Main tool: Games w/ vector payoff.

Def: $\mathcal{G} = (k, \{A_i\}_{i\in[k]}, \{\bar{u}_i\}_{i\in[k]})$

$\bar{u}_i : \prod_{j\in[k]} A_j \to \mathbb{R}^n$

Today: $\boxed{k=2}$

Analyze from player 1's perspective

$\bar{u}_1 \to \bar{u}$

Approachable Set

$\mathcal{G} = (2, \{A_1, A_2\}, \bar{u})$ $\quad \bar{u} : A_1 \times A_2 \to \mathbb{R}^n$

We say a set $S \subseteq \mathbb{R}^n$ is appraoachable if $\exists$ a randomized algo for player 1 to choose $i_1, i_2 \ldots \in A_1$ s.t. $\forall$ choices $j_1, j_2, \ldots \in A_2$

$$\text{dist}\left(\frac{1}{T}\sum_{t=1}^{T} u(i_t, j_t), S\right) \to 0 \text{ as } T \to \infty$$

$$\text{dist}(x, S) = \min_{y \in S} \|x - y\|_2$$

No (external) regret algo (n experts learning)

(max.)

$i_1, i_2, \ldots$

$g_t : [n] \to [0,1]$

$$\max_{i \in [n]} \frac{1}{T} \sum_{t=1}^{T} \left(g_t(i) - g_t(i_t) \right) \to 0 \text{ as } T \to \infty \quad (**)$$

$A_1 = [n]$, $A_2 = \{ g : [n] \to [0,1] \}$

$$\bar{u}(i,g) = \begin{pmatrix} g(1) - g(i) \\ g(2) - g(i) \\ \vdots \\ g(n) - g(i) \end{pmatrix}$$

$(**) \Leftarrow i_1, \ldots, i_t$ approaches S.

$S = \{ (x_1, \ldots, x_n) \mid x_i \le 0 \}$

$$\mathrm{dist}\left(\frac{1}{T} \sum_t \bar{u}(i_t, g_t), S \right) \to 0$$

No internal regret learning algo

$\hookrightarrow \max_{\substack{i \neq j \\ \in [n]}} \frac{1}{T} \sum_{t=1}^{T} \left(g(j) - g(i)\right) \cdot \mathbb{1}_{i_t = i}$ (*)

$\longrightarrow 0$ as $T \to \infty$

$A_1 = [n]$, $A_2 \{ g : [n] \to \{0,1\} \}$ $\quad \bar{u} : A_1 \times A_2 \to \mathbb{R}^{n \times n}$

$$u(i,g) = \begin{array}{c} \\ \left(\begin{array}{ccc} 0 & \begin{array}{c} i \\ g(1)-g(i) \\ g(2)-g(i) \\ \vdots \\ g(n)-g(i) \end{array} & 0 \end{array} \right) \end{array}$$

$S \doteq \{ (x_1, \ldots, x_n) \mid x_i \leq 0 \}$

$\text{dist}\left(\frac{1}{T} \sum_t \bar{u}(i_t, g_t), S \right) \to 0 \quad (\Rightarrow (*))$

(j,i) entry $\to$ average regret of $(i;j)$

Blackwell's Approachability Thm

Let $\mathcal{G}$ be a 2player normal form vector payoff game ($\bar{u} \to$ payoff for P1)

Let S be a non-empty closed convex set s.t. $\forall$ halfspaces $H \supseteq S$, H is approachable. Then S is approachable.

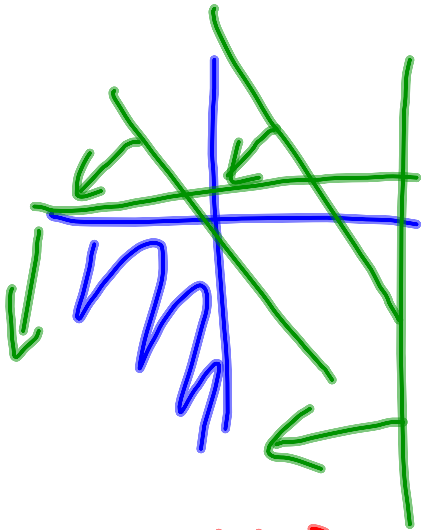
S_n be the -ve orthant in $\mathbb{R}^n$

To show S_n is approachable

show $\{a^T x \le b\}$

$a_i \geq 0, b \geq 0$

by showing can approach $\{a^T x \le 0\}$



Show $\forall g : [n] \to [0,1]$, $\exists\, p \in \Delta(A_1)$

s.t. $a^T u(p,g) = 0$.

$\exists$ No external regret learning algo

$$u(i, g) = \begin{pmatrix} g(1) - g(i) \\ \vdots \\ g(n) - g(i) \end{pmatrix} = g - \mathbb{1}(e_i^T g)$$

$$g = \begin{pmatrix} g(1) \\ g(2) \\ \vdots \\ g(n) \end{pmatrix}$$

$p \in \Delta([n])$

$$u(p, g) = g - \mathbb{1}(p^T g)$$

$\forall \bar{a} \in \mathbb{R}^n_{\geq 0}, \exists p \in \Delta([n]), \forall g : [n] \to [0,1]$

$$a^T(g - \mathbb{1}(p^T g)) = 0 \qquad \text{Take } p = \frac{a}{a^T \mathbb{1}}$$

$$\hookrightarrow a^T g - (p^T g)(a^T \mathbb{1})$$

<u>Lemma</u>: If M is a $n \times n$ matrix s.t

(i) $M_{ij} \geq 0 \quad \forall i \neq j$

(ii) $\mathbb{1}^T M = 0$

$\Rightarrow \exists p \in \mathbb{R}^n_{\geq 0}$ s.t $Mp = 0$

$\exists$ No internal regret algo

$$u(i,g) = \begin{array}{c} \begin{array}{ccc} & i & \end{array} \\ \begin{pmatrix} & g(1)-g(i) & \\ 0 & \vdots & 0 \\ & g(n)-g(i) & \end{pmatrix} \end{array}$$

$$\left[\; v \in \mathbb{R}^n \quad D(v) = \begin{pmatrix} v_1 & & 0 \\ & v_2 & \\ & & \ddots \\ 0 & & & v_n \end{pmatrix} \right.$$

Claim: $p \in \Delta([n])$

$$u(p,g) = g\,p^T - \mathbb{1}\,\underbrace{g^T D(p)}$$

Let $p \in e_i$

$$g\,p^T = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} \begin{array}{c} \begin{array}{ccc} & i & \end{array} \\ (0 \cdots 1 \cdots 0) \end{array} = \begin{array}{c} \begin{array}{ccc} & i & \end{array} \\ \begin{pmatrix} & g_1 & \\ 0 & \vdots & 0 \\ & g_n & \end{pmatrix} \end{array} \quad \begin{array}{c} \begin{array}{ccc} & i & \end{array} \\ \begin{pmatrix} & +g(i) & \\ 0 & \vdots & 0 \\ & +g(i) & \end{pmatrix} \end{array}$$

$$g^T D(e_i)$$

$$\underbrace{\begin{pmatrix} \vdots \\ \vdots \\ \vdots \end{pmatrix} (g_1, \dots, g_n)}_{\begin{pmatrix} g_1 & \cdots & g_n \\ g_1 & \vdots & g_n \\ \vdots & \vdots & \vdots \\ g_1 & & g_n \end{pmatrix}} \begin{array}{cc} & \begin{array}{cccc} & & i & \end{array} \\ i & \begin{pmatrix} 0 & & & & 0 \\ & 0 & & & \\ & & \ddots & & \\ & & & 1 & \\ 0 & & & & 0 \ddots 0 \end{pmatrix} \end{array} = \begin{array}{c} i \\ \begin{pmatrix} & g_i & \\ & \vdots & \\ 0 & \vdots & 0 \\ & \vdots & \\ & g_i & \end{pmatrix} \end{array}$$

$$u(p,g) = g p^T - \bar{1}\, g^T D(p).$$

A halfspace over $\mathbb{R}^{n\times n}$

is $Tr(A^T X) \le b$.

$\uparrow \in \mathbb{R}$

$$Tr(C) = \sum_i c_{ii}$$

Properties: (·) $Tr(A+B) = Tr(A) + Tr(B)$

(·) $Tr(AB) = Tr(BA) \Rightarrow Tr(ABC) = Tr(CAB)$

(·) $Tr(A^T) = Tr(A)$

Will prove

$\forall A \in \mathbb{R}^{n\times n}_{\geq 0}, \exists p \in \Delta([n]), \forall g : [n] \to [0,1]$

$$\mathrm{Tr}\left(A^T \left(g p^T - \bar{1} g^T D(p)\right)\right) = 0$$

$$= \underbrace{\mathrm{Tr}\left(A^T g p^T\right)}_{= \mathrm{Tr}(p^T A^T g) = \mathrm{Tr}(g^T A p)} - \underbrace{\mathrm{Tr}\left(A^T \bar{1} g^T D(p)\right)}_{= \mathrm{Tr}(g^T D(p) A^T \bar{1})} = 0$$

$D(v)w = D(w)v$

$\exists p$ s.t. $Ap = D(p) A^T \bar{1}$

Want $A p = D(p) A^T \bar{1}$

$$= D(A^T \bar{1}) p$$

$$\Leftarrow \underbrace{\overbrace{\left(A - D(A^T \bar{1})\right)}^{M}}_{\text{Use lemma here.}} p = 0$$

$$\mathbb{1}^T M = \mathbb{1}^T A - \mathbb{1}^T D(A^T \mathbb{1})$$

$$= \left[A^T \mathbb{1} - \underbrace{D(A^T \mathbb{1}) \mathbb{1}}_{D(\mathbb{1}) \cdot A^T \mathbb{1}}\right]^T$$

M $n \times n$ (i) $M_{ij} \geq 0$ $i \neq j$ $L_{ii} > 0$

(ii) $\mathbb{1}^T M = \mathbf{0}$

$\exists\, p \in \mathbb{R}^n_{\geq 0}$ s.t. $Mp = 0$

(i) $\Rightarrow \exists\, \varepsilon > 0$ s.t. $L = I + \varepsilon M \in \mathbb{R}^{n \times n}_{\geq 0}$

$\Rightarrow \forall\, q \in \mathbb{R}^n_{\geq 0} \setminus \{0\}$, $Lq \in \mathbb{R}^n_{\geq 0} \setminus \{0\}$

$q \longrightarrow \dfrac{Lq}{\mathbb{1}^T Lq}$ is continuous

$\Delta([n]) \longrightarrow \Delta([n])$

By BFPT $\Rightarrow \exists\, p \in \Delta([n])$

$$p = \frac{Lp}{\mathbb{1}^T Lp}$$

$$p\,(\underbrace{\mathbb{1}^T L p}_{=1}) = Lp$$

$$\mathbb{1}^T L p = (\mathbb{1}^T I + \varepsilon \underbrace{\mathbb{1}^T M}_{\substack{(ii)\\ =0}})\,p = \underbrace{\mathbb{1}^T p}_{=1}$$

$$\Rightarrow Lp = p$$

$$\Rightarrow (I + \varepsilon M)\,p = p + \varepsilon \underbrace{Mp}_{=0} = p$$