# Unstructured integrator using MPI

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# Motivation

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- Numerical solution

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- Analytic solution
  - Some functions have nonelementary antiderivatives, e.g., $\exp(x^2)$
- Numerical solution
  - Convergence, accuracy, and conservation

# Integration review

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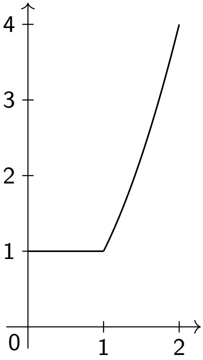
**Idea**

To compute $\int_I f \, d(\sigma)$, divide $I$ into "equal" disjoint subsets.

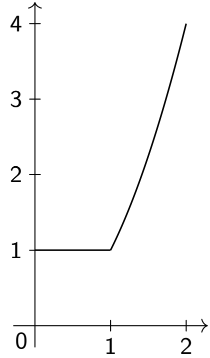
# Naïve (numerical) solution

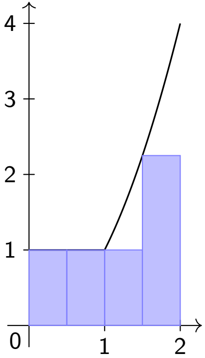
We want to integrate the function below



# Naïve (numerical) solution

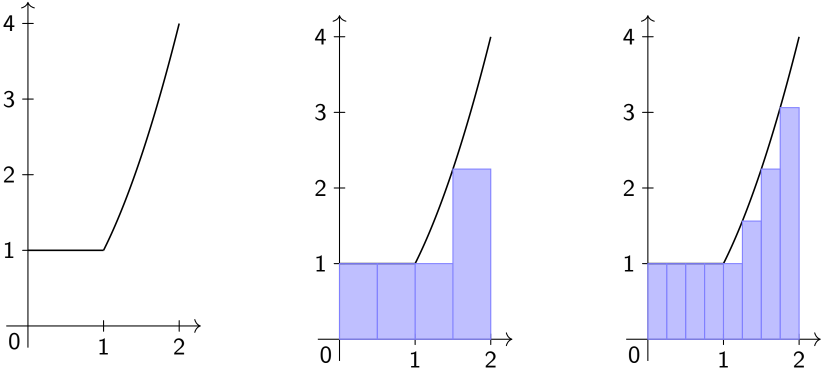
Divide the domain uniformly and calculate the area of each individual bar



# Naïve (numerical) solution

Refine the domain



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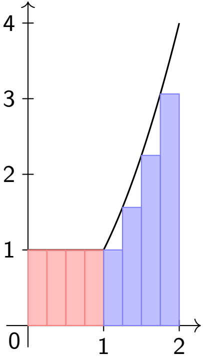
2. Coarse mesh is sufficient for constant functions
   - Refining the red region does not improve accuracy



# Unstructured integrator

An unstructured integrator using unstructured domain

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**Output:** integral of a given function over the given mesh elements

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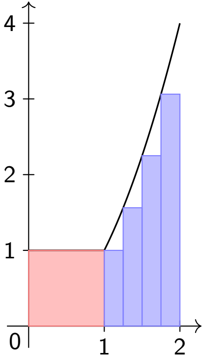

# Sequential algorithm

Suppose that the input contains chunks $M_1, M_2, \ldots, M_n$, where each $M_i$ is a (disjoint) collection of mesh elements.

1. Compute the integration of the given function $f$ over each $M_i$ and store the result in $s_i$

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1. Compute the integration of the given function $f$ over each $M_i$ and store the result in $s_i$
2. Compute the sum of $s_i$

# Parallel algorithm

Given $n$ processors, suppose that the input contains chunks $M_1, M_2, \ldots, M_n$, where each $M_i$ is a (disjoint) collection of mesh elements.

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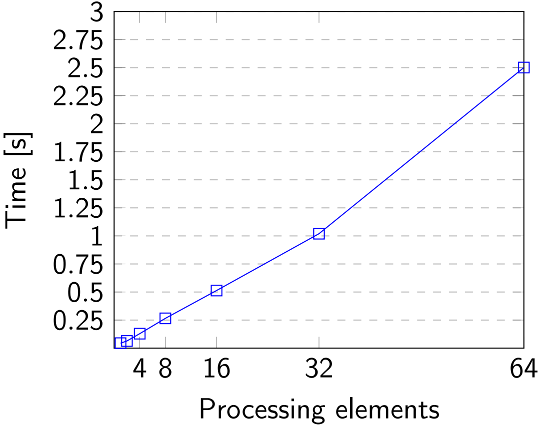
1. Distribute each chunk $M_i$ to a processor $P_i$
2. In parallel, each processor $P_i$ computes the integration of the given function $f$ over $M_i$
3. Compute the sum of the results via collective communication, and store the final result in the master processor $P_0$

# Strong scaling

Each process element computes 10 million mesh elements



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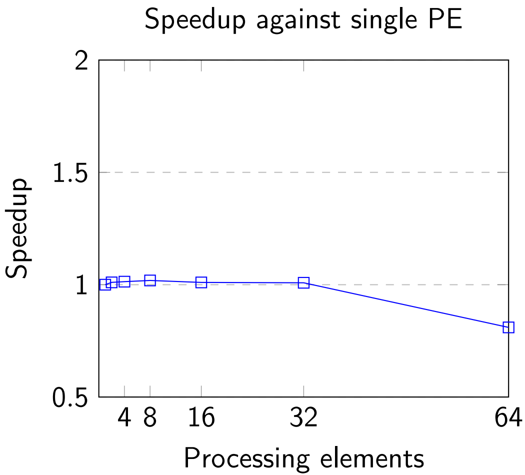
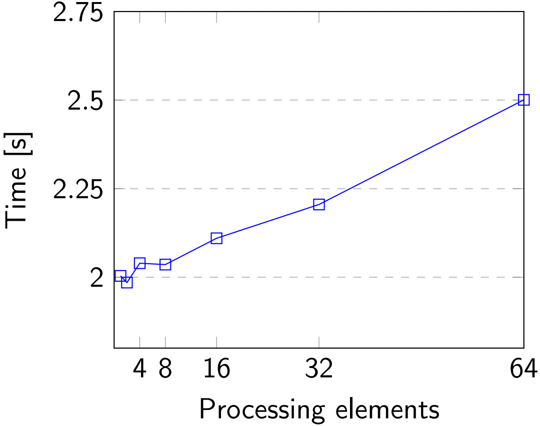


# Weak scaling

All processing elements collectively compute 640 million elements



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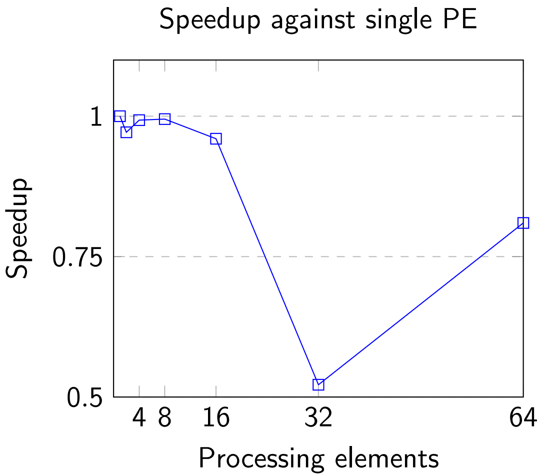

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  - Similar work has been done in [Jac06]

# Bibliography I


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