Coalitional games on graphs: core structure, substitutes and frugality

Rahul Garg[1], Vijay Kumar[2], Atri Rudra[3] *, and Akshat Verma[1]

[1] IBM India Research Lab,
Block I, Indian Institute of Technology,
Hauz Khas, New Delhi, 110016, India
{grahul,akshatverma}@in.ibm.com
[2] Amazon.com,
605 5^{th} Ave South,
Seattle, WA 98104, U.S.A.
vijayk@amazon.com
[3] Dept. of Computer Sciences,
University of Texas at Austin
Austin, TX, 78712, U.S.A.
atri@cs.utexas.edu

Abstract. We study mechanisms that can be modelled as coalitional games with transferrable utilities, and apply ideas from mechanism design and game theory to problems arising in a network design setting. We establish an equivalence between the game-theoretic notion of agents being *substitutes* and the notion of *frugality* of a mechanism. We characterize the core of the network design game and relate it to outcomes in a sealed bid auction with VCG payments. We show that in a game, agents are substitutes if and only if the core of the forms a complete lattice. We look at two representative games – Minimum Spanning Tree and Shortest Path in this light. Finally, we design an ascending price mechanism for such games and study the strategic behavior of agents.

1 Introduction

The Internet brings together a large, diverse collection of autonomous entities to interact, collaborate and compete. Game theory has emerged as an important tool to understand the complex interplay of the interests of these autonomous agents, and thus model and analyze the architecture and the functioning of the Internet [26, 18, 27, 29, 12]. Ideas from game theory have been used to design protocols [18, 20, 11, 29] and to gain insights into basic computer science problems [23, 1, 3, 21]. Some of the application areas that have received extensive interest include problems relating to routing protocols for networks [27], such as congestion control[29, 12], bandwidth pricing [20], multicasting [11], and design of auction mechanisms for various settings [5, 17, 23].

When the interaction of autonomous agents and conflicting interests is modelled as a game, the possible outcomes depend on the preferences of the agents as well as the structure of the game. The field of *mechanism design* concerns itself with the design of games that realize certain "socially desirable" objectives or outcomes that are desirable from the designer's perspective. A good deal of recent work addresses the design of mechanisms for the underlying graph problems in communication networks [23, 1, 24, 5]. Such work offers insight into the so-called "price of anarchy" — the cost of a solution arrived at through the distributed and decentralized decision-making of selfish agents, rather than through the global

* This work was done while the author was at IBM India Research Lab.

optimization and centralized decision-making associated with the conventional RAM model of computation. The distributed approach arguably makes for a more realistic modelling of the Internet.

In addition to maximizing the overall utility or surplus, the designer of a mechanism must often consider whether distribution of this surplus among participating agents is fair and competitive, particularly in settings where such utility is transferrable among agents. Otherwise, a subset of the agents may have an incentive to deviate towards a different outcome. *Coalitional game theory* considers games where the players are involved in splitting some aggregate payoff among themselves, and may group themselves into coalitions to maximize their share of the payoff. An important solution concept is the *core* of a coalitional game. A proposed splitting of the total payoff is said to be in the core of the game if every possible set of players receives a total payoff no smaller than the payoff they can achieve by forming a coalition (for a good exposition, see [25]).

One of the problem that have been well-studied in the coalitional framework is the Assignment Problem. Shapley and Shubik [28] have characterized the core of the assignment problem and have shown that it forms a complete lattice. Demange and Gale [9] have given an ascending price auction mechanism for the problem that converges to an outcome in the core. Leonard [21] has proposed an incentive compatible sealed bid auction mechanism for the assignment problem that leads to an outcome in the core. Both these mechanisms lead to an outcome that is favorable to the buyers. Mishra and Garg [22] have considered a descending price auction mechanism that leads to an outcome in the core that is favorable to sellers. Crawford and Knoer [8] have proposed mechanisms with similar properties.

While several game-theoretic analyses of graph problems have been offered ([23, 1]), most of these problems have not been adequately characterized as coalitional games. Gul and Stacchetti [15] explore the core of a generic class of problems that satisfy a certain *gross substitutes* property. However, basic problems such as Minimum Spanning Tree and Shortest Path lie outside this class. Bikhchandani et al. [5] have modelled a class of such problems that satisfy the *agents are substitutes* condition and studied the sealed bid VCG auction [30, 6, 14] mechanism for such problems. They propose a descending price auction mechanism for the spanning tree problem that converges to VCG payments. Bikhachandani and Ostroy [**?**] relate the VCG payments with the structure of the core. Archer and Tardos [1] introduce the notion of *frugality* in the context of the Shortest Path problem.

In this paper, we study the coalitional game formulations of graph or network problems, and characterize the cores of those games. We correlate the concepts of frugality, studied in mechanism design, with the concept of agents being *substitutes*, studied in game theory. We show that the core of a game is a lattice if and only if agents are substitutes. We give an ascending price auction mechanism for such games that may converge to outcomes that are favorable to the auctioneer. We also show that the lattice structure of the core simplifies the strategy computation for an agent even in an Ascending Price Mechanism.

2 Preliminaries

While we consider minimization games on graphs, our results extend to maximization games in a straightforward fashion. We consider a graph $G = (V, E)$, each of whose edges is controlled by an autonomous agent. The *auctioneer* (also termed the *buyer*) is endowed with a sum U of money. The auctioneer wishes to purchase a collection S of edges from the respective agents such that S induces a subgraph with some desirable property P (we refer to S as a *solution*). For instance, in the Minimum Spanning Tree (or MST) game, the auctioneer wishes to acquire a collection of edges that constitute a spanning tree of G. Each edge e_i, and thus the corresponding agent i, has an associated cost C_i, such that the agent incurs cost C_i if e_i is (bought and) used by the auctioneer. The value of C_i is known only to i. Each agent receives a payment P_i from the auctioneer, such that $\sum_i P_i \leq U$. Clearly P_i must be no less than C_i if $e_i \in T$; WLOG, we assume that $P_i = 0$ otherwise. All parties aim to maximize their *payoff* – the payoff of the auctioneer is $U - \sum_i P_i$, while that of any other agent i is $P_i - C_i$. Thus, the auctioneer's objective is to construct a solution at the lowest possible cost. We assume the sum U is no less than the cost of the second best solution.

First, we set out some definitions and formalism. After [25], we define

Definition 1. *A* Coalitional game $< N, V >$ with transferrable payoff *consists of a finite set* N*, the* set of players*, and a function* V *that associates with every nonempty subset (*coalition*)* S *of* N*, a real number* $V(S)$ *(the* worth *of* S*).*

For each coalition S, $V(S)$ is the total payoff that is available for division among the members of S. For any vector $(x_i)_{i \in N}$ of real numbers (*profile*), let $x(S) = \sum_{i \in S} x_i$. $(x_i)_{i \in S}$ is an S*-feasible payoff vector* if $x(S) = V(S)$. An N-feasible payoff vector is a *feasible payoff profile.*

Definition 2. *The* core *of coalitional game* $< N, v >$ *is the set of feasible payoff profiles* $(x_i)_{i \in N}$ *for which* $V(S) \leq x(S)$ *for every coalition* S*.*

As noted above, in our model, there is one auctioneer and multiple agents; and each agent owns a resource and the auctioneer needs a collection of resources for forming a solution S. The set N of all players consists of the auctioneer and the agents. We denote the auctioneer as agent 0. We note that the *worth* of a subset of S may be zero (if that subset does not by itself constitute a solution). In other words, the function V may exhibit *complementarity.* We assume that for any $S_1 \subseteq S_2$, $V(S_2) \geq V(S_1)$; that is, V possesses the "*zero cost of disposal*" property. We specify $V(S)$ for all $S \subseteq N$ in the following manner. $V(S) = 0$ if $0 \notin S$, or if $(S - \{0\})$ does not contain a feasible solution. Otherwise, $V(S) = U - C(S)$, where $C(S)$ is the cost of the best solution contained in S. Let the cost of an optimal solution be $\mathcal{C}$; that is, $V(N) = U - \mathcal{C}$.

Let $\{O_1, \cdots, O_m\}$ be the set of all optimal solutions [1]. Define $\mathcal{O} = \cap_{l=1}^{m} O_l$, the set of agents that are contained in every optimal solution. Thus, if optimum solution is unique, $\mathcal{O}$ denotes the set of agents in the optimal solution. Note that in any payoff vector which is in the core, any agents that receive non-zero payoff are in $\mathcal{O}$. To see this consider an agent a such that $a \in O_i$ and $a \notin O_j$. Now, $V(N - \{a\}) \geq V(\{0\} \cup O_j) = V(N)$. The inequality comes from the "zero cost of disposal" property of V. Note that this implies that the Vickrey payoff of a is zero.

The "*agents are substitutes*" property (or the *substitutes* property, for short) is a commonly employed characterization. After [5], we define.

[1] Note that $\forall i \in O_l$, $O_l - \{i\}$ is not a solution

Definition 3. *We say that* agents are substitutes *if* $V(N) - V(N-K) \geq \sum_{i\in K}(V(N) - V(N-\{i\}))$ *for all* $K \subset N$ *such that* $0 \notin K$*.*

3 Cost of VCG Payment

The VCG mechanism [30, 6, 14] is a celebrated incentive-compatible (truth-revealing) strategy which is widely used. The costliness of the VCG solution, though, varies with the problem in hand. Archer and Tardos [1] showed that the VCG payment in the Shortest Path game can be very high as compared to the cost of second best solution. On the other hand (as we shall show in Section 5), the VCG payment in games like MST coincide with the cost of the second best solution.

Characterizing the *frugality* of the VCG payment is important to determine whether a VCG based strategy is practical or not. If the VCG payment is very high and exceeds the auctioneer's budget, then there may not be an agreement between the auctioneer and the agents. In such scenarios, the auctioneer may want to consider other mechanisms — one of which we explore in Section 6. Hence, it is important to determine if the VCG solution is *frugal* with respect to the second-best solution.

The concept of frugality was introduced (albeit without a formal definition) in [1]. They show that in certain classes of graphs, the VCG *payoff* for the Shortest path game is k times the difference of the costs of second best and the optimal solution, where k is the number of edges in the shortest path. We formalize the notion of frugality and relate it to well-studied concepts in game theory.

Definition 4. *We define the* frugality ratio *of a payoff vector* π *as* $\mathcal{F}(\pi) = \max_{O\subseteq\mathcal{O}} \frac{\sum_{i\in O}\pi_i}{V(N)-V(N-O)}$. *We say that* π *is* frugal *if* $\mathcal{F}(\pi) \leq 1$*.*

Consider π^V, the VCG payoff for an optimal solution $\mathcal{O}$.

$$\pi_i^V = \begin{cases} V(N) - V(N-\{i\}) & \text{if } i \in \mathcal{O}, \\ 0 & \text{otherwise.} \end{cases}$$

The frugality of VCG payoffs is examined in [1], where it is shown that VCG payoffs aren't always frugal, and in fact, the VCG payoff for the Shortest path game has a worst-case frugality ratio of k, where k is the the number of edges in the shortest path. Below, we relate the frugality of VCG payoffs to the substitutes property.

Theorem 1. π^V *is* frugal *if and only if agents are substitutes.*

Proof. It follows from Definitions 3 and 4 and the definition of π^V that the substitutes property is a sufficient condition for the frugality of the VCG payoff. To see that it is necessary, assume that π^V is frugal, that is, $\forall A \subseteq \mathcal{O}$, $V(N) - V(N-A) \geq \sum_{i\in A}(V(N) - V(N-\{i\}))$. Consider any set $K \subseteq N$. Let $A = K \cap \mathcal{O}$. Now,

$$\begin{aligned} V(N) - V(N-K) &\geq V(N) - V(N-A), && \text{due to "zero cost of disposal"} \\ &\geq \textstyle\sum_{i\in A}(V(N) - V(N-\{i\})) && \text{since } \pi^V \text{ is frugal} \\ &= \textstyle\sum_{i\in K}(V(N) - V(N-\{i\})) && \text{since } V(N) = V(N-\{i\}) \text{ for } i \notin \mathcal{O}. \ \blacksquare \end{aligned}$$

Thus, in order to verify if π^V is frugal, it is sufficient to check if agents satisfy the substitutes condition. However, in certain situations, the auctioneer may be willing to pay more than the cost of the second best solution. Still, (s)he may not be willing to make arbitrarily high payoffs due to budget constraints. For such situations, we introduce a generalization of the substitutes concept and then relate it to the frugality ratio of π^V.

Definition 5. *Agents are c-substitutes if for all $K \subset N$ such that $0 \notin K$, $c(V(N) - V(N - K)) \geq \sum_{i \in K} V(N) - V(N - \{i\})$ and $\forall c' < c$, $\exists K' \subset N$ such that $c'(V(N) - V(N - K')) < \sum_{i \in K'} V(N) - V(N - \{i\})$*

Arguments similar to proof of Theorem 1 gives:

Theorem 2. *$\mathcal{F}(\pi^{\mathcal{V}}) \leq c$ if and only if agents are c-substitutes.* ■

In the next section, we explore the relationship between the concept of substitutes and the structure of the core.

4 Structure of the Core

The structure of the core has been extensively studied in literature. Bikhchandani and Ostroy [4] show that if buyers are substitutes, then the core is a lattice with respect to the buyers. They also show the equivalence of "buyers are substitutes", π^V being in the core, and π^V being the maximum of all payoffs in the core. Shapley and Shubik [28] have characterized the core of the assignment problem and have shown that it forms a complete lattice. Gul and Stacchetti [15] explore the core of a generic class of problems that satisfy *gross substitutes* [19]. Gul et al. [15] show that if the valuations of the agents satisfy *gross substitutes*, then the core is a lattice. However, most network connectivity problems, such as Minimum Spanning Tree and Shortest Path, lie outside this class (see Section 5 for details). In the following, we show the in our model, the substitutes property is **equivalent** to the core being a lattice.

Let $CORE$ denote the core of the game, and $\pi \in CORE$. Then $\forall S \subseteq N$, $x_\pi(S) = \sum_{i \in S} \pi_i$. As noted earlier, if $j \notin \mathcal{O}$ then $\pi_j = 0$. Also we define $\pi_0 = U - \mathcal{C} - \sum_{j \in N - \{0\}} \pi_j$; recall that $\mathcal{C}$ is the cost of an optimal solution.

Let $\pi^1, \pi^2 \in CORE$, and let $\overline{\pi}$ be defined as follows – $\forall i \in (N - \{0\})$, $\overline{\pi}_i = \max(\pi_i^1, \pi_i^2)$; and $\overline{\pi}_0 = U - \mathcal{C} - \sum_{j \in N - \{0\}} \overline{\pi}_j$. It can be shown that $\overline{\pi}_0$ is always non-negative. The following result is due to Bikhchandani and Ostroy [4]:

Lemma 1. *If $V(N) - V(N - \mathcal{O}) \geq \sum_{i \in \mathcal{O}}(V(N) - V(N - \{i\}))$ and $\pi^1, \pi^2 \in CORE$, then $\overline{\pi} \in CORE$.*

Roughly speaking, if agents are substitutes then the "maximum" of two core payoffs is also in the core. Let us similarly consider a "minimum" of π^1 and π^2. If $\pi^1, \pi^2 \in CORE$, then $\forall i \in (N - \{0\})$, $\underline{\pi}_i = \min(\pi_i^1, \pi_i^2)$; and $\underline{\pi}_0 = U - \mathcal{C} - \sum_{j \in N - \{0\}} \underline{\pi}_j$.

Next, we see that the "minimum" of two core payoffs is (unconditionally) in the core. The absence of any condition (in contrast with the dependence on the substitutes property in Lemma 1) is a consequence of the fact that we are considering minimization games.

Lemma 2. *If $\pi^1, \pi^2 \in CORE$ then $\underline{\pi} \in CORE$.*

Proof. It follows from the definition of $\underline{\pi}_0$ that $x_{\underline{\pi}}(N) = V(N)$. Next, we show that $\forall S \subset N$, $x_{\underline{\pi}}(S) \geq V(S)$. Let $\mathcal{O} \cap S = A$.

$$
\begin{aligned}
x_{\underline{\pi}}(S) &= \underline{\pi}_0 + \textstyle\sum_{i \in A} \underline{\pi}_i + \sum_{i \in (S-A)} \underline{\pi}_i \\
&= U - \mathcal{C} - \textstyle\sum_{i \in \mathcal{O}} \underline{\pi}_i + \sum_{i \in A} \underline{\pi}_i && \text{since agents not in } \mathcal{O} \text{ get 0 payoff} \\
&= U - \textstyle\sum_{i \in (\mathcal{O}-A)} \underline{\pi}_i - \mathcal{C} \\
&\geq U - \textstyle\sum_{i \in (\mathcal{O}-A)} \pi_i^1 - \mathcal{C} && \text{as } min(\pi_i, \pi_j) \leq \pi_i) \\
&= x_{\pi^1}(S) \\
&\geq V(S) && \text{since } \pi^1 \text{ is in the core.}
\end{aligned}
$$

■

Lemma 1 and Lemma 2 together imply

Lemma 3. *If $\pi^1, \pi^2 \in CORE$ and agents are substitutes, then $\overline{\pi}, \underline{\pi} \in CORE$.* ■

Lemma 4. *If $\pi^1, \pi^2 \in CORE \Rightarrow \overline{\pi}, \underline{\pi} \in CORE$, then agents are substitutes.*

Proof. $\forall i \in N - \{0\}$, let ξ^i be a payoff vector such that $\xi_i^i = V(N) - V(N - \{i\})$, $\xi_j^i = 0$ $\forall j \notin \{0, i\}$, and $\xi_0^i = U - \mathcal{C} - \xi_i^i$. It is easy to see that $\xi^i \in CORE$. Let $\xi^{N-\{0\}}$ be the "maximum" of all such vectors ξ^i – that is, $\forall i \neq 0$, $\xi_i^{N-\{0\}} = V(N) - V(N - \{i\})$, and $\xi_0^{N-\{0\}} = U - \mathcal{C} - \sum_{i \neq 0} \xi_i^{N-\{0\}}$.

Now, we are given that $\pi^1, \pi^2 \in CORE \Rightarrow \overline{\pi}, \in CORE$. By repeated application of this argument, it follows that $\xi^{N-\{0\}} \in CORE$, since each $\xi^i \in CORE$. Consider any $S \subseteq N$ such that $0 \in S$. Let $A = S \cap \mathcal{O}$. It follows from the definition of the core that $\sum_{j \in S} \xi_j^{N-\{0\}} \geq V(S)$. In other words,

$\xi_0^{N-\{0\}} + \sum_{j \in S-\{0\}} \xi_j^{N-\{0\}} \geq V(S)$, that is,
$U - \mathcal{C} - \sum_{j \in N-\{0\}} \xi_j^{N-\{0\}} \sum_{j \in S-\{0\}} \xi_j^{N-\{0\}} \geq V(S)$.

Since $V(N) = U - \mathcal{C}$, we have
$V(N) - \sum_{j \in N-S} \xi_j^{N-\{0\}} \geq V(S)$, or,
$V(N) - V(S) \geq \sum_{j \in N-S} \xi_j^{N-\{0\}} = \sum_{j \in N-S}(V(N) - V(N - \{j\}))$.

Since this is true for any S, it follows that agents are substitutes. ■

We say $\pi^1 \preceq \pi^2$ if $\forall i \in N - \{0\}, \pi_i^1 \leq \pi_i^2$.

Lemma 5. *$(CORE, \preceq)$ is a lattice if and only if $\pi^1, \pi^2 \in CORE \Rightarrow \overline{\pi}, \underline{\pi} \in CORE$.*

Proof. We show that if $\overline{\pi}, \underline{\pi} \in CORE$ for any $\pi^1, \pi^2 \in CORE$, then $(CORE, \preceq)$ is a lattice. The proof is the other direction is obvious and is omitted.

By definition of $\preceq$, $(CORE, \preceq)$ is a partial order. Let $\pi^1, \pi^2 \in CORE$, then it follows from the definition of $\underline{\pi}$ that there cannot exist $\pi^* \in CORE$ such that $\underline{\pi} \preceq \pi^*$, $\pi^* \preceq \pi^1$ and $\pi^* \preceq \pi^2$. Assume that $\exists \pi' \in CORE$ such that π' and $\underline{\pi}$ are incomparable and $\pi' \preceq \pi^1$, $\pi' \preceq \pi^2$. As π' and $\underline{\pi}$ are incomparable, $\exists i, j$ such that $\underline{\pi}_i < \pi'_i$ and $\underline{\pi}_j > \pi'_j$, i.e., $\min(\pi_i^1, \pi_i^2) < \pi'_i$.

Now $\min(\pi_i^1, \pi_i^2)$ is either π_i^1 or π_i^2, thus contradicting the assumption $\pi' \preceq \pi^1$ or $\pi' \preceq \pi^2$ respectively. Thus, $\underline{\pi}$ is the unique infimum of π^1 and π^2.

Similarly one can show that $\overline{\pi}$ is the unique supremum of π^1 and π^2. It follows that $(CORE, \preceq)$ is a lattice. ■

Lemmas 3, 4 and 5 imply

Theorem 3. *A core is a lattice if and only if the agents are substitutes.* ■

We define P^{min} as the minimum element of the lattice and P^{max} as the maximum. As shown in [4], P^{max} is the VCG payoff.

5 Network design problems and the structure of the core

In general, connectivity problems do not satisfy the gross substitutes property [15] and hence, the results of Gul and Stacchetti [15] about the structure of the core and its relationship with the gross substitutes property do not apply to them. The essential reason is the complementarity inherent in connectivity problems. For instance, in a connected network, if the removal of an edge disconnects the network, then in many network design problems the remaining network has no utility. It is not hard to see that this, coupled with the fact that the auctioneer has a budget constraint, leads to the violation of the gross substitutes property.

We illustrate the above in case of the MST problem. Consider a spanning tree T such that $Cost(T) \leq U$. We increase the price of all edges not in T to $U + 1$. Hence, all edges in T are desirable to the auctioneer. Now if we increase the price of any edge, $e \in T$, to $U + 1$ the edges in $T - \{e\}$ are no longer desirable. This violates the gross substitutes property, which implies that raising the price of an edge should not cause a different edge to drop out of the set with optimal value. One can argue similarly in the case of other graph problem, such as Shortest Path.

The MST game has received a lot of attention. However, in most cases the problem is modelled such that the agents own the nodes of the graph. It is an easy algorithmic task to determine a core allocation – a simple greedy approach works [2]. Faigle et al [10] prove that core membership testing is co-NP-Complete. Another model of the game where the agents own the edges are also received a lot of attention in the recent past. Bikhchandani et al. [5] give a descending-price auction mechanism for this problem – this is the same as the VCG mechanism proposed in [23].

Bikhchandani et al. [5] show that MST satisfies the buyers are substitutes condition. The equivalence of frugality and substitutes (Section 3) imply that the MST game is frugal. We would like to point out that in a somewhat different valuation model, the results of [4] imply that the core of the game is a lattice with respect to the sellers when sellers are substitutes. Applied to the MST game, their valuation model requires the buyer's valuations to be additive with respect to the edges, while in our model that is clearly not the case. However, the relevant proof in [4] does not seem to depend on this limitation of the valuations. Our own proofs have a similar structure.

Bikhchandani et al. [5] also show that gross substitutes property implies the "agents are substitutes" property. The discussion above in the context of the MST game shows that the converse may not be true.

Nisan and Ronen[23] applied the VCG mechanism to the Shortest Path game where the agents own edges. Archer et al.[1] proved that the VCG payment is not "frugal". Bikhchandani et al.[5] show that the Shortest Path game does not satisfy the substitutes property. It follows from the discussion in the previous section that both the results are equivalent and any one of them coupled with our results imply the other.

Sections 4 and 3 characterize the games and their outcomes in which a VCG mechanism leads to a frugal payment. The equivalence of the three concepts – the substitutes property, the lattice structure of the core, and frugality – contributes insights in designing frugal mechanisms, and provides an alternative interpretation as well as perhaps a general framework for the results in [1].

The lattice structure of the core suggests some possibilities for mechanism design. Along with the fact that the VCG payment is the maximum of all payments in the core, it suggests the possibility of designing mechanism which improve the payoff of the auctioneer, compared to the VCG auction. Also, the well-defined structure of the core provides two natural entry points — the maximum and minimum elements of the lattice. The descending price mechanism terminates at the maximum and does not explore any other points in the lattice. This leaves the entire lattice unexplored, whereas if the auction were to terminate at any other point, the auctioneer's payoff would be higher. Next, we propose a mechanism which achieves that. This mechanism allows the auctioneer to pay possibly lower prices to the other players, depending on how well they play the game. In this light, we explore the ascending price auction mechanism [9] in the next section.

6 Ascending Price Auction

The Revenue Equivalence theorem [13] sheds light on equivalence of mechanisms with respect to revenue. Any incentive-compatible strategy may not lead to increased payoff for the auctioneer as compared to the VCG payoff[13]. Hence, we look at Ascending Price mechanism, which is not truth-telling (incentive-compatible).

Ascending Price Auctions are usually not the mechanism of choice because it requires complicated decision models for the agents. On the other hand, mechanisms based on VCG mechanism, like the Descending Price Mechanism, lead to the highest payment (lowest profit to the auctioneer) in the core as noted earlier [5]. From a game-theoretic point of view, they also suffer from the drawback of being strategy-proof. There is no incentive for better players to choose strategies to improve their returns. This is taken care of in the *Ascending Price mechanism* as the payoffs of an agent are determined by her strategy. However, as noted, the agent while disclosing his bid has to guess the bid of all other agents in a general game which is a hard decision problem. We now show that if the core is a lattice, then the decision is dependent only the valuation of the Vickrey substitute, which simplifies the decision problem considerably.

Definition 6. *A set of strategies $R^i = \{r^i_{low}, \cdots, r^i_{high}\}$ is a range strategy for agent i if the agent plays according to any strategy in R^i.*

Definition 7. *A range strategy R^{i^*} for an agent i with range $[r^i_{low}, r^i_{high}]$ is said to be* optimal *if $r^i_{low} = 0$ and $r^i_{high} = \pi^V_i$. R^i is* sub-optimal *if $R^i \subset R^{i^*}$.*

Theorem 4. *If* $(CORE, \preceq)$ *is a lattice, then every agent can independently follow any* sub-optimal *range strategy and not effect the outcome as well as the payment of other bidders, that is, each player can play the game independently.*

Proof. It is easy to see that $\pi \in (CORE, \preceq)$ iff $P^{max} \preceq \pi \preceq P^{min}$. Hence, the proof. ∎

This implies that as long as an agent follows an optimal range strategy, she will not effect the payment of other players.

If we define the rules of the game as each agent i following a sub-optimal range strategy R^i, then each agent i can follow any of the strategies in R_i and get a payment which is determined entirely by how well he plays the game. In a non-lattice core (like shortest path), this may not be possible.

We now describe the ascending price auction which allows an agent to play a range strategy as opposed to the descending price auction in which the range is a single point. The auctioneer starts inviting bids at a low price and keeps on increasing them depending upon the agents' responses. Specifically, let $\mathcal{P}(t)$ be the price which the auctioneer offers at time t[2] and $\mathcal{A}$ be the set of agents in the intermediate solution. At any time t, agent i uses some bidding strategy to decide whether to accept at $\mathcal{P}(t)$. If i accepts, the auctioneer includes it in $\mathcal{A}$, if its inclusion does not invalidate the current solution. Auctioneer increases $\mathcal{P}(t)$ after he has processed all the responses.

Let $S = (s_1, \cdots, s_m)$, where $m = |N| - 1$.

We now define bidding strategy for an agent i, $BS(s_i)$ –

Definition 8. *The agent accepts auctioneer's offer* $\mathcal{P}$ *iff* $\mathcal{P} \geq s_i + C_i$.

Note that the payoff for i by following $BS(s_i)$ is s_i.

Also let $\mathcal{BS}(S) = \{BS(s_i) | i \in (N - \{0\})\}$, $S_0 = (0, \cdots, 0)$ and $S' = (\pi_1^V, \cdots, \pi_m^V)$.

Theorem 5. *If all agents follow* $\mathcal{BS}(S_0)$ *then the payoff is* P^{min}.

Proof. By the definitions of $\mathcal{BS}(.)$, S_0 and P^{min}. ∎

Note that following $\mathcal{BS}(S_0)$ results in zero payoff for each agent.

Lemma 6. *The strategy* $\mathcal{BS}(S')$ *is a Nash equilibrium.*

Proof. Pick any $j \in (N - \{0\})$. Assume $N - \{0, j\}$ follow $\mathcal{BS}(S')$ while j does not follow $BS(S'_j)$. Let j follow $BS(s_j)$.

- Case 1 : $\exists l \in [1, m]$ such that $j \in O_l$.
  - Case 1.1 : $S_j < (C_j + \pi_j^V)$. In this case j would have a higher payoff if $S_j = (C_j + \pi_j^V)$.
  - Case 1.2 : $S_j > (C_j + \pi_j^V)$. In this case $(O_l - \{j\}) \cup \{j'\})$, where j' is the Vickrey substitute of j, would be the optimum solution and thus, payoff for $j = 0$.
- Case 2 : $\forall l \in [1, m]$ $j \notin O_l$. j would never get a chance to bid. ∎

Theorem 6. *If all agents follow* $\mathcal{BS}(S')$ *then the payment is* P^{max}.

[2] $\mathcal{P}(0)$ would typically be 0

Proof. By the definitions of $\mathcal{BS}(.)$, S' and P^{max}. ∎

Note that if other agents are following a sub-optimal strategy and one deviant agent i follows a different strategy, he will decrease his payoff because the Vickrey substitute of i may follow the minimum sub-optimal range strategy.

7 Conclusion and Future research

In this paper we correlate some concepts explored in mechanism design to those studied in game theory. We formalize the notion of frugality [1] and extend it to define frugality ratio. We establish the equivalence of *frugality* and a slightly different version of *agents are substitutes* [5] condition. We study the properties of the *core*, which defines the set of stable outcomes in coalition games with transferrable utilities. We show that if the core forms a a lattice, then the agents satisfy the substitutes condition. Hence, we establish the equivalence of core being a lattice and agents satisfying the substitutes condition. We study the implications of our result using two representative graph problems- MST and shortest path. We observe that since MST satisfies agents are substitutes condition, the VCG payment is frugal. Also, we observe that the two results: (1) shortest path does not satisfy agents are substitutes and (2) VCG payment for shortest path is not frugal are equivalent in light of our results.

Finally, we observe that the descending price auction mechanism of Bikhchandani et al. and sealed bid auctions with VCG payments lead to the maximum element of the lattice [5]. Since the maximum is the most unfavorable outcome in the core for the auctioneer, we propose an Ascending Price mechanism that allows the auction to terminate at other payments, depending on the knowledge of the agents. We show that if the core is a lattice, then computing a strategy for an agent in this mechanism is considerably simplified. We also show that in the case where the agents have complete knowledge (i.e in perfect information), the outcome is the same as achieved by the descending price mechanism. However, one would expect, in a real scenario, that the auction would terminate much earlier, thus increasing the auctioneer payoffs.

Our proposed *Ascending Price Mechanism*, with very conservative sellers terminates close to the minimum element in the lattice whereas the *descending price mechanism* terminates at the maximum. This opens several unanswered questions. Is it possible to design mechanisms which obtain a balance between these two extremes and allow the agents and the auctioneer to share the payoffs in a proportional manner? Is it possible to characterize the "value of information" in this setting, i.e., the increase in the expected payoff of an agent as a function of the information it possess about its competitors?

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