Coalitional Games on Graphs: Core Structure, Susbtitutes and Frugality

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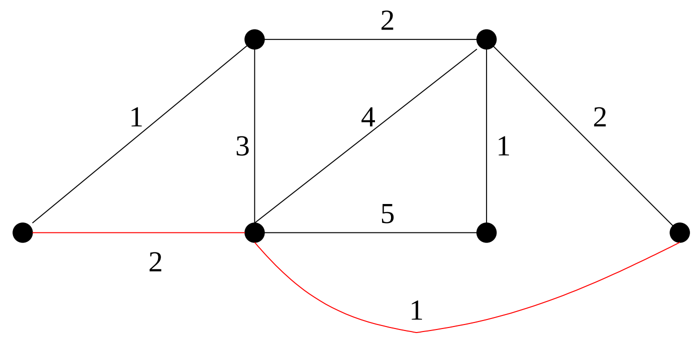
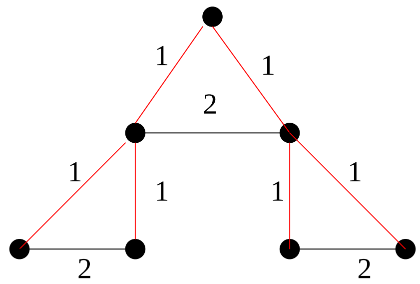
The Model

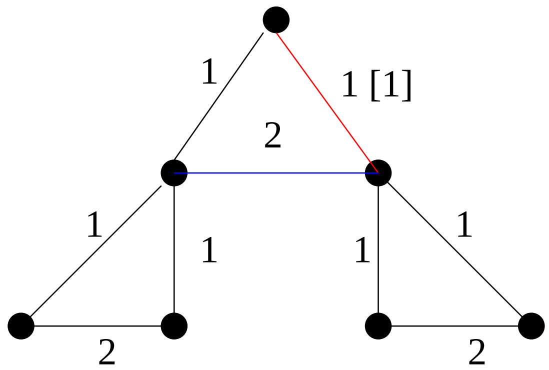
- Every edge in graph G owned by a (selfish) agent.
- Auctioneer wants a structure.
  - ◊ Shortest Path.
  - ◊ Minimum Spanning Tree.
- Auctioneer has budget U.
- Each agent i has (private) cost C_i.
- Each agent i gets payment P_i
  - ◊ Max payoff: $P_i - C_i$.
- Minimization Games.



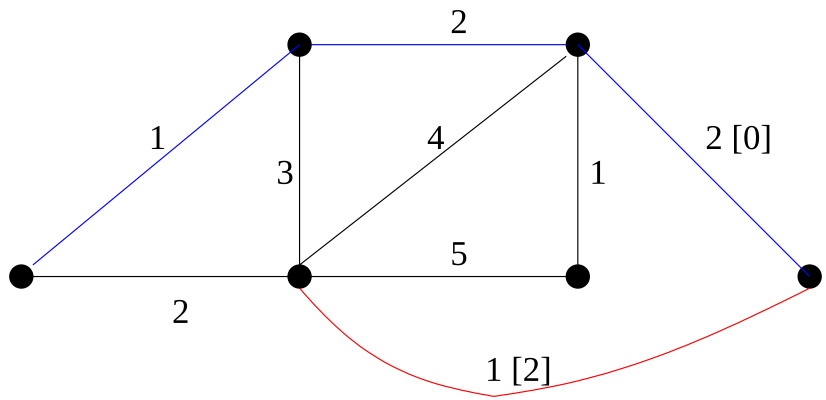


VCG Payoffs

- $\pi_i^V = \mathtt{cost}(\mathtt{Opt}(G - \{i\})) - \mathtt{cost}(\mathtt{Opt}(G))$.
  - ◇ π^V is the VCG payoff vector.



MST

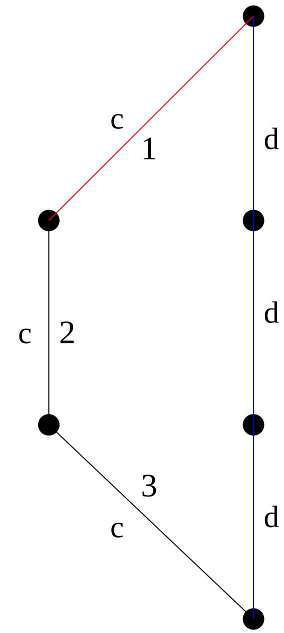


Shortest Path

For edge i the label is $C_i[\pi_i^V]$.

Frugality

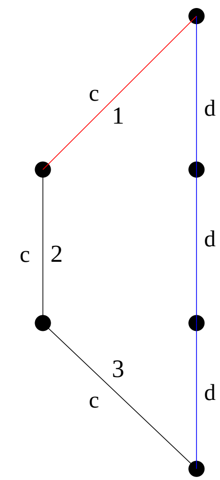
- Introduced by Archer and Tardos.
- $\sum_{i=1}^{3} \pi_i^V = 9(d-c)$.
- In general, VCG payoff can be k-times the difference in the second shorest and shortest path costs.
  - Shortest path has k edges.
- No bad example for MST.



$c < d.$

Agents are Substitutes

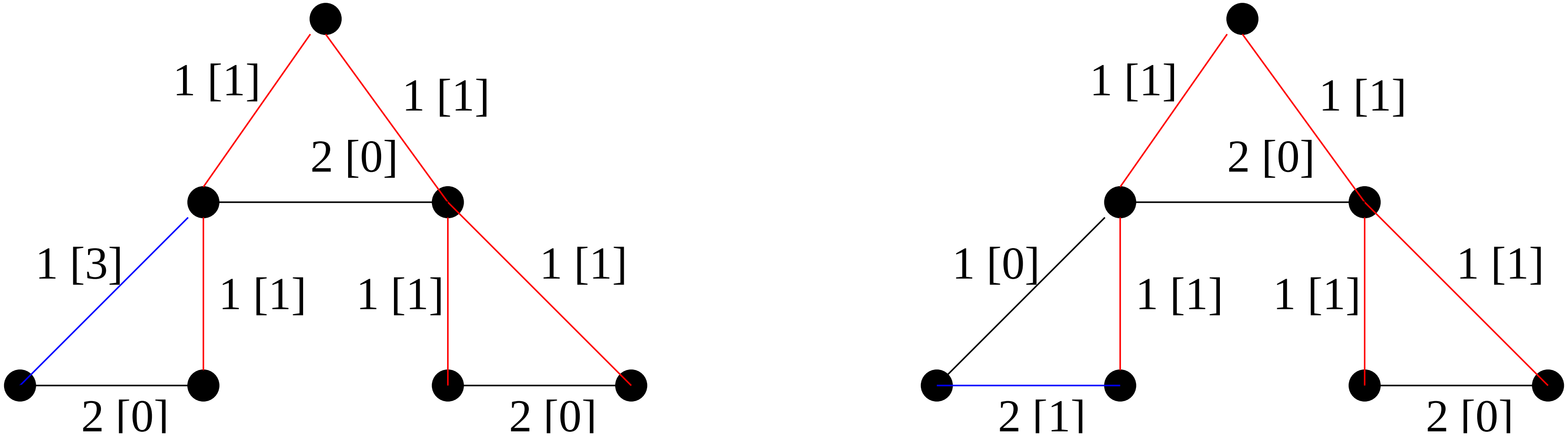
- $\sum_{i \in K} \pi_i^V \leq \mathsf{cost}(\mathtt{Opt}(G - K)) - \mathsf{cost}(\mathtt{Opt}(G))$.
  - Holds for any subset of agents K.
- Not true for Shortest Path.



  - $\sum_{i=1}^{3} \pi^V i = 9(d - c)$.
  - RHS= $3(d - c)$.
- True for MST.
  - Bikhchandani et. al.

The Core

- Set of "Stable" payoffs to agents.



- First payoff not stable.
  - Red+Blue edges in second case form a new coalition.
  - Auctioneer gets a "better" deal in the second case.

Our Results

Equivalent notions:

- Frugality.
- Agents are Substitutes (AS).
- Core being a lattice (wrt agents).

More on the Results

- AS $\Leftrightarrow$ Frugality.
  - A Formal definition for Frugality.
  - Proof follows from the definition.
  - Talwar has an alternate formulation.
- AS $\Leftrightarrow$ Core is a lattice.
  - AS and $\pi^1, \pi^2 \in CORE$ $\Rightarrow \max(\pi^1, \pi^2), \min(\pi^1, \pi^2) \in CORE$.
    - Bikhchandani and Ostroy.
  - $(\pi^1, \pi^2 \in CORE \Rightarrow \max(\pi^1, \pi^2), \min(\pi^1, \pi^2) \in CORE) \Rightarrow$ AS.
  - $(CORE, \preceq)$ is a lattice $\Leftrightarrow$ $(\pi^1, \pi^2 \in CORE$ $\Rightarrow \max(\pi^1, \pi^2), \min(\pi^1, \pi^2) \in CORE)$.

Current and Future Work

- UTCS Tech Report TR-02-60
  - ◊ `http://www.cs.utexas.edu/users/atri/papers/core.ps`
- Design of auctions where core is a lattice.
  - ◊ Truthfulness.
  - ◊ Other useful properties.