CSE 720: Sparse Approximation

Jan 20, 2012

Course blog

csandgt.wordpress.com

Course notes

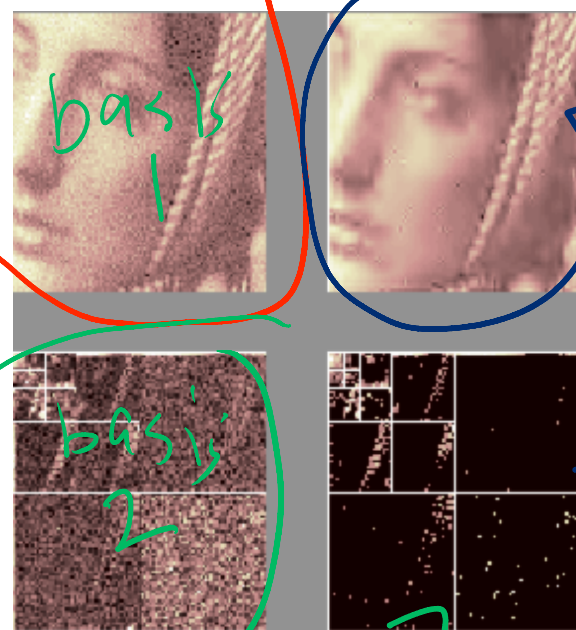
Email me comments
(pass onto Anna)

Case 1

original

"compressed" image

transform (T)



decompress (T^{-1})

(compression)

zero out small components

Sparse Approximation

given $Ax = b$ $(d \leq n)$

want sparse x.

$A \to$ given

Ex1: Image Compression:

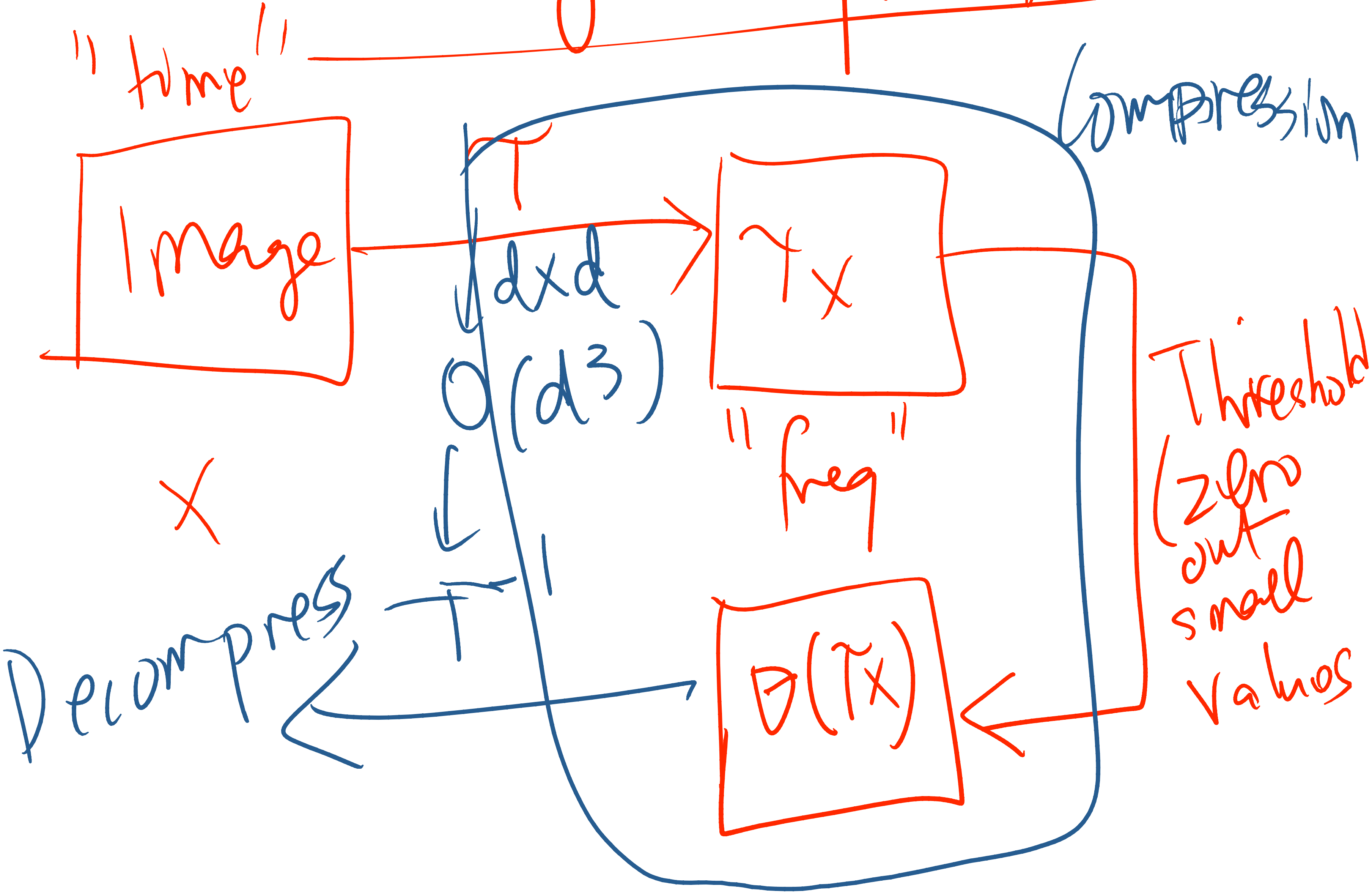
$b \to$ image $\quad A \to$ transform

(compressed) $x \to$ sparse (as possible)

Overview:

- → Background
- → Tools
- → Algorithms
- → Applications

Image compression



Claim Inverse in $O(d^2)$ time, "Orthogonal matrix" (over $\mathbb{C}$)

$$\boxed{\Phi^{-1} = \Phi^*}$$

(Hermetian)

(over $\mathbb{R}$) $= \Phi^T$

$$\boxed{\Phi^* = \left(\bar{\Phi}\right)^T}$$

Def

$$(a+ib)^* = (a-ib)$$

$$\begin{pmatrix} 1+2i & 2+3i \\ 3+4i & 4+5i \end{pmatrix}^* = \begin{bmatrix} 1-2i & 3-4i \\ 2-3i & 4-5i \end{bmatrix}$$

$\bar{x} = (x_1, \ldots, x_d) \in \mathbb{C}^d \quad \bar{y}$

$\rightarrow \langle \bar{x}, \bar{y} \rangle = \sum_{i=1}^{d} x_i \cdot (y_i)^*$

$\rightarrow \|\bar{x}\|_2 = \sqrt{\langle \bar{x}, \bar{x} \rangle}$ ("Euclidean" norm)

$\rightarrow$ $\bar{x}$ is unit if $\|x\|_2 = 1$

$\rightarrow$ $\bar{x}, \bar{y}$ are orthogonal $\langle \bar{x}, \bar{y} \rangle = 0$

Orthonormal basis $\mathbb{C}^d$

$\vec{v}_1, \ldots, \vec{v}_d \;\downarrow\;$ iff
(1) $\forall i, \|\vec{v}_i\|_2 = 1$
(2) $\forall i \neq j \; \langle \vec{v}_i, \vec{v}_j \rangle = 0$

$$\underbrace{\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}}_{d=4}$$

Orthogonal matrix

$$\begin{pmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \cdots & \vec{v}_d \\ \downarrow & & \downarrow \end{pmatrix} \quad I_d$$

(Discrete) Fourier Trans (DFT)

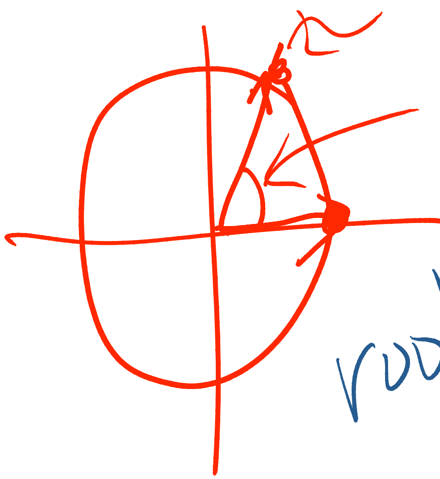
$$\begin{array}{cc} & \downarrow t \\ \ell \to & \left(\dfrac{e^{\frac{-2\pi i \ell t}{d}}}{\sqrt{d}} \right) \end{array} \updownarrow d$$

$$\xleftrightarrow{\quad d \quad}$$

$$e^{2\pi i x} = \cos 2\pi x + i \sin 2\pi x$$

Roots of unity (d^{th} roots of unity)

$$e^{\frac{2\pi i}{d}} = \omega_d$$



$\frac{2\pi}{d}$

roots of 1 $= \{\omega_d^0, \omega_d, \ldots, \omega_d^{d-1}\}$

DFT

$$M^{-1} = \left(\frac{\omega^{-\ell t}}{\sqrt{d}}\right)$$

$$M = \begin{array}{c c c} & t & \\ \ell & \left(\dfrac{\omega^{\ell t}}{\sqrt{d}}\right) & \begin{array}{c} 0 \\ \vdots \\ d-1 \end{array} \\ & 0 \; \cdots \; d-1 & \end{array}$$

(Vandermonde matrix)

Claim

$$\sum_{t=0}^{d-1} \left(\omega_d^{\ell}\right)^t = 0$$

t^{th} clm

$$M_t = \frac{1}{\sqrt{d}} \begin{pmatrix} 1 \\ w_d^t \\ \vdots \\ (w_d^t)^{d-1} \end{pmatrix}$$

$$\|M_t\|_2^2 \quad \| 1$$

$$= \frac{1}{d} \sum_{\ell=0}^{d-1} (w_d^t)^{\ell} \cdot (w_d^t)^{-\ell} = 1$$

$$w_d^{-t\ell} \qquad w_d^{t\ell} = e^{\pm 2\pi i \frac{t\ell}{d}}$$

$$= \cos \frac{2\pi t\ell}{d} + i \sin \frac{2\pi t\ell}{d}$$

$$\langle M_t, M_u \rangle = \frac{1}{d} \sum_{\ell=0}^{d-1} (w_d^t)^{\ell} (w_d^u)^{-\ell} = \frac{1}{d} \sum (w_d^{t-u})^{\ell} = 0$$

MATHEMATICS AWARENESS WEEK 1998

ORIGINAL

WAVELET
300:1 compression
Preserves all but fine structure

WAVELET PACKET BASIS
30:1 compression
Preserves fine structure of [illegible]

RESIDUAL

Mathematics & Imaging

The original image is the sum of the three. Each pulls out different features of the original. We can think of the image as being synthesized by three different instruments, in a way similar to a musical orchestration where the final sound is the sum of the notes from each instrument.

This mathematical transcription is useful for a more efficient and accurate storage and processing of imaging data, but also provides tools for denoising and identifying structures in images. For example, it can sharpen detail in medical images, such as MRI, and be used to identify particular objects for diagnostic purposes.

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Images provided by Ronald Coifman, Yale University