Efficient Memoization for Approximate Function Evaluation over Sequence Arguments

AAIM 2014

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6. And we need good approximation to $\mu(\cdots)$ (only) under distributions $D(x)$ controlled by a few model-specific parameters,

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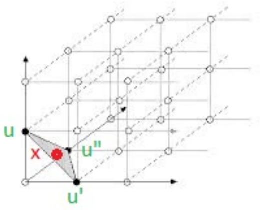

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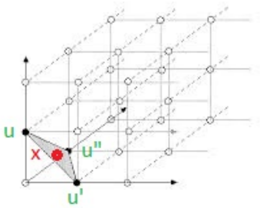

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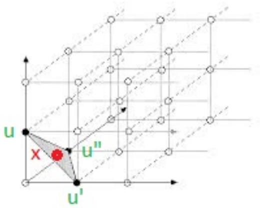

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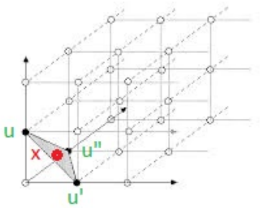

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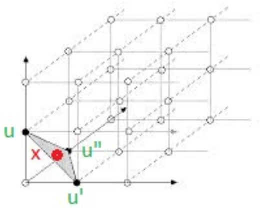

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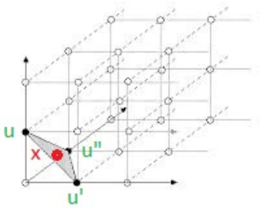

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5. What if the grid is warped "similarly" to f?

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6. Derived Outputs:
   - Aggregate statistics: *move-match* MM, *average error* AE, ...
   - Projected confidence intervals for those statistics.
   - "Intrinsic Performance Ratings" (IPR's).

How the Model Operates

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6. Note f is *symmetric*, so x can be an ordered sequence.

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7. Also: Current Expansion uses data for each depth d.

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6. **"Universal Guess"**: In the first Taylor term, use

$$\frac{\partial f}{\partial x_i} \approx \frac{1}{i} a_i = \frac{1}{i}(1 - x_i).$$

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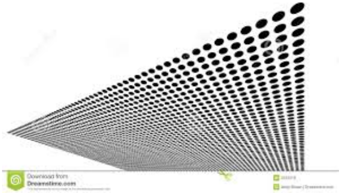

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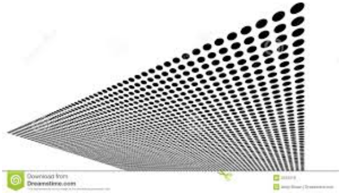

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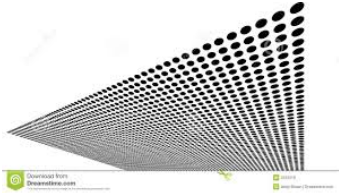

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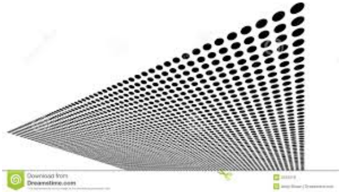

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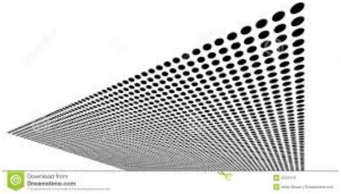

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5. How to make the computation of nearby gridpoints efficient?

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2. "**Nearest Neighbor**" is defined by a *nondecreasing* sequence using only values from x^+ and x^-. Always $u_1 = 0.0 = x_1^+ = x_1^-$.

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1. **Bounds** x^+ and x^- are well-defined by rounding each coordinate up/down to a gridpoint.
2. "**Nearest Neighbor**" is defined by a *nondecreasing* sequence using only values from x^+ and x^-. Always $u_1 = 0.0 = x_1^+ = x_1^-$.
3. Start with x^-, but "round up" when the rounding-down deficiency exceeds some weighted threshold.

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7. Combine with "universal gradient" idea, or even ignore said idea.

Results So Far...

1. We ran experiments under a randomized distribution D_ϵ in which $r \in [x_{i-1}, 1]$ is sampled uniformly and

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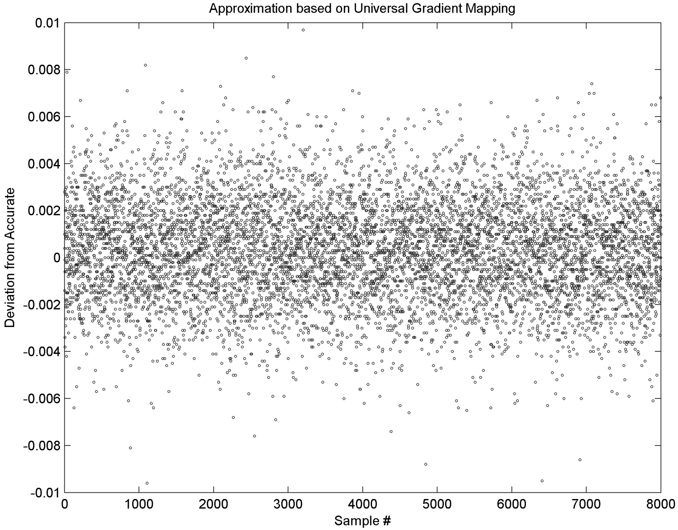
Results for NN+UG



Results for Just NN