1)$

$\Pr[X \leq z] \leq e^{-\frac{z^2}{2}}$

Central limit theorem: iid $Y_1, \ldots, Y_T$, $\mathbb{E}[Y_1] = \mu$, $\mathrm{Var}[Y_1] = \sigma^2$

$$\lim_{t \to \infty} \frac{\sum_{t=1}^{T} Y_t - \mu T}{\sigma \sqrt{T}} \to \mathcal{N}(0,1)$$

$$\Pr\left[\frac{\sum Y_t - \mu T}{\sigma\sqrt{T}} \le z\right] = e^{-z^2/2} \pm o_T(1)$$

$\forall z \in \mathbb{R}$

$$\frac{T}{2} - \Theta\left(\sqrt{T \ln n}\right) \ge OPT = \mathbb{E}\left[\min_{x\in[n]} \sum_{t=1}^{T} c_t(x)\right]$$

Fix $x \in [n]$ $\quad Y_t = c_t(x) \quad \mu = \frac{1}{2}, \ \sigma^2 = \frac{1}{8}$

$$z = -\sqrt{2\alpha \ln n}$$

$(\alpha < 1)$

$$\Pr\left[\sum_{t=1}^{T} c_t(x) \le \frac{T}{2} - \sqrt{2\alpha T \ln n}\right] = e^{-\alpha \ln n} - o_T(1)$$

$$= \frac{1}{n^\alpha} - o_T(1)$$

$$\ge \frac{1}{2n^\alpha} \quad \text{for large enough } T$$

$$o_T(1) \le \frac{1}{2n^\alpha}$$

$$\Pr\left[\underbrace{\exists x}_{\min_{x\in[n]}} \text{ s.t. } \right] \ge 1 - \left(1 - \frac{1}{n^\alpha}\right)^n$$

$$1 - e^{-\theta(1-\alpha)n}$$

$$\mathbb{E}\left[\max_{x\in[n]} \sum_{t=1}^{T} c_t(x)\right] \leq \sum_{t=1}^{T} \mathbb{E}\left[\max_{x\in[n]} c_t(x)\right]$$

$$= \frac{T}{2}.$$

$$\boxed{Y_t = c_t(x) - \frac{1}{2}}$$

for large enough T, $R(A) \geq \Omega\left(\sqrt{T \ln n}\right)$

BERRY-ESSEEN

i.i.d. $Y_1, \dots, Y_T$ $\quad \mathbb{E}[Y_1] = 0$, $\mathrm{Var}[Y_1] = \mathbb{E}[X_1^2] = \sigma^2 < \infty$, $\sigma^2 = \frac{1}{4}$

$\exists$ an absolute constant $c > 0$ $\quad \mathbb{E}[|X_1|^3] = \rho < \infty$, $\rho = \frac{1}{8}$

$$\Pr\left[\frac{\sum_{t=1}^{T} Y_t}{\sigma\sqrt{T}} \leq z\right] \geq e^{-z^2/2} - \frac{c\rho}{\sigma^3\sqrt{T}}$$

$$z = -\sqrt{2\alpha \ln n} \qquad \frac{1}{n^\alpha} - \theta\left(\frac{1}{\sqrt{T}}\right) \qquad T \geq \Omega\left(n^{2\alpha}\right)$$

For our distribution

$\sqrt{2\pi m}\left(\frac{m}{e}\right)^m (1+o_m(1)) = m!$

$\text{Prob}[\ \ldots\ ] \le \frac{1}{n^\alpha}$ provided $T \geq \Omega(\ln n \cdot n)$

$= \#$ strings of wt $\le \frac{T}{2} - \theta(\sqrt{T \ln n})$

$$\frac{\sum_{l=\frac{T}{2}-2a\sqrt{T\ln n}}^{\frac{T}{2}-\sqrt{T\ln n}} \binom{T}{\frac{T}{2}-l}}{2^T}$$

$\frac{T}{2} - a\sqrt{T \ln n}$

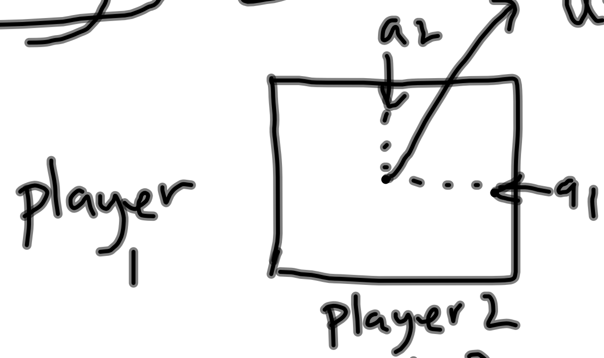
$$\boxed{\frac{T!}{\left(\frac{T}{2}-b\sqrt{T\ln n}\right)!\,\left(\frac{T}{2}+b\sqrt{T\ln n}\right)!}}$$

$$\binom{T}{\frac{T}{2}-b\sqrt{T\ln n}} = \frac{1}{\sqrt{T}\,n^{\theta(b^2)}}$$ provided $T \geq \Omega(\ln n \cdot n)$

Game theory $\boxed{n=2}$

$\left(u_1(a_1,a_2), \; u_2(a_1,a_2) \right)$

Normal-form game



→ n players

→ $\forall\, i \in [n]$, strategy set A_i (finite)

→ $\forall\, i \in [n]$, utility function

$$u_i(a_1, \dots, a_n) \in \mathbb{R}$$

$$a_i \in A_i$$

(ii) Cannot predict utility value. (i) Which of multiple NE to pick?

Bob Marley OR Shakira

NE: expected payoff $\boxed{\frac{2}{3}}$

P1: $\begin{cases} B & \text{wp } 2/3 \\ S & \text{wp } 1/3 \end{cases}$

P2: $\begin{cases} B & \text{wp } 1/3 \\ S & \text{wp } 2/3 \end{cases}$

P1 \ P2	B	S
B	(2,1)	(0,0)
S	(0,0)	(1,2)

Pure NE: (B,B)
(S,S)

Bob Marley AND Shakira

Prisoner's Dilema problem

	H	L
H	(3,3)	(1,4)
L	(4,1)	(1,1)

L is dominant strategy NE

Penalty kick

	L	R
L	$(-1,1)$	$(1,-1)$
R	$(1,-1)$	$(-1,1)$

Rows: striker. Columns: Goalie.

Matching Pennies

<u>Only</u> NE: Both players

L w.p. $\frac{1}{2}$

R — $\frac{1}{2}$

0 sum game

$$u_1(a_1,a_2) = -u_2(a_1,a_2)$$

Mixed strategy (strategy set A)

$$\Delta(A) = \{ p : A \to [0,1] \mid \sum_{a_i \in A} p(a_i) = 1 \}$$

every $(P_1, \ldots, P_n) \in \prod_{i \in [n]} \Delta(A_i) \equiv \Delta(A_1) \times \Delta(A_2) \cdots \times \Delta(A_n)$

mixed strategy profile

$$u_i(P_1, \ldots, P_n) = \sum_{\bar{a} \in \prod_j A_j} u_i(\bar{a}) \cdot p_1(a_1) \cdot p_2(a_2) \cdots p_n(a_n)$$

$\bar{a} = (a_1, \ldots, a_n)$

$\in \prod_{i \in [n]} A_i$ pure strategy profile

Nash Equilibrium (NE)

$\bar{a} \in \prod_{j \in [n]} \Delta(A_j) \quad b \in \Delta(A_i)$

<u>Def</u>: $(b, a_{-i}) \overset{def}{=} (a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n)$

<u>Def</u>: A mixed (pure) strategy profile $\bar{p} = (p_1, \ldots, p_n)$ is a mixed (pure) NE (or NE) if $\forall i \in [n]$ $\forall q_i \in \Delta(A_i)$

$$u_i(q_i, p_{-i}) \leq u_i(p_i, p_{-i})$$

~~Next (class)~~

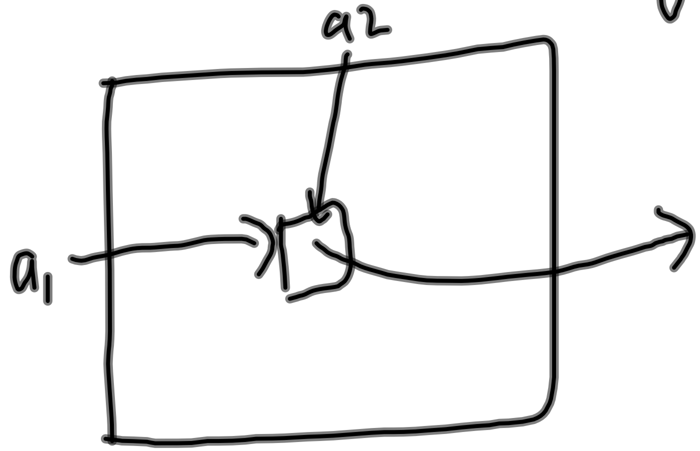
for player 2:

$$\min_q \max_p u_1(p,q)$$

for player 1

$$\max_p \min_q u_1(p,q)$$

von Neumann's Minimax theorem for 0 sum games



$$u_2(a_1,a_2) = -u_1(a_1,a_2)$$