Floodlight Illumination of Infinite Wedges

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The floodlight illumination problem asks whether there exists a one-to-one placement of n floodlights illuminating infinite wedges of angles $\alpha_1, \ldots, \alpha_n$ at n locations $p_1, \ldots, p_n$ in a plane such that a given infinite wedge W of angle θ located at point q is completely illuminated by the floodlights. We prove that this problem is NP-hard, closing an open problem from CCCG 2001 [2]. In fact, we show that the problem is NP-complete even when $\alpha_i = \alpha$ for all $1 \le i \le n$ (the *uniform* case) and $\theta = \sum_{i=0}^{n} \alpha_i$ (the *tight* case). We discuss various approximate solutions and show that computing any *finite* approximation is NP-hard while ε*-angle* approximations can be obtained efficiently. Most proofs are omitted in this abstract due to lack of space. Interested readers are referred to [1].

1 Preliminaries

A *generalized wedge* is a wedge with a continuous finite region adjacent to its apex removed. The FLOODLIGHT ILLUMINATION problem on generalized wedges is as follows: given n sites $p_1, \ldots, p_n$, n angles $\alpha_1, \ldots, \alpha_n$, and a generalized wedge W, determine whether there is an assignment of angles to sites along with angle orientations that illuminates W. In the *tight* illumination problem, the sum of floodlight spans $\sum \alpha_i$ equals the wedge angle. A different specialization is the *uniform* problem, where in addition to being tight, $\alpha_i = \alpha_j$ for all i, j.

We now look at the related problem of MONOTONE MATCHING Suppose we are given n lines in the plane, $n + 1$ vertical lines defining n finite width vertical slabs, and two points, one on the leftmost vertical line and one on the rightmost. Call this an *arrangement of lines and slabs*, and denote it by (L, S, λ, ρ), where L is the set of lines, $S \equiv \{s_1, \ldots, s_{n+1}\}$ is the set of vertical lines $x = s_i$ forming slabs, and λ and ρ are the two special points on the lines $x = s_1$ and $x = s_{n+1}$, respectively. A *monotone matching* in (L, S, λ, ρ) is a set of n line segments, each a portion of a unique line and spanning a unique slab, such that the following holds: (1) the left end point of the first segment is above λ, (2) the left endpoint of each subsequent segment is above the right endpoint of the segment in the previous slab, and (3) ρ is above the right endpoint of the last segment. In the more general problem of PSEUDOLINE MONOTONE MATCHING, one has to check whether a given arrangement of pseudolines[1] and slabs admits a monotone matching.

The floodlight illumiination problem can be related to the monotone matching problem through duality as described by Steiger and Streinu [3].

As a tool for our main result, we use NP-completeness of the problem of finding whether a given directed graph has a directed disjoint cycle cover.

Theorem 1. DIRECTED DISJOINT CYCLE COVER *is NP-complete, even for graphs with indegree and outdegree each bounded above by 3, as well as for graphs with outdegree exactly 3 and indegree at most 4.*

2 Floodlight Illumination is NP-Hard

To give a flavor of the proof of our main result, we prove the following result in this abstract:

Theorem 2. PSEUDOLINE MONOTONE MATCHING *is NP-complete.*

The most important gadget is the *forcing gadget*, shown in Figure 1. This is a sequence of slabs associated with pseudolines that forces the line used previous to the gadget to end below a chosen point, and the line used after the gadget to start above another chosen point.

Proof of Theorem 2. As a potential matching can easily be verified in polynomial time, this problem is in NP. The proof of NP-hardness is by a reduction from the bounded degree version of DIRECTED DISJOINT CYCLE COVER (see Theorem 1).

[1] A *pseudoline* is a curve in $\mathbb{R}^2$ that intersects any vertical line in exactly one point. A *collection of pseudolines* is a set of pseudolines no two of which intersect more than once.

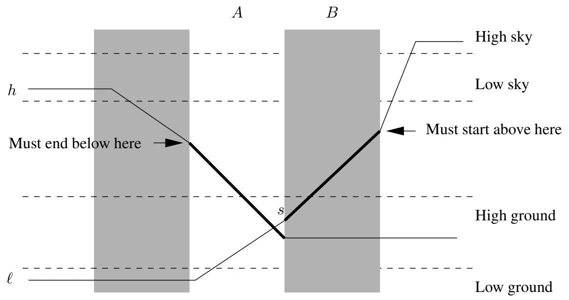

Figure 1: The forcing gadget. The arrows show how any lines used before or after the gadget are constrained.

Suppose we are given a directed graph $G = (V, E)$ with the outdegree of all vertices exactly 3 and the indegree at most 4. We will have gadgets $\mathbf{In}(v)$ and $\mathbf{Out}(u)$ for $u, v \in V$ as shown in Figure 2. Let $\mathcal{I}(v) \subset E$ be in the in-edges of v, and let $\mathcal{O}(u) \subset E$ be the out-edges of u. By our choice of G, $|\mathcal{I}(v)| \leq 4$ and $|\mathcal{O}(u)| = 3$.

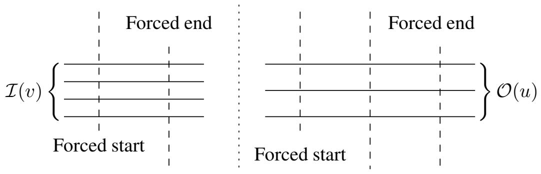

Figure 2: Graph gadgets $\mathbf{In}(v)$ and $\mathbf{Out}(u)$.

Let $n = |V|$ and $m = |E|$. We will use m primary pseudolines, each corresponding to an edge in E. There will be a number of auxiliary pseudolines used in forcing gadgets. The $\mathbf{Out}(\cdot)$ gadgets and $\mathbf{In}(\cdot)$ gadgets will be arranged in sequence as shown in Figure 3. The primary pseudoline corresponding to edge (u, v) will first pass through $\mathbf{Out}(u)$, and then pass through $\mathbf{In}(v)$.

We claim that when arranged as in Figure 3 along with appropriate forcing gadgets for each $\mathbf{In}(\cdot)$ and $\mathbf{Out}(\cdot)$ gadget, exactly one $e \in \mathcal{I}(v_i)$ is used in $\mathbf{In}(v_i)$ and exactly one $e \in \mathcal{O}(v_i)$ is not used in $\mathbf{Out}(v_i)$, $1 \leq i \leq n$.

A directed disjoint cycle cover of G is equivalent to a permutation π on the vertices, where $\pi(v)$ is the predecessor of v in the cycle containing v. If such a permutation exists, then a monotone matching exists, by not selecting the edge at $\mathbf{Out}(u)$ corresponding to $\pi^{-1}(u)$, and selecting the edge corresponding to $\pi(v)$ at $\mathbf{In}(v)$. Conversely, if a monotone matching exists, then the permutation π can be recovered by setting $\pi(v)$ equal to the edge that is used in $\mathbf{In}(v)$. This completes the reduction. □

The proof of NP-hardness of MONOTONE MATCHING with straight lines is based on the same idea but is somewhat more involved. The details can be found in [1]. From the duality between monotone matching and floodlight illumination, we have

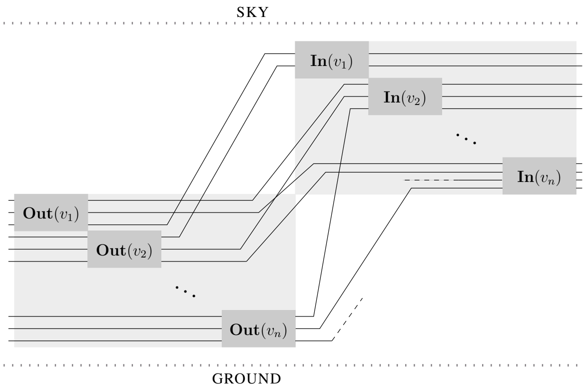

Figure 3: Overall view of the reduction from DIRECTED DISJOINT CYCLE COVER.

Theorem 3. FLOODLIGHT ILLUMINATION *is NP-hard. The tight, restricted, and uniform versions of the problem are NP-complete.*

3 Approximate Illumination

We now look at approximation algorithms to solve the floodlight illumination problem in the tight case. Let $\mathcal{F}$ be an illumination of a wedge W. $\mathcal{F}$ is a *finite-approximation* if it illuminates $W \setminus S$, where S is a finite region. $\mathcal{F}$ is an ε *angle-approximation* if it illuminates $W \setminus S_\varepsilon$, where S_ε is a union of wedges whose total angle is at most ε. We have the following result:

Theorem 4. *For the tight floodlight illumination problem, computing a finite-approximation is NP-hard, where as for any ε an ε angle-approximation can be found in polynomial time.*

References

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