CSE 736

Online

Learning

RECAP

Criticism of NE

(i) If multiple NEs, which one will occur?

GAMES: (i) n players (ii) Strategy sets: $A_1, \dots, A_n$

(iii) Utility functions: $\forall i,\ u_i : \prod_{j=1}^{n} A_j \to \mathbb{R}$

MIXED STATEGIES: $\Delta(A_i)$: set of all prob. dist. over A_i ← pure strategy

$$u_i(p_1, \dots, p_n) = \sum_{(a_1, \dots, a_n) \in \prod_j A_j} u_i(a_1, \dots, a_n) \cdot p_1(a_1) \cdot p_2(a_2) \cdots p_n(a_n)$$

NE: $(p_1, \dots, p_n) \in \prod_j \Delta(A_j)$ is a NE if $\forall i$ & $\forall p_i' \in \Delta(A_i)$

$$u_i(p_i', p_{-i}) \leq u_i(p_i, p_{-i})$$

(ii) May not even be able to state the stable payoff.

(iii) Sort of circular definition

2 player 0-sum game

(i) $n=2$ (ii) A_1, A_2 (iii) $u_2(a_1,a_2) = -u_1(a_1,a_2)$

$\forall\ (a_1,a_2) \in A_1 \times A_2$

Given $q \in \Delta(A_2)$, player 1 would like to

$$\arg\max_{p \in \Delta(A_1)} \ u_1(p,q)$$

Given $p \in \Delta(A_1)$, player 2 would like to

$$\arg\min_{q \in \Delta(A_2)} \ u_1(p,q)$$

von Neumann's Minimax THM:

$\forall$ 2 player 0-sum games with $u_1 : A_1 \times A_2 \to \mathbb{R}$,
$\exists$ an $v \in \mathbb{R}$ (value of the game) s.t.

(i) $v = \max\limits_{\underbrace{p \in \Delta(A_1)}} \min\limits_{\underbrace{q \in \Delta(A_2)}} u_1(p,q) = \min\limits_{q} \max\limits_{p} u_1(p,q)$

(ii) $\exists$ a mixed NE. $\forall$ NE $(p,q) \in \Delta(A_1) \times \Delta(A_2)$ iff

(1) $p \in \arg\max\limits_{p'} \min\limits_{q'} u_1(p',q')$

(2) $q \in \arg\min\limits_{\hat{q}} \max\limits_{\hat{p}} u_1(\hat{p},\hat{q})$

(iii) $\forall$ NE (p,q), $v = u_1(p,q)$

Main Lemma: (...) $\boxed{\max_{p'} \min_{q'} u_1(p',q')} = L$

$\geq \boxed{\min_{\hat{q}} \max_{\hat{p}} u_1(\hat{p},\hat{q})} = R$

Pf of (i) of THM:

$(\geq)$: Main Lemma

$(\leq)$: for any $(\hat{p},\hat{q})$ $\quad u_1(\hat{p},\hat{q}) \leq \max_{p'} u_1(p',\hat{q})$

$q^* \in \arg\min_{q'} \max_{p'} u_1(p',q')$

$$\underline{\min_{q'} u_1(\hat{p},q')} \leq u_1(\hat{p},q^*) \leq \max_{p'} u_1(p',q^*) = R$$

↑ As true $\forall \hat{p}$,

$L \leq R$

Pf of (ii) & (iii)

$B_1 \stackrel{\text{def}}{=} \arg\max_{p'} \min_{q'} u_1(p',q')$

$B_2 \stackrel{\text{def}}{=} \arg\min_{q'} \max_{p'} u_1(p',q')$

Claim: $B_1, B_2 \neq \emptyset$

$B_1 \neq \emptyset \Leftarrow$ for every fixed q, $\exists$ a p that $\max u_1(p,q)$ (pure strategy)

$\sim B_2 \neq \emptyset$ (Formal: Compactness of $\Delta(A_1), \Delta(A_2)$, A_1 & A_2 are finite & u_1 is continuous.)

$(p,q) \in B_1 \times B_2 \overset{①}{\Rightarrow} (p,q)$ is NE ✓

② $v = u_1(p,q)$

① player 1 plays $p' \neq p$

$u_1(p',q) \leq u_1(p,q)$ (as $p \in B_1$)

~ player 2 $\forall q' \neq q$, $u_1(p,q') \geq u_1(p,q)$

② $v = \max_{p'} \min_{q'} u_1(p',q') \underset{\text{as } p \in B_1}{=} \min_{q'} u_1(p,q')$

$\underset{=}{\leq} u_1(p,q) \underset{=}{\leq} \max_{\hat{p}} u_1(\hat{p},q) \underset{q \in B_2}{=} \min_{\hat{q}} \max_{\hat{p}} u_1(\hat{p},\hat{q}) \underset{(i)}{=} v$

(p,q) NE $\Rightarrow (p,q) \in B_1 \times B_2$

& $u_1(p,q) = v \Rightarrow q \in B_2$

$$u_1(p,q) = \max_{p'} u_1(p',q) \underset{=}{\geq} \min_{q'} \max_{p'} u_1(p',q') = v$$

$$u_1(p,q) = \min_{\hat{q}} u_1(p,\hat{q}) \underset{=}{\leq} \max_{\hat{p}} \underbrace{\min_{\hat{q}} u_1(\hat{p},\hat{q})}_{=v}$$

$\Rightarrow u_1(p,q) = v$

$\Downarrow$

$p \in B_1$

Pf of Main Lemma (2 proofs)

Regret of (Max)Hedge $\left(\varepsilon = \sqrt{\frac{\ln n}{T}}\right)$

$$\mathbb{E}\left[\sum_{t=1}^{T} g_t(x_t)\right] \geq \mathbb{E}\left[\underbrace{\max_{x \in [n]} \sum_{t=1}^{T} g_t(x)}_{\leq\, \mathbb{E}\left[\max_{x \in \Delta([n])} \sum_{t=1}^{T} g_t(x)\right]}\right] - \underbrace{2\sqrt{T \ln n}}_{T\delta}$$

$$T = \left\lceil \frac{4 \ln n}{\delta^2} \right\rceil$$

$$\max_{p'} \min_{q'} u_1(p', q') \geq \min_{\hat{q}} \max_{\hat{p}} u_1(\hat{p}, \hat{q})$$

$+\delta \qquad -\delta \qquad +\delta$

W.l.o.g. $u_1(a_1, a_2) \in [0,1] \quad \forall (a_1, a_2) \in A_1 \times A_2$

(pick b & c s.t. $u_1' = b\, u_1 + c$)

$n = \max\{|A_1|, |A_2|\}$

Player 1 runs Hedge(ε)
each $a_1 \in A_1$ is an expert

$p_1, \ldots, p_T$

$g_t(x) = u_1(x, q_t)$

Player 2 runs Hedge(ε)
each $a_2 \in A_2$ is an expert

$q_1, \ldots, q_T$

$g_t(x) = 1 - u_1(p_t, x)$

Player 1

By Hedge's regret thm:

$$\frac{1}{T}\sum_{t=1}^{T} u_1(p_t,q_t) \geq \boxed{\max_p \frac{1}{T}\sum_{t=1}^{T} u_t(p,q_t) - \delta}$$

$$\bar{p} = \frac{1}{T}\sum_{t=1}^{T} p_t$$

for any fixed q

$$\frac{1}{T}\sum_{t=1}^{T} u_1(p_t,q) = u_1(\bar{p},q)$$

Player 2

$$\frac{1}{T}\sum_{t=1}^{T} 1-u_1(p_t,q_t)$$

$$\geq \max_q \frac{1}{T}\sum_{t=1}^{T} 1-u_1(p_t,q) - \delta$$

$$\Rightarrow \frac{1}{T}\sum_{t=1}^{T} u_1(p_t,q_t) \leq \min_q \frac{1}{T}\sum_{t=1}^{T} u_1(p_t,q) + \delta$$

for any fixed p

$$\frac{1}{T}\sum_{t=1}^{T} u_1(p,q_t) = u_1(p,\bar{q})$$

$$\bar{q} = \frac{1}{T}\sum q_t$$

$$\min_q \frac{1}{T}\sum_{t=1}^{T} u_1(p_t,q)+\delta \geq \frac{1}{T}\sum_{t=1}^{T} u_1(p_t,q_t)$$

$$\geq \max_p \frac{1}{T}\sum_{t=1}^{T} u_1(p,q_t)-\delta$$

$\forall \hat{q}$ $\geq$

$$\frac{1}{T}\sum u_1(p_t,\hat{q})+\delta = u_1(\bar{p},\hat{q})+\delta$$

$$\geq \frac{1}{T}\sum u_1(\hat{p},q_t)-\delta = u_1(\hat{p},\bar{q})-\delta$$

$\forall \hat{p}$

$$\Rightarrow \underbrace{\min_q u_1(\bar{p},q)}_{\leq\, \max_p \min_q u_1(p,q)}+\delta \geq \underbrace{\max_p u_1(p,\bar{q})}_{\geq\, \min_q \max_p u_1(p,q)}-\delta$$

$$\overbrace{\max_p}^{=\,v+\delta} \min_q \; u_1(p,q) + \delta$$

$$\geq \frac{1}{T} \sum_{t=1}^{T} u_1(p_t, q_t)$$

$$\geq \min_q \max_p \; \underbrace{u_1(p,q)}_{=\,v-\delta} - \delta$$

$$T = \Theta\left(\frac{\ln n}{\delta^2}\right)$$

Yao's lemma (prove lower bounds for randomized algo).

$\mathcal{I} \to$ set of all inputs $\quad \mathcal{D} = \Delta(\mathcal{I})$

$\mathcal{A} \to$ ——— deterministic algorithms.

$i \in \mathcal{I}, a \in \mathcal{A} \Rightarrow t(i,a) \to$ cost of running a on i.

$\mathcal{R} = \Delta(\mathcal{A})$ = set of all randomized algos.

$$\min_{r \in \mathcal{R}} \max_{i \in \mathcal{I}} t(i,r) = \min_{r \in \mathcal{R}} \max_{d \in \mathcal{D}} t(d,r)$$

$$\underset{\text{vN}}{=} \max_{d \in \mathcal{D}} \min_{r \in \mathcal{R}} t(d,r) = \max_{d \in \mathcal{D}} \min_{a \in \mathcal{A}} t(d,a) \geq \min_{a \in \mathcal{A}} t(d^*,a)$$

pick $d^* \in \mathcal{D}$