DECISION TREE MODEL

$f : \{0,1\}^n \longrightarrow \{0,1\}$

$f(x_1, \ldots, x_n)$

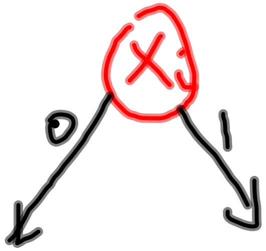
Goal: given x compute $f(x)$

Decision trees

→ Binary tree

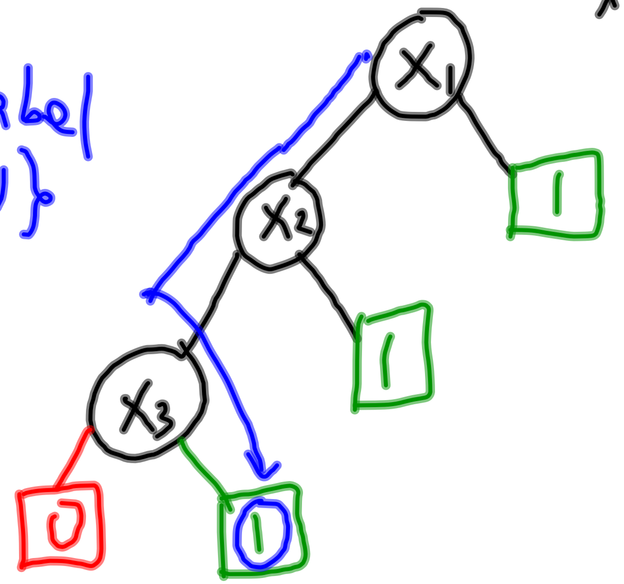
→ Each leaf has a label $\in \{0,1\}$

→ Internal node

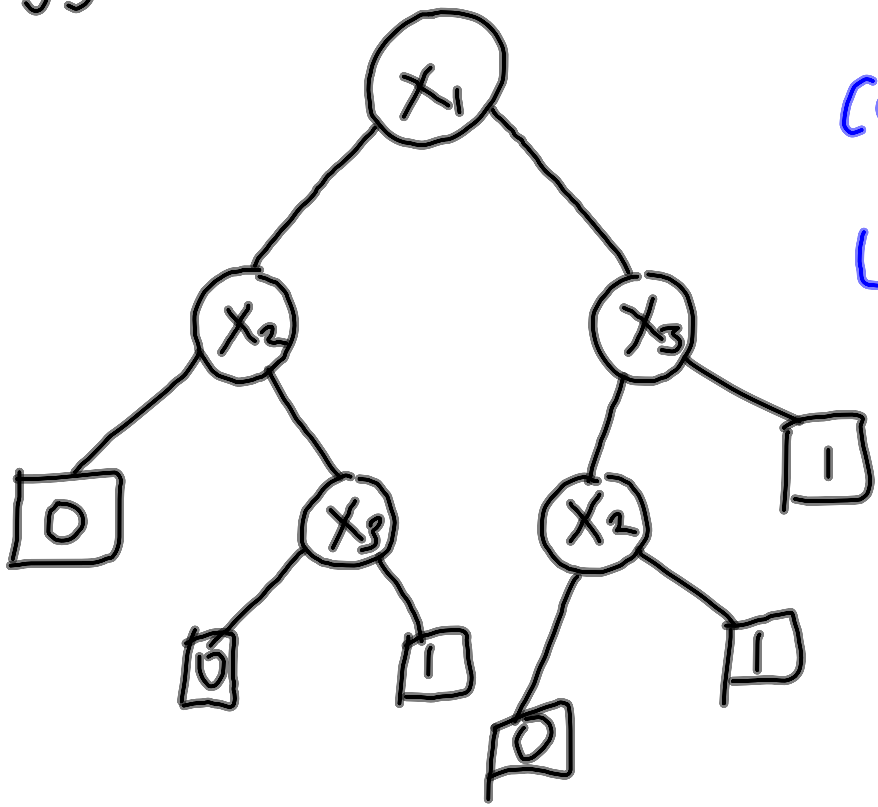


OR_3 function

$x = (0,0,1)$



Maj_3



$\mathrm{cost}(0,0,1) = 2$

$\mathrm{cost}(1,0,1) = 3$

→ A decision tree computes f if
$\forall x, \ T(x) = f(x)$

value of leaf for x

cost(x, T) = depth of leaf in X

\# queries need to make according to T.

Deterministic decision tree complexity of f

$$D(f) = \min_{T \in \mathcal{T}_f} \max_{x \in \{0,1\}^n} \text{cost}(x, T)$$

all trees that compute f.

$D(f) \leq n$

Claim: $D(OR_n) = n$

$D(Maj_n) = n$

$D(Parity_n) = n$

Pf: Adversarial argument:

→ Give me any T for OR_n

→ give you an input x s.t $cost(x, T) = n$.

For all first $n-1$ (distinct) queries answer 0

⇒ last bit decides OR_n ⇒ T needs to query all n bits

Randomized decision trees (0-error)

$\hookrightarrow \; \in \Delta(\mathcal{T}_f)$ $\quad \mathrm{cost}(x,\mathcal{P})$

$$R(f) = \min_{\mathcal{P}\in\Delta(\mathcal{T}_f)} \; \max_{x\in\{0,1\}^n} \mathrm{cost}(x,\mathcal{P}) = \sum_{t\in\mathcal{T}_f} \mathrm{cost}(x,t)\,\mathcal{P}(t).$$

$$R(f) \le D(f) \le n$$

Lemma: $R(\mathrm{Maj}_3) \le \frac{8}{3}$ (<3)

Pf: Non-adaptive decision trees
(fix order of bits to query)

$3! = 6$ choices

$P \to$ uniform distribution over 3 choices (x_1, x_2, x_3)

show $\boxed{\max_{x \in \{0,1\}^3} \mathrm{cost}(x,P) \le 8/3}$

Case 1: All bits of x are the same
$\Rightarrow \mathrm{cost}(x,P) \le 2$

Case 2: 2 bits are same & 3^{rd} is diff

$\Pr[\text{ come first}] = \frac{1}{3}$ $\quad \mathrm{cost}(x,P) \le \frac{1}{3}\cdot 2 + \frac{2}{3}\cdot 3 = 8/3$

Q: If $R(\mathrm{Maj}_3) < \frac{8}{3}$? NO

Yao's lemma: $\min_{P \in \Delta(\mathcal{T}_{\mathrm{Maj}_3})} \max_{x \in \{0,1\}^n} \mathrm{cost}(x, P)$

$= \max_{d \in \Delta(\{0,1\}^3)} \min_{T \in \Delta(\mathcal{T}_{\mathrm{Maj}_3})} \mathrm{cost}(d, T)$

(Pick $d^* \in \Delta(\{0,1\}^3)$)

$\geq \min_{T \in \Delta(\mathcal{T}_{\mathrm{Maj}_3})} \mathrm{cost}(d^*, T)$

Lemma: $R(\mathrm{Maj}_3) \geq \frac{8}{3}$

("only" for non-adaptive trees) can remove this assumption

Pf: By Yao's lemma

d^*: prob 0 on $(0,0,0), (1,1,1)$

prob $\frac{1}{6}$ on the rest

Show: For all orderings of x_1, x_2, x_3 $(\Rightarrow T)$

$\mathrm{cost}(d^*, T) \geq 8/3$

$\Pr_{x \leftarrow_{d^*} \{0,1\}^n}[1^{st} \text{ two bits are the same}] = \frac{1}{3}$

($\cancel{\geq}$ 2 queries in this case)

$\Pr[\,1^{st}$ two bits are not same$\,] = \frac{2}{3}$

(3 queries in this case)

$$\mathrm{cost}(d^*, T) = \frac{1}{3}\cdot 2 + \frac{2}{3}\cdot 3$$

$$= \frac{8}{3}. \quad \blacksquare$$

Lemma: $R(\text{Parity}_n) = n$

Claim: For any T that computes Parity_n, the MINIMUM depth is n.

Ex. Prove
$R(OR_n) \geq \Omega(n)$
$R(\text{Maj}_n) \geq \Omega(n)$

Conj. ?..?
$R_\varepsilon(f) \geq D(f)$

THM 1 (Nash) For any normal form game $\mathcal{G} = ([K], \{A_1, \dots, A_K\}, \{u_1, \dots, u_K\})$ s.t. $K, |\bigcup_{i=1}^{K} A_i| < \infty$, there $\boxed{\text{exists}}$ a mixed NE.

Belief: $P \neq PPAD$

(Pf is non-constructive)

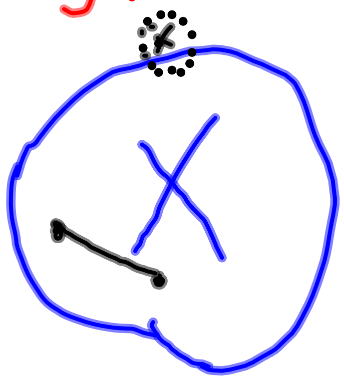
Necessary (2006-07, Dasalakis, Goldberg, Papadimitriou; Chen, Deng)

Computing NE is PPAD-hard (even $K=2$)

[Lipton, Marakakis, Mehta '03] ε-NE in $n^{\mathrm{poly}_\varepsilon(\log n)}$

Brower's fixed pt thm: If $X \subseteq \mathbb{R}^m$ s.t. X is compact (closed & bounded) and convex then for ANY continuous function $f: X \longrightarrow X$ has a fixed point, i.e. $\exists\, y \in X$ s.t. $f(y) = y$.

Bounded: $X \subseteq B(\bar{0}, r)$



Closed:

X convex

Proof idea

$\mathcal{G} \longrightarrow X = \underbrace{\prod_{i=1}^{k} \Delta(A_i)}$ closed, bounded, convex

"Somehow" pick $f: X \to X$ (continuous)

s.t any fixed pt $\bar{p} \in X$ of f is a NE of $\mathcal{G}$

$\xrightarrow{\text{Brouwer's}}$ THM 1.

Attempt 1: (will fail)

$$f(p_1, \ldots, p_K) = (q_1, \ldots, q_K)$$

$$\text{s.t. } \forall i \quad q_i \in \underset{q \in \Delta(A_i)}{\arg\max} \; u_i(q, p_{-i})$$

→ f may not be a function as multiple q_i's possible

→ Tie breaking mechanism

Problem 2: f might not be continuous!

Counter-ex: Penalty-kick game

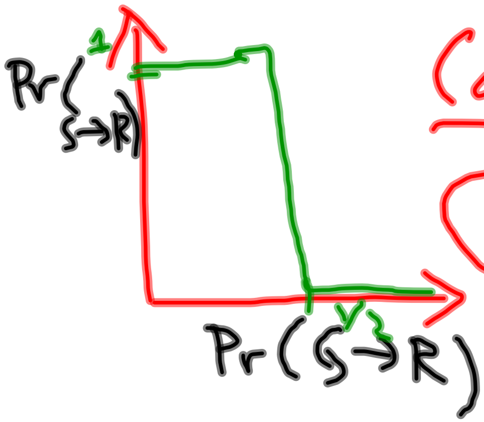
Striker

Goalie	L	R
L	1, −1	−1, 1
R	−1, 1	1, −1

Let (P_L, P_R) be Goalie's strategy

Case 1: $P_L > \frac{1}{2} \Rightarrow$ best response for striker is $(0,1)$

Case 2: $P_L < \frac{1}{2} \Rightarrow$ best response $(1,0)$



NOT continuous!

Nash: $f: X \to 2^X$

→ Analog of Brower's thm
Kakutani's fixed pt. thm for correspondences

→ best response $f \Rightarrow$ Nash's thm.

Final f:

$$f(p_1, \ldots, p_k) = (q_1, \ldots, q_k)$$

s.t. $\forall i$

$$q_i \in \underset{q \in \Delta(A_i)}{\arg\max} \left(u_i(q, p_{-i}) - \|q - p_i\|_2^2 \right)$$

NEXT LECTURE:

(i) f is a function

(ii) f is cont.

(iii) every fixed pt of f is a NE of $\mathcal{G}$.