# Coalitional games on graphs: core structure, substitutes and frugality

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**Abstract**

We study mechanisms that can be modelled as coalitional games with transferable utilities, and apply ideas from mechanism design and game theory to problems arising in a network design setting. We establish an equivalence between the game-theoretic notion of agents being *substitutes* and the notion of *frugality* of a mechanism. We characterize the core of the network design game and relate it to outcomes in a sealed bid auction with VCG payments. We show that in a game, agents are substitutes if and only if the core of the forms a complete lattice. We look at two representative games – Minimum Spanning Tree and Shortest Path – in this light.

## 1 Introduction

The Internet brings together a large, diverse collection of autonomous entities to interact, collaborate and compete. Game theory has emerged as an important tool to understand the complex interplay of the interests of these autonomous agents, and thus model and analyze the architecture and the functioning of the Internet [26, 18, 27, 29, 12]. Ideas from game theory have been used to design protocols [18, 20, 11, 29] and to gain insights into basic computer science problems [23, 1, 3, 21]. Some of the application areas that have received extensive interest include problems relating to routing protocols for networks [27], such as congestion control [29, 12], bandwidth pricing [20], multicasting [11], and design of auction mechanisms for various settings [5, 17, 23].

When the interaction of autonomous agents and conflicting interests is modelled as a game, the possible outcomes depend on the preferences of the agents as well as the structure of the game. The field of *mechanism design* concerns itself with the design of games that realize certain "socially desirable" objectives or outcomes that are desirable from the designer's perspective. A good deal of recent work addresses the design of mechanisms for the underlying graph problems in communication networks [23, 1, 24, 5]. Such work offers insight into the so-called "price of anarchy" — the cost of a solution arrived at through the distributed and decentralized decision-making of selfish agents, rather than through the global optimization and centralized decision-making associated with the conventional RAM model of computation. The distributed approach arguably makes for a more realistic modeling of the Internet.

In addition to maximizing the overall utility or surplus, the designer of a mechanism must often consider whether distribution of this surplus among participating agents is fair and competitive,

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particularly in settings where such utility is transferable among agents. Otherwise, a subset of the agents may have an incentive to deviate towards a different outcome. *Coalitional game theory* considers games where the players are involved in splitting some aggregate payoff among themselves, and may group themselves into coalitions to maximize their share of the payoff. An important solution concept is the *core* of a coalitional game. A proposed splitting of the total payoff is said to be in the core of the game if every possible set of players receives a total payoff no smaller than the payoff they can achieve by forming a coalition (for a good exposition, see [25]).

One of the problem that have been well-studied in the coalitional framework is the Assignment Problem. Shapley and Shubik [28] have characterized the core of the assignment problem and have shown that it forms a complete lattice. Demange and Gale [9] have given an ascending price auction mechanism for the problem that converges to an outcome in the core. Leonard [21] has proposed an incentive compatible sealed bid auction mechanism for the assignment problem that leads to an outcome in the core. Both these mechanisms lead to an outcome that is favorable to the buyers. Mishra and Garg [22] have considered a descending price auction mechanism that leads to an outcome in the core that is favorable to sellers. Crawford and Knoer [8] have proposed mechanisms with similar properties.

While several game-theoretic analyses of graph problems have been offered ([23, 1]), most of these problems have not been adequately characterized as coalitional games. Gul and Stacchetti [15] explore the core of a generic class of problems that satisfy a certain *gross substitutes* property. However, basic problems such as Minimum Spanning Tree and Shortest Path lie outside this class. Bikhchandani et al. [5] have modelled a class of such problems that satisfy the *agents are substitutes* condition and studied the sealed bid VCG auction [31, 6, 14] mechanism for such problems. They propose a descending price auction mechanism for the spanning tree problem that converges to VCG payments. Bikhchandani and Ostroy [4] relate the VCG payments with the structure of the core. Archer and Tardos [1] introduce the notion of *frugality* in the context of the Shortest Path problem.

In this paper, we study the coalitional game formulations of graph or network problems, and characterize the cores of those games. We correlate the concepts of frugality, studied in mechanism design, with the concept of agents being *substitutes*, studied in game theory. We show that the core of a game is a lattice if and only if agents are substitutes.

## 2 Preliminaries

While we consider minimization games on graphs, our results extend to maximization games in a straightforward fashion. We consider a graph $G = (V, E)$, each of whose edges is controlled by an autonomous agent. The *auctioneer* (also termed the *buyer*) is endowed with a sum $U$ of money. The auctioneer wishes to purchase a collection $S$ of edges from the respective agents such that $S$ induces a subgraph with some desirable property $P$ (we refer to $S$ as a *solution*). For instance, in the Minimum Spanning Tree (or MST) game, the auctioneer wishes to acquire a collection of edges that constitute a spanning tree of $G$. Each edge $e_i$, and thus the corresponding agent $i$, has an associated cost $C_i$, such that the agent incurs cost $C_i$ if $e_i$ is (bought and) used by the auctioneer. The value of $C_i$ is known only to $i$. Each agent receives a payment $P_i$ from the auctioneer, such that $\sum_i P_i \leq U$. Clearly $P_i$ must be no less than $C_i$ if $e_i \in S$; WLOG, we assume that $P_i = 0$ otherwise. All parties aim to maximize their *payoff* – the payoff of the auctioneer is $U - \sum_i P_i$, while that of any other agent $i$ is $P_i - C_i$. Thus, the auctioneer's objective is to construct a solution at the lowest possible cost. We assume the sum $U$ is no less than the cost of the second best solution.

First, we set out some definitions and formalism. After [25], we define

**Definition 1** *A* Coalitional game $< N, V >$ with transferable payoff *consists of a finite set* $N$*, the* set of players*, and a function* $V$ *that associates with every nonempty subset (*coalition*)* $S$ *of* $N$*, a real number* $V(S)$ *(the* worth *of* $S$*).*

For each coalition $S$, $V(S)$ is the total payoff that is available for division among the members of $S$. For any vector $(x_i)_{i\in N}$ of real numbers (*profile*), let $x(S) = \sum_{i\in S} x_i$. $(x_i)_{i\in S}$ is an $S$*-feasible payoff vector* if $x(S) = V(S)$. An $N$-feasible payoff vector is a *feasible payoff profile.*

**Definition 2** *The* core *of coalitional game* $< N, v >$ *is the set of feasible payoff profiles* $(x_i)_{i\in N}$ *for which* $V(S) \leq x(S)$ *for every coalition* $S$*.*

As noted above, in our model, there is one auctioneer and multiple agents; and each agent owns a resource and the auctioneer needs a collection of resources for forming a solution $S$. The set $N$ of all players consists of the auctioneer and the agents. We denote the auctioneer as agent 0. We note that the *worth* of a subset of $S$ may be zero (if that subset does not by itself constitute a solution). In other words, the function $V$ may exhibit *complementarity.* We assume that for any $S_1 \subseteq S_2$, $V(S_2) \geq V(S_1)$; that is, $V$ possesses the "*zero cost of disposal*" property. We specify $V(S)$ for all $S \subseteq N$ in the following manner. $V(S) = 0$ if $0 \notin S$, or if $(S - \{0\})$ does not contain a feasible solution. Otherwise, $V(S) = U - C(S)$, where $C(S)$ is the cost of the best solution contained in $S$. Let the cost of an optimal solution be $\mathcal{C}$; that is, $V(N) = U - \mathcal{C}$.

Let $\{O_1, \cdots, O_m\}$ be the set of all optimal solutions. [1] Define $\mathcal{O} = \cap_{l=1}^{m} O_l$, the set of agents that are contained in every optimal solution. Thus, if optimum solution is unique, $\mathcal{O}$ denotes the set of agents in the optimal solution. Note that in any payoff vector which is in the core, any agents that receive non-zero payoff are in $\mathcal{O}$. To see this consider an agent $a$ such that $a \in O_i$ and $a \notin O_j$. Now, $V(N - \{a\}) \geq V(\{0\} \cup O_j) = V(N)$. The inequality comes from the "zero cost of disposal" property of $V$. Note that this implies that the Vickrey payoff of $a$ is zero.

The "*agents are substitutes*" property (or the *substitutes* property, for short) is a commonly employed characterization. After [5], we define.

**Definition 3** *We say that* agents are substitutes *if* $V(N) - V(N-K) \geq \sum_{i\in K}(V(N) - V(N - \{i\}))$ *for all* $K \subset N$ *such that* $0 \notin K$*.*

## 3 Cost of VCG Payment

The VCG mechanism [31, 6, 14] is a celebrated incentive-compatible (truth-revealing) strategy which is widely used. The costliness of the VCG solution, though, varies with the problem in hand. Archer and Tardos [1] showed that the VCG payment in the Shortest Path game can be very high as compared to the cost of second best solution. On the other hand (as we shall show in Section 5), the VCG payment in games like MST coincide with the cost of the second best solution.

Characterizing the *frugality* of the VCG payment is important to determine whether a VCG based strategy is practical or not. If the VCG payment is very high and exceeds the auctioneer's budget, then there may not be an agreement between the auctioneer and the agents. In such scenarios, the auctioneer may want to consider other mechanisms. Hence, it is important to determine if the VCG solution is *frugal* with respect to the second-best solution.

[1] Note that $\forall i \in O_l$, $O_l - \{i\}$ is not a solution.

The concept of frugality was introduced (albeit without a formal definition) in [1]. They show that in certain classes of graphs, the VCG *payoff* for the Shortest path game is $k$ times the difference of the costs of second best and the optimal solution, where $k$ is the number of edges in the shortest path. We formalize the notion of frugality and relate it to well-studied concepts in game theory.

**Definition 4** *We define the* frugality ratio *of a payoff vector* $\pi$ *as* $\mathcal{F}(\pi) = \max_{O \subseteq \mathcal{O}} \frac{\sum_{i \in O} \pi_i}{V(N) - V(N-O)}$. *We say that* $\pi$ *is* frugal *if* $\mathcal{F}(\pi) \leq 1$.

Consider $\pi^V$, the VCG payoff for an optimal solution $\mathcal{O}$.

$$\pi_i^V = \begin{cases} V(N) - V(N - \{i\}) & \text{if } i \in \mathcal{O}, \\ 0 & \text{otherwise.} \end{cases}$$

We note that $\forall i \in O \subseteq \mathcal{O}$, by "zero cost of disposal" property of $V$, $\pi^V(i) \leq V(N) - V(N-O)$. This along with Definition 4 implies the following proposition-

**Proposition 1** $\mathcal{F}(\pi^V) \leq |\mathcal{O}|$.

The frugality of VCG payoffs is examined in [1], where it is shown that VCG payoffs aren't always frugal, and in fact, the VCG payoff for the Shortest path game has a frugality ratio of $k$, where $k$ is the the number of edges in the shortest path. Below, we relate the frugality of VCG payoffs to the substitutes property.

**Theorem 1** $\pi^V$ *is* frugal *if and only if agents are substitutes.*

*Proof*: It follows from Definitions 3 and 4 and the definition of $\pi^V$ that the substitutes property is a sufficient condition for the frugality of the VCG payoff. To see that it is necessary, assume that $\pi^V$ is frugal, that is, $\forall A \subseteq \mathcal{O}$, $V(N) - V(N-A) \geq \sum_{i \in A}(V(N) - V(N - \{i\}))$. Consider any set $K \subseteq N$. Let $A = K \cap \mathcal{O}$. Now,

$$\begin{aligned} V(N) - V(N-K) &\geq V(N) - V(N-A), && \text{due to "zero cost of disposal"} \\ &\geq \textstyle\sum_{i \in A}(V(N) - V(N - \{i\})) && \text{since } \pi^V \text{ is frugal} \\ &= \textstyle\sum_{i \in K}(V(N) - V(N - \{i\})) && \text{since } V(N) = V(N - \{i\})) \text{ for } i \notin \mathcal{O}. \end{aligned}$$

■

Thus, in order to verify if $\pi^V$ is frugal, it is sufficient to check if agents satisfy the substitutes condition. However, in certain situations, the auctioneer may be willing to pay more than the cost of the second best solution. Still, she may not be willing to make arbitrarily high payoffs due to budget constraints. For such situations, we introduce a generalization of the substitutes concept and then relate it to the frugality ratio of $\pi^V$.

**Definition 5** *Agents are c-substitutes if for all* $K \subset N$ *such that* $0 \notin K$, $c(V(N) - V(N-K)) \geq \sum_{i \in K} V(N) - V(N - \{i\})$ *and* $\forall c' < c$, $\exists K' \subset N$ *such that* $c'(V(N) - V(N-K')) < \sum_{i \in K'} V(N) - V(N - \{i\})$

Arguments similar to proof of Theorem 1 gives:

**Theorem 2** $\mathcal{F}(\pi^{\mathcal{V}}) \leq c$ *if and only if agents are c-substitutes.* ■

### 3.1 Examples and Related Work

Bikhchandani et al. [5] show that MST satisfies the buyers are substitutes condition which along with Theorem 1 implies that the MST game is frugal. As mentioned earlier, [1] shows that the shortest path game has frugality ratio of $k$, where $k$ is the number of edges in the shortest path. We show in Appendix A that Minimum cut has frugality ratio of $k$, where $k$ is the number of edges in the cut set.

Using a similar argument on the dual of $G$,[2] we can argue that the Minimum Vertex cut has frugality ratio of $k$, where $k$ is the number of vertices in the cut set.

Independently, Talwar in [30] has an alternate formulation of *frugality*. However the results hold for *canonical* cost functions whereas our results are valid for all possible cost functions. [30] defines *frugality ratio*, $\phi(\pi^V)$ as $\frac{\sum_{i \in \mathcal{O}} \pi^V(i) + C(\mathcal{O})}{C(\mathcal{O})}$ and *marginal frugality*, $\phi'(\pi^V)$ as $\frac{\sum_{i \in \mathcal{O}} \pi^V(i)}{V(N) - V(N - \mathcal{O})}$. Further, [30] shows that $\phi'(\pi^V) \geq \phi(\pi^V)$ and $\phi(\pi^V) \leq 1$ if and only if the agents satisfy the *frugoids* condition. Note that by Definition 4, we have $\mathcal{F}(\pi^V) \geq \phi'(\pi^V)$ and thus, Theorem 1 is a superset of the main result of that work. It is however an interesting question if infact the two results are the same.

The above discussion along with results in [30] give an alternate proof that Minimum cut has frugality ratio of $k$, where $k$ is the size of the cut set. In the next section, we explore the relationship between the concept of substitutes and the structure of the core.

## 4 Structure of the Core

The structure of the core has been extensively studied in literature. Bikhchandani and Ostroy [4] show that if buyers are substitutes, then the core is a lattice with respect to the buyers. They also show the equivalence of "buyers are substitutes", $\pi^V$ being in the core, and $\pi^V$ being the maximum of all payoffs in the core. Shapley and Shubik [28] have characterized the core of the assignment problem and have shown that it forms a complete lattice. Gul and Stacchetti [15] explore the core of a generic class of problems that satisfy *gross substitutes* [19]. Gul et al. [15] show that if the valuations of the agents satisfy *gross substitutes*, then the core is a lattice. However, most network connectivity problems, such as Minimum Spanning Tree and Shortest Path, lie outside this class (see Section 5 for details). In the following, we show that in our model, the substitutes property is **equivalent** to the core being a lattice.

Let $CORE$ denote the core of the game, and $\pi \in CORE$. Then $\forall S \subseteq N$, $x_\pi(S) = \sum_{i \in S} \pi_i$. As noted earlier, if $j \notin \mathcal{O}$ then $\pi_j = 0$. Also we define $\pi_0 = U - \mathcal{C} - \sum_{j \in N - \{0\}} \pi_j$; recall that $\mathcal{C}$ is the cost of an optimal solution.

Let $\pi^1, \pi^2 \in CORE$, and let $\overline{\pi}$ be defined as follows – $\forall i \in (N - \{0\})$, $\overline{\pi}_i = \max(\pi_i^1, \pi_i^2)$; and $\overline{\pi}_0 = U - \mathcal{C} - \sum_{j \in N - \{0\}} \overline{\pi}_j$. It can be shown that $\overline{\pi}_0$ is always non-negative. The following result is due to Bikhchandani and Ostroy [4]:

**Lemma 1** *If* $V(N) - V(N - \mathcal{O}) \geq \sum_{i \in \mathcal{O}} (V(N) - V(N - \{i\}))$ *and* $\pi^1, \pi^2 \in CORE$*, then* $\overline{\pi} \in CORE$*.*

Roughly speaking, if agents are substitutes then the "maximum" of two core payoffs is also in the core. Let us similarly consider a "minimum" of $\pi^1$ and $\pi^2$. If $\pi^1, \pi^2 \in CORE$, then $\forall i \in (N - \{0\})$, $\underline{\pi}_i = \min(\pi_i^1, \pi_i^2)$; and $\underline{\pi}_0 = U - \mathcal{C} - \sum_{j \in N - \{0\}} \underline{\pi}_j$.

[2]Note that here we are talking about a model where the agents sit on the nodes of the graph

Next, we see that the "minimum" of two core payoffs is (unconditionally) in the core. The absence of any condition (in contrast with the dependence on the substitutes property in Lemma 1) is a consequence of the fact that we are considering minimization games.

**Lemma 2** *If* $\pi^1, \pi^2 \in CORE$ *then* $\underline{\pi} \in CORE$.

*Proof*: It follows from the definition of $\underline{\pi}_0$ that $x_{\underline{\pi}}(N) = V(N)$. Next, we show that $\forall S \subset N$, $x_{\underline{\pi}}(S) \geq V(S)$. Let $\mathcal{O} \cap S = A$.

$$
\begin{array}{lll}
x_{\underline{\pi}}(S) & = \underline{\pi}_0 + \sum_{i\in A} \underline{\pi}_i + \sum_{i\in(S-A)} \underline{\pi}_i & \\
& = U - \mathcal{C} - \sum_{i\in\mathcal{O}} \underline{\pi}_i + \sum_{i\in A} \underline{\pi}_i & \text{since agents not in } \mathcal{O} \text{ get 0 payoff} \\
& = U - \sum_{i\in(\mathcal{O}-A)} \underline{\pi}_i - \mathcal{C} & \\
& \geq U - \sum_{i\in(\mathcal{O}-A)} \pi^1_i - \mathcal{C} & \text{as } min(\pi_i, \pi_j) \leq \pi_i) \\
& = x_{\pi^1}(S) & \\
& \geq V(S) & \text{since } \pi^1 \text{ is in the core.}
\end{array}
$$

■

Lemma 1 and Lemma 2 together imply

**Lemma 3** *If* $\pi^1, \pi^2 \in CORE$ *and agents are substitutes, then* $\overline{\pi}, \underline{\pi} \in CORE$. ■

**Lemma 4** *If* $\pi^1, \pi^2 \in CORE \Rightarrow \overline{\pi}, \underline{\pi} \in CORE$*, then agents are substitutes.*

*Proof*: $\forall i \in N - \{0\}$, let $\xi^i$ be a payoff vector such that $\xi^i_i = V(N) - V(N - \{i\})$, $\xi^i_j = 0 \; \forall j \notin \{0, i\}$, and $\xi^i_0 = U - \mathcal{C} - \xi^i_i$. It is easy to see that $\xi^i \in CORE$. Let $\xi^{N-\{0\}}$ be the "maximum" of all such vectors $\xi^i$ – that is, $\forall i \neq 0$, $\xi^{N-\{0\}}_i = V(N) - V(N - \{i\})$, and $\xi^{N-\{0\}}_0 = U - \mathcal{C} - \sum_{i\neq 0} \xi^{N-\{0\}}_i$.

Now, we are given that $\pi^1, \pi^2 \in CORE \Rightarrow \overline{\pi}, \in CORE$. By repeated application of this argument, it follows that $\xi^{N-\{0\}} \in CORE$, since each $\xi^i \in CORE$. Consider any $S \subseteq N$ such that $0 \in S$. Let $A = S \cap \mathcal{O}$. It follows from the definition of the core that $\sum_{j\in S} \xi^{N-\{0\}}_j \geq V(S)$. In other words,

$\xi^{N-\{0\}}_0 + \sum_{j\in S-\{0\}} \xi^{N-\{0\}}_j \geq V(S)$, that is,
$U - \mathcal{C} - \sum_{j\in N-\{0\}} \xi^{N-\{0\}}_j \sum_{j\in S-\{0\}} \xi^{N-\{0\}}_j \geq V(S)$.

Since $V(N) = U - \mathcal{C}$, we have
$V(N) - \sum_{j\in N-S} \xi^{N-\{0\}}_j \geq V(S)$ , or,
$V(N) - V(S) \geq \sum_{j\in N-S} \xi^{N-\{0\}}_j = \sum_{j\in N-S}(V(N) - V(N - \{j\}))$.

Since this is true for any $S$, it follows that agents are substitutes. ■

We say $\pi^1 \preceq \pi^2$ if $\forall i \in N - \{0\}, \pi^1_i \leq \pi^2_i$.

**Lemma 5** $(CORE, \preceq)$ *is a lattice if and only if* $\pi^1, \pi^2 \in CORE \Rightarrow \overline{\pi}, \underline{\pi} \in CORE$.

*Proof*: We show that if $\overline{\pi}, \underline{\pi} \in CORE$ for any $\pi^1, \pi^2 \in CORE$, then $(CORE, \preceq)$ is a lattice. The proof is the other direction is obvious and is omitted.

By definition of $\preceq$, $(CORE, \preceq)$ is a partial order. Let $\pi^1, \pi^2 \in CORE$, then it follows from the definition of $\underline{\pi}$ that there cannot exist $\pi^* \in CORE$ such that $\underline{\pi} \preceq \pi^*$, $\pi^* \preceq \pi^1$ and $\pi^* \preceq \pi^2$.

Assume that $\exists \pi' \in CORE$ such that $\pi'$ and $\underline{\pi}$ are incomparable and $\pi' \preceq \pi^1$, $\pi' \preceq \pi^2$. As $\pi'$ and $\underline{\pi}$ are incomparable, $\exists i, j$ such that $\underline{\pi}_i < \pi'_i$ and $\underline{\pi}_j > \pi'_j$, i.e., $\min(\pi^1_i, \pi^2_i) < \pi'_i$. Now $\min(\pi^1_i, \pi^2_i)$ is either $\pi^1_i$ or $\pi^2_i$, thus contradicting the assumption $\pi' \preceq \pi^1$ or $\pi' \preceq \pi^2$ respectively. Thus, $\underline{\pi}$ is the unique infimum of $\pi^1$ and $\pi^2$.

Similarly one can show that $\overline{\pi}$ is the unique supremum of $\pi^1$ and $\pi^2$. It follows that $(CORE, \preceq)$ is a lattice. ■

Lemmas 3, 4 and 5 imply

**Theorem 3** *A core is a lattice if and only if the agents are substitutes.* ■

We define $P^{min}$ as the minimum element of the lattice and $P^{max}$ as the maximum. As shown in [4], $P^{max}$ is the VCG payoff.

## 5 Network design problems and the structure of the core

In general, connectivity problems do not satisfy the gross substitutes property [15] and hence, the results of Gul and Stacchetti [15] about the structure of the core and its relationship with the gross substitutes property do not apply to them. The essential reason is the complementarity inherent in connectivity problems. For instance, in a connected network, if the removal of an edge disconnects the network, then in many network design problems the remaining network has no utility. It is not hard to see that this, coupled with the fact that the auctioneer has a budget constraint, leads to the violation of the gross substitutes property.

We illustrate the above in case of the MST problem. Consider a spanning tree $T$ such that $Cost(T) \leq U$. We increase the price of all edges not in $T$ to $U + 1$. Hence, all edges in $T$ are desirable to the auctioneer. Now if we increase the price of any edge, $e \in T$, to $U + 1$ the edges in $T - \{e\}$ are no longer desirable. This violates the gross substitutes property, which implies that raising the price of an edge should not cause a different edge to drop out of the set with optimal value. One can argue similarly in the case of other graph problem, such as Shortest Path.

The MST game has received a lot of attention. However, in most cases the problem is modelled such that the agents own the nodes of the graph. It is an easy algorithmic task to determine a core allocation – a simple greedy approach works [2]. Faigle et al [10] prove that core membership testing is co-NP-Complete. Another model of the game where the agents own the edges has also received a lot of attention in the recent past. Bikhchandani et al. [5] give a descending-price auction mechanism for this problem – this is the same as the VCG mechanism proposed in [23].

Bikhchandani et al. [5] show that MST satisfies the buyers are substitutes condition. The equivalence of frugality and substitutes (Section 3) imply that the MST game is frugal. We would like to point out that in a somewhat different valuation model, the results of [4] imply that the core of the game is a lattice with respect to the sellers when sellers are substitutes. Applied to the MST game, their valuation model requires the buyer's valuations to be additive with respect to the edges, while in our model that is clearly not the case. However, the relevant proof in [4] does not seem to depend on this limitation of the valuations. Our own proofs have a similar structure.

Bikhchandani et al. [5] also show that gross substitutes property implies the "agents are substitutes" property. The discussion above in the context of the MST game shows that the converse may not be true.

Nisan and Ronen [23] applied the VCG mechanism to the Shortest Path game where the agents own edges. Archer et al. [1] proved that the VCG payment is not "frugal". Bikhchandani et al.

[5] show that the Shortest Path game does not satisfy the substitutes property. It follows from the discussion in the previous section that both the results are equivalent and any one of them coupled with our results imply the other.

Sections 4 and 3 characterize the games and their outcomes in which a VCG mechanism leads to a frugal payment. The equivalence of the three concepts – the substitutes property, the lattice structure of the core, and frugality – contributes insights in designing frugal mechanisms, and provides an alternative interpretation as well as perhaps a general framework for the results in [1].

### 5.1 Algorithmic Issues

In this section we look at how efficiently can the three conditions: frugality, agents being substitutes and core being a lattice be evaluated in light of the results in Section 4 and 3.

There does not seem to be an easy way to check if the substitutes condition hold. [4] shows that the VCG payoff being in the core and the agents being substitutes are equivalent conditions, which coupled with results in Section 4 reduces the problem of checking if the core is a lattice to checking if the Vickrey payoff is in the core or not. Another seemingly attractive possibility is to check for frugality. Note that the frugality ratio can be computed by determining the VCG payoffs. In general this is an optimization problem. However, as shown in [4], when agents are substitutes, the computation of the VCG payoffs can be done by solving a single linear program and its dual.

## 6 Conclusion and Future research

In this paper we correlate some concepts explored in mechanism design to those studied in game theory. We formalize the notion of frugality [1] and extend it to define frugality ratio. We establish the equivalence of *frugality* and a slightly different version of *agents are substitutes* [5] condition. We study the properties of the *core*, which defines the set of stable outcomes in coalition games with transferable utilities. We show that if the core forms a lattice, then the agents satisfy the substitutes condition. Hence, we establish the equivalence of the core being a lattice and agents satisfying the substitutes condition. We study the implications of our result using two representative graph problems- MST and shortest path. We observe that since MST satisfies the agents are substitutes condition, the VCG payment is frugal. Also, we observe that the two results: (1) shortest path does not satisfy agents are substitutes and (2) VCG payment for shortest path is not frugal are equivalent in light of our results.

An interesting open problem is whether frugality ratio and marignal frugality defined in [30] are equal, which in turn would imply that the frugoids condition in [30] and substitutes condition are equivalent.

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# A Frugality of Minimum cut

In this section we show that the Minimum cut has a frugality ratio of $k$, where $k$ is the number of edges in the cut set.

Let $n = 3k$, $k > 3$. Consider the graphs $G_1, G_2, G_3$, each being a $K_k$ and all the edges having weight $c_1$. We construct a graph $G$ on $n$ vertices from $G_1, G_2$ and $G_3$ in the following manner. Nodes in $G_1$ and $G_2$ are connected by a matching where each edge in the matching is of weight $c_3$ (let this matching be denoted by $M_1$). Further, nodes in $G_2$ and $G_3$ are connected by a matching where these new edges (denoted by $M_2$) have weight $c_2$. Finally, we have the constraint that $c_1 > c_2 > c_3 \geq 1$. Figure A shows the graph $G$ for $k = 4$, $c_3 = 1$, $c_2 = 2$ and $c_1 = 3$. Now, the

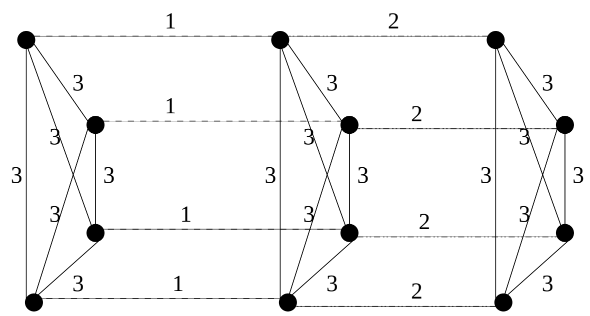


Figure 1: Construction of "bad case" for Minimum cut

optimal solution, $\mathcal{O}$ is $M_1$, while the second best solution is $= M_2$.[3] It can be seen that $\forall i \in \mathcal{O}$, $\pi^V(i) = c_2 k = C(M_2)$. Thus, the frugality ratio is $\geq k$. Proposition 1 implies that the Minimum cut has a frugality ratio of $k$.

[3] Note that all the vertices of $G_1, G_2, G_3$ have to be in one of the partitions induced by the min-cut as otherwise the weight of the cut would be $\geq c_1(k-1)$.