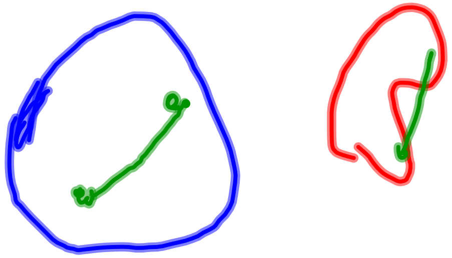
Dick Lipton (3pm, here)

Quantum Computing

Nash's Thm: Every normal form game $\mathcal{G} = (k, \{A_i\}_{i\in[k]}, \{u_i\}_{i\in[k]})$ $k, \bigcup_i A_i) < \infty$ has a mixed NE.

Brower's fixed pt thm: Let $X \subseteq \mathbb{R}^n$ be a compact (closed, bounded) convex set. Then every $f: X \to X$ that is continuous has a fixed pt

i.e. $y \in X$ s.t. $f(y) = y$



Proof framework

$\mathcal{G} \xrightarrow{\text{Step 1}}$ (i) define $f : X \longrightarrow X$

$$X = \prod_{i=1}^{k} \Delta(A_i)$$

s.t.

(1) f is continuous

(2) Every fixed pt. of f is a NE of $\mathcal{G}$

Nash's thm $\xleftarrow[\text{thm}]{\text{Brower's}}$

$$f(p_1, \ldots, p_k) = (q_1, \ldots, q_k) \text{ s.t. }$$

$\forall i \in [k]$

$$q_i = \arg\max_{q \in \Delta(A_i)} \left\{ u_i(q, p_{-i}) - \|q - p_i\|_2^2 \right\}$$

$\rightarrow$ f is a well-defined function

$\rightarrow$ f is cont.

$\rightarrow$ Every f.p. of f is a NE

(<u>Technical Lemma</u>) If $A \subseteq \mathbb{R}^n$ that is convex & closed, then $\forall x \in \mathbb{R}^n$, let $y = g(x) = \arg\min_{y \in A} \|x-y\|$ is (1) unique (2) g is cont.

① f is cont. & well-defined

$$\Big(X = \prod_{i\in[k]} \Delta(A_i)\Big)$$ compact & convex.

$$u_i(q, p_{-i}) = \sum_{a\in A_i} q(a)\, u_i(a, p_{-i})$$

$$\boxed{-\|q - (v_i(p) + p_i)\|_2^2 + \text{const}} = \langle q, v_i(p)\rangle$$

a^{th} value is $u_i(a, p_{-i})$

$$u_i(q, p_{-i}) - \|q - p_i\|_2^2 = \langle q, v_i(p)\rangle - \langle q - p_i, q - p_i\rangle$$

$$= \langle q, v_i(p)\rangle - \|q\|^2 + 2\langle q, p_i\rangle + \|p_i\|$$

$$= -\|q\|^2 + \langle q, v_i(p) + 2p_i\rangle - \|p_i\|^2 - \|p_i\|^2$$

Thus,

$$q_i = \arg\min_{q \in \underbrace{\Delta(A_i)}_{\text{convex \& compact.}}} \left\| q - \boxed{\left(\frac{v_i(p)}{2} + p_i\right)} \right\|$$

for each dim

By techincal lemma:

(i) q_i is unique $\Rightarrow$ f is well-defined

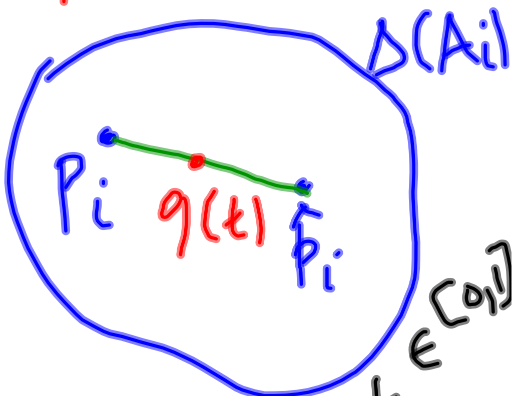
(ii) q_i "function" is cont $\Rightarrow$ f is cont.

Every fixed pt. of f is a NE

$$f(p_1,\dots,p_k)=(p_1,\dots,p_k)$$

fix $i\in[k]$ consider any $\hat p_i \neq p_i \in \Delta(A_i)$

$$\underbrace{u_i(q(t),p_{-i}) - \|q(t)-p_i\|^2}_{\text{max at } q(0)=p_i \text{ (by def. of } f)}$$



$$q(t)=p_i+t(\hat p_i-p_i), \quad t\in[0,1]$$

$$\Rightarrow \frac{d}{dt}\left(u_i(q(t),p_{-i}) - \|q(t)-p_i\|^2\right)\Big|_{t=0} \le 0$$

$$= u_i(\hat p_i,p_{-i}) - u_i(p_i,p_{-i}) \le 0$$

$$\|q(t)-p_i\|^2 = \|t(\hat{p}_i - p_i)\|^2 = t^2\|\hat{p}_i - p_i\|^2$$

$$q(t) = p_i + t(\hat{p}_i - p_i)$$

$$\Rightarrow \frac{d}{dt}\left(\|q(t)-p_i\|^2\right)\Big|_{t=0} = 0.$$

$$u_i(q, p_{-i}) = \langle q, v(p)\rangle$$

$$\frac{d}{dt}\left(u_i(q(t), p_{-i})\right)$$

$$= \left\langle \frac{d}{dt}q(t), v(p)\right\rangle + \left\langle q(t), \frac{d}{dt}(v(p))\right\rangle = 0$$

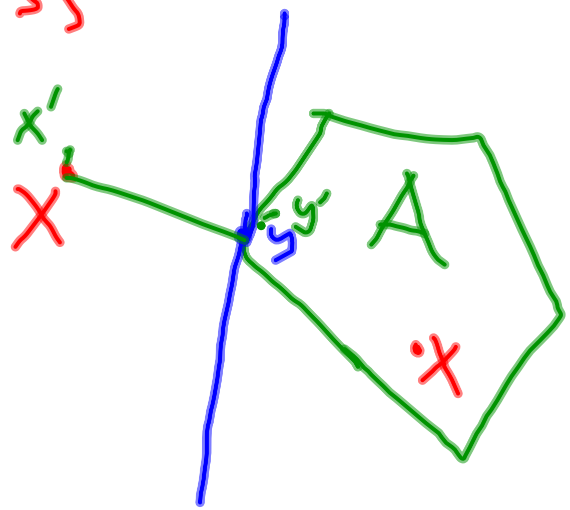
$$= \langle \hat{p}_i - p_i, v(p)\rangle = \langle \hat{p}_i, v(p)\rangle - \langle p, v(p)\rangle$$

$$= u_i(\hat{p}_i, p_{-i}) - u_i(p, p_{-i})$$

Technical Lemma: $A \subseteq \mathbb{R}^n$ compact + convex

Then $\forall x \in \mathbb{R}^n$, $\arg\min_{y \in A} \|x-y\|$ is unique & cont.

Uniqueness obvious for $x \in A$.

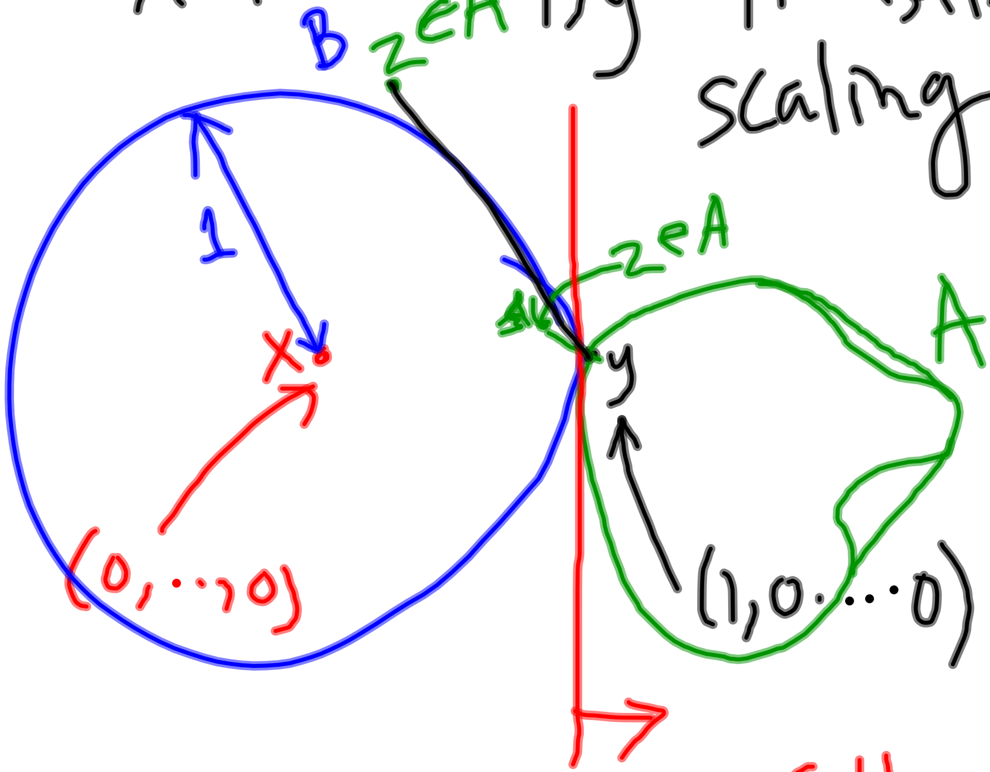


$d(y) = \|x-y\|$ $(y \in A)$

$\Rightarrow$ min of $\|x-y\|$ attained at ≥ 1 $y \in A$.

(A is compact)

$x \notin A$ By translation, rotation & scaling



if $z = (<1, \cdots)$
$\Rightarrow z \notin A$

No interior pt in B is in A

(oth if $z \neq y \in B \cap A$
$\Rightarrow$ y is not the closest pt.)

$\Rightarrow \exists$ hyperplane H

$A \subseteq H$

Closest pt to x on H is y (& is unique)

Continuity

$\forall \varepsilon > 0, \exists \delta > 0$ s.t

$\forall x, x'$ s.t $\|x - x'\| \leq \delta$

$\Rightarrow \|y - y'\| \leq \varepsilon$

$\left(\begin{array}{l} x \leftarrow y \\ x' \leftarrow y' \end{array} \right)$

$\|x - y'\| \leq \|x - x'\| + \|x' - y'\|$

$\leq \|x - x'\| + \|x' - y\|$

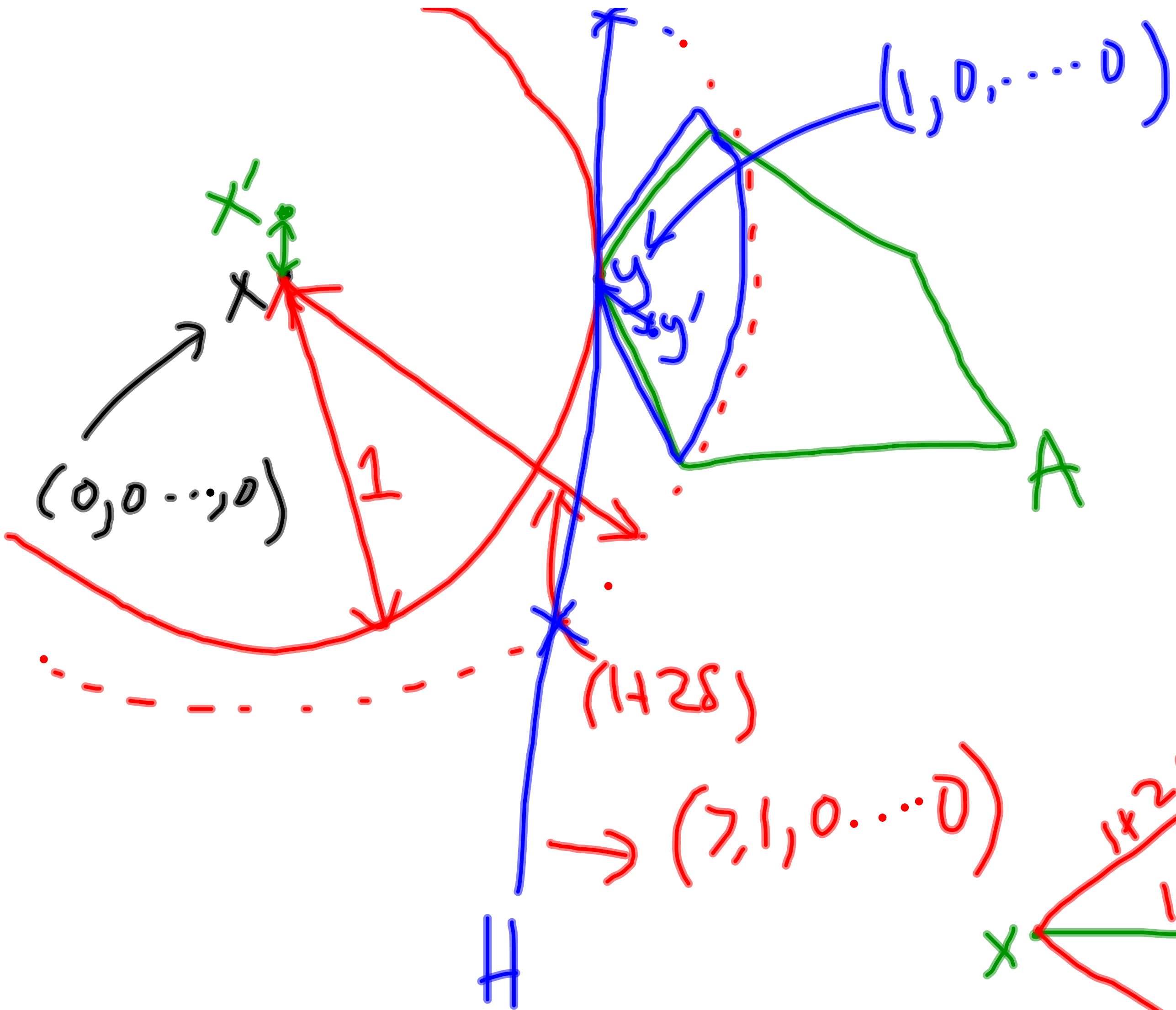
$\leq \|x - x'\| + \|x' - x\| + \|x - y\|$

$\leq \overbrace{\|x - y\|}^{r_0} + 2\delta$

$y, y' \in B(x, r_0 + 2\delta)$

Pick δ small s.t $\mathrm{diam}(B(x, 2\delta)) \leq \varepsilon$

If $r_0 = 0$ $\Rightarrow \mathrm{diam}(B(x, 2\delta)) \to 0$ as $\delta \to 0$



$r_0 = 1$

$A \subseteq H$

$\operatorname{dia}(B(x, 1+2\delta) \cap H) \to 0$ as $\delta \to 0$

Next two weeks

Blackwell's approachability thm

"No" internal regret

Correlated Equilibria

NE is PPAD-hard

EP