Tracking Quantum Circuits By Polynomials

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[1]Includes joint work with Amlan Chakrabarti, U. Calcutta, and prospectively students...

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- Central notion: **Quantum Circuits**.
- Hardly the only player—quantum adiabatic machines are closer to built and quantum communication systems are already deployed.

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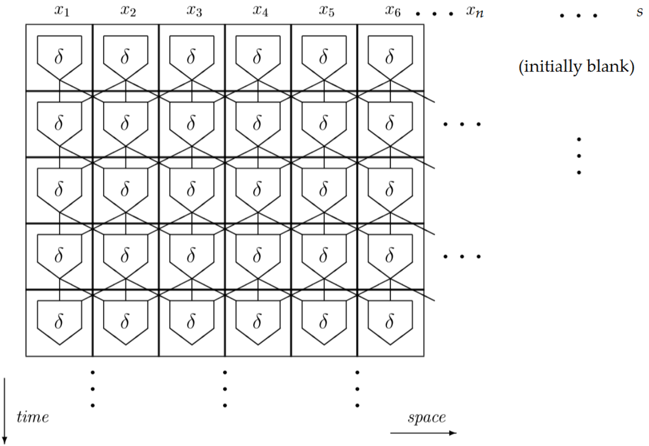
- Bits have no common identity across wires, but we can create a universal layout for Boolean circuits in which they *do* retain identity...

Turing "Cue Bits"



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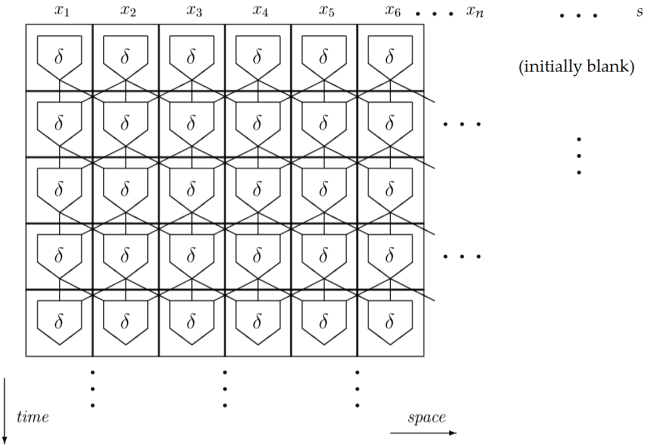
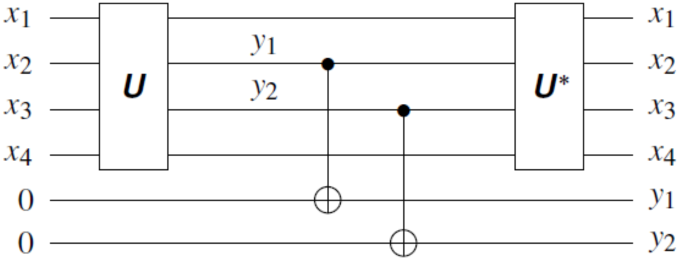


Space s, so $n - s$ "ancillary" cells. Can also be made *reversible*.

Quantum Circuits: similar picture, 90° flipped.



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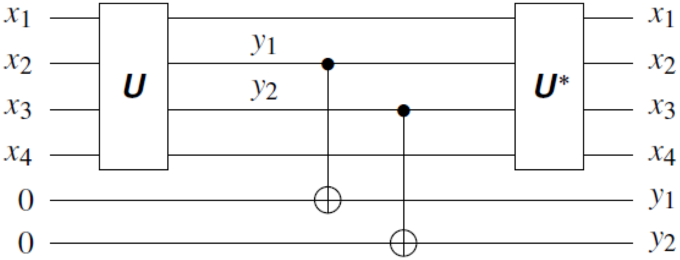

Must be reversible. In certain circumstances values $y_1, y_2, \ldots$ can be copied to extra lines as "$f(x)$." Then reversing the gates re-creates the input x so the whole mapping is invertible ("copy-uncompute trick").

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- Under the hood are $S = 2^s$ complex entries of a unit **state vector**.

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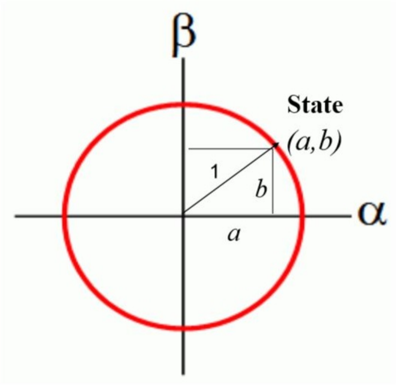

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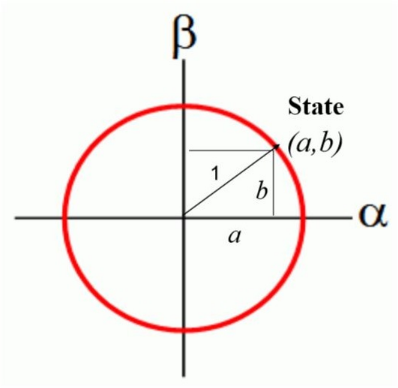

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- A *qubit* is a physical entity that combines as described by a 2-vector $(a, b) = ae_0 + be_1$ over $\mathbb{C}$ and yields two *observations*: e_0 with probability $|a|^2$ or e_1 with probability $|b|^2$.

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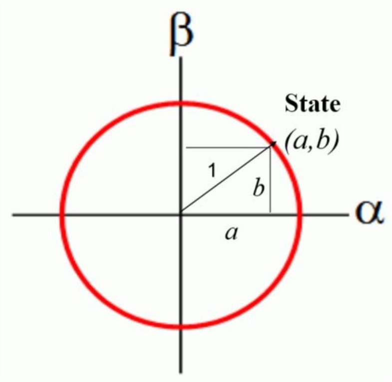

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- A *qutrit* yields 3 results and is (a, b, c) where $|a|^2 + |b|^2 + |c|^2 = 1$.

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- But also common: $\mathsf{Y} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $\mathsf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\mathsf{S} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$.

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With $k = 2$ qubits, $K = 4$. "Controlled Not" showing quantum coordinates:

$$\mathsf{CNOT} = \begin{array}{c|cccc|} & 00 & 01 & 10 & 11 \\ \hline 00 & 1 & 0 & 0 & 0 \\ 01 & 0 & 1 & 0 & 0 \\ 10 & 0 & 0 & 0 & 1 \\ 11 & 0 & 0 & 1 & 0 \\ \hline \end{array}$$

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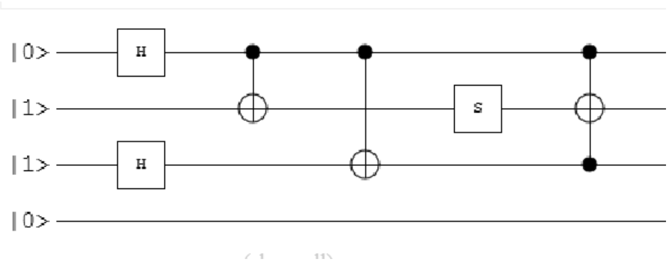
- $\mathsf{H} + \mathsf{CNOT}$ is not quantum universal; it recognizes a proper subclass of P.

Example Quantum Circuit



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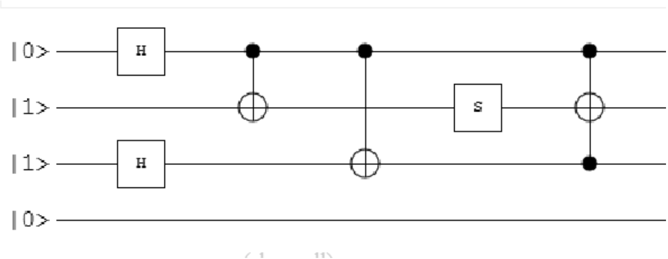

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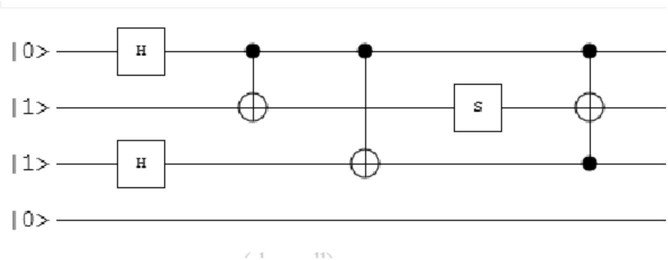

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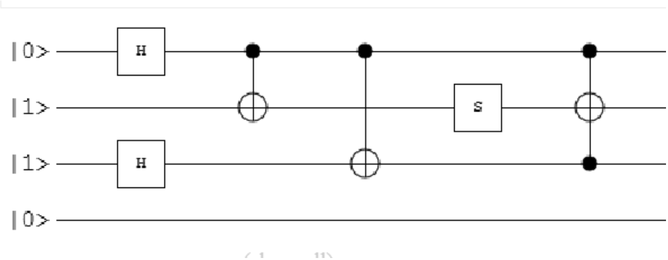

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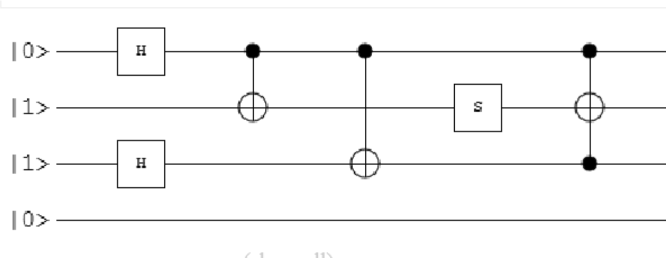

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4. After the S in stage 4, a TOF with controls on 1,3 and target on 2. The whole C computes a unitary U_C.

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- Can project as amplitudes: $C(x) \mapsto (z'_0, \ldots, z'_{2^r-1})$ where $|z_d|^2$ is the probability of outcome $d \in \{0,1\}^r$.

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- For outcome $d \in \{0,1\}^r$ on r-many designated qubit lines, $\Pr[C(x) \to d] = \sum_{b \sqsupseteq d} |z_b|^2$.
- Can project as amplitudes: $C(x) \mapsto (z'_0, \ldots, z'_{2^r-1})$ where $|z_d|^2$ is the probability of outcome $d \in \{0,1\}^r$.
- Call this amplitude z_d as $A[C(x) \mapsto d]$.

BQP

Definition

A language L belongs to BQP if there are poly-time uniform quantum circuits C_n for each n such that forall n and inputs $x \in \{0,1\}^n$, designating qubit 1 for yes/no output:

$$\begin{aligned} x \in L &\quad\Longrightarrow\quad \Pr[C_n(x) \mapsto 1] > \frac{3}{4}, \\ x \notin L &\quad\Longrightarrow\quad \Pr[C_n(x) \mapsto 1] < \frac{1}{4}, \end{aligned}$$

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- But no classical way known without paying $\approx 2^n$ time overhead as for factoring.

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- But in many cases, **#SAT solvers** may be effective.
- Computation by solver more "offline" than managing 2^n-sized arrays.
- P_C may yield other algebraic and physical info about C, including about entanglement.

The Basic Theorem

Theorem (Dawson et al.. 2004, implicitly before?)

Given C built from TOF *gates and h-many* H *gates, we can efficiently compute a polynomial $P_C(x_1, \ldots, x_n, y_1, \ldots, y_h, z_1, \ldots, z_r)$ and a constant R (here, $R = \sqrt{2^h}$) such that for all $x \in \{0,1\}^n$ and $z \in \{0,1\}^r$,*

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- Hence cannot approximate the difference by approximating each term. Need exact solution counting.

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How it Works: The "Quantum Grille"

Consider a general 2×2 matrix A. Assign an indicator variable u to its input and y to its output:

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- If gate is deterministic, can *substitute* y by expression in u.
- Every qubit at every stage has a well-defined *local* "algebraic value."

How it Works: Two-Qubit Gate

Now let A be a general 4×4 matrix. Assign indicator variables u_1, u_2 to the two incoming qubits and y_1, y_2 to their outgoing selves:

	$(1-y_1)(1-y_2)$	$(1-y_1)y_2$	$y_1(1-y_2)$	$y_1 y_2$
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- The effect on P_C is then how the substituted terms are input to subsequent Hadamard and other kinds of gates.

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	$2 - 2y^2$	$y^2 + y$	$y^2 - y$
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Now P_A has 36 terms. But it simplifies for certain matrices:

$$A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}; \qquad P_A = 4 + 2\sqrt{3}iuy + 2u^2y^2.$$

Here $\omega = \frac{1}{2}(-1 + \sqrt{3}i)$, and the "$\mathsf{H}_3$ multiplier" is $2/\sqrt{3}$.

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- Whereas $c = 0$ zeroes all such terms, so he only effect is to inflate R.

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Theorem

Given any C of minphase K, we can efficiently compute a polynomial $Q_C(x_1, \ldots, x_n, y_1, \ldots, y_h, z_1, \ldots, z_r, w_1, \ldots, w_t)$ over $\mathbb{Z}_K$ and a constant R such that for all $x \in \{0,1\}^n$ and $z \in \{0,1\}^r$,

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where $Q_C = \sum_{gates\ g} q_g + \sum_{constraints\ c} q_c$ has bounded degree.

Thus we can do all calculations using Q_C over $\mathbb{Z}_K$.

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Open: replace K by $\log K$ in the time? Affirmative for $A[C(x) \mapsto z]$.

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- Hence [CCLL] gives poly-time simulation by soution counting in $\mathbb{Z}_4$.

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