Combinatorial Invariants and Quantum Circuits

(With speculation on the status of "quantum supremacy")

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[1]Joint work with Amlan Chakrabarti, University of Calcutta, and Chaowen Guan, University of Cincinnati

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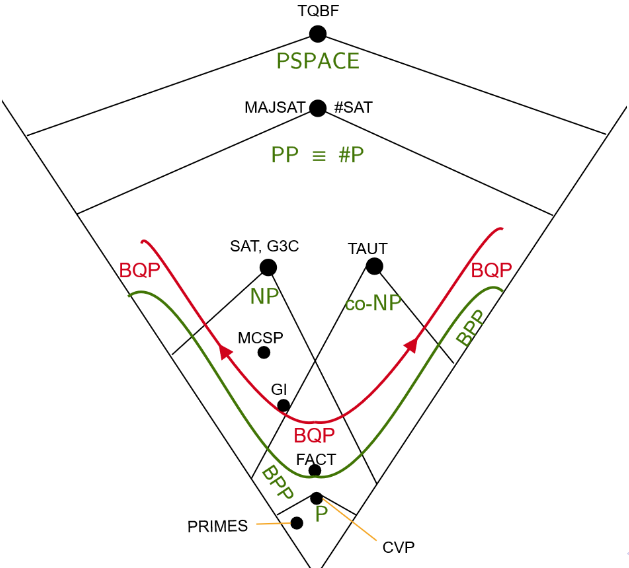
The Complexity Class Neighborhood...



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Can we capture **quantum circuits** by combinatorial invariants that lead to new heuristics for *classically* simulating them?

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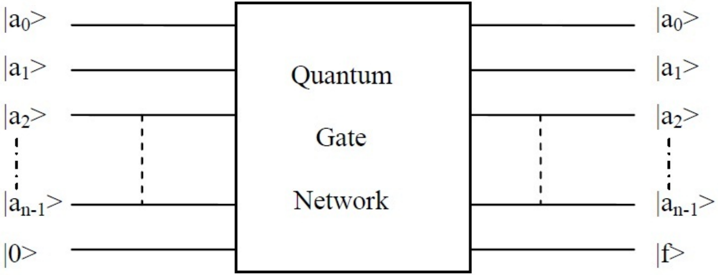
Three kinds of combinatorial invariants for these circuits:

1. Phase-and-location ("Feynman Path") polynomials.
2. Graphs, and their generalization to *graphical 2-polymatroids*.
3. Versions of the Tutte Polynomial associated to such graphs and matroids.

Quantum Circuits

Quantum circuits look more constrained than Boolean circuits:

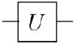


But Boolean circuits look similar if we do Savage's TM-to-circuit simulation and call each *column* for each tape cell a "cue-bit."

Quantum Gates—three slides by M. Rötteler

Quantum gates

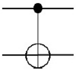
single qubit operation:



controlled-NOT:

control

target



unitary matrix $= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

controlled-U:

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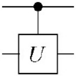
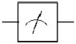


unitary matrix $= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & U_{00} & U_{01} \\ 0 & 0 & U_{10} & U_{11} \end{pmatrix}$

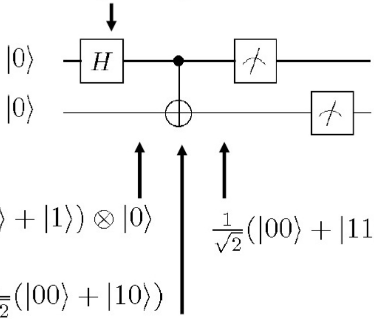
measurement in the $|0\rangle, |1\rangle$ **basis:**



[Slides concept by D. Bacon, U Washington]

Quantum circuit example

$$H \otimes \mathbf{1}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \mathbf{1}_2$$



$\text{Prob}(\text{meas. } |00\rangle) = 0.5$

$\text{Prob}(\text{meas. } |11\rangle) = 0.5$

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$$

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

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$$|00\rangle \to |00\rangle$$
$$|01\rangle \to |01\rangle$$
$$|10\rangle \to |11\rangle$$
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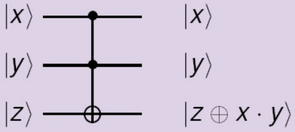
Toffoli Gate

The Toffoli gate "TOF"

x	y	z	x'	y'	z'
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	0



Theorem (Toffoli, 1981)

Any reversible computation can be realized by using TOF gates and ancilla (auxiliary) bits which are initialized to 0.

Slides by Martin Rötteler

Some More Gates

$$\mathsf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathsf{Y} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \mathsf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\mathsf{S} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad \mathsf{T} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}, \quad \mathsf{R_8} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/8} \end{bmatrix},$$

$$\mathsf{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathsf{CZ} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \mathsf{CS} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{bmatrix}.$$

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- Note: $\mathsf{T}^2 = \mathsf{S}$, $\mathsf{S}^2 = \mathsf{Z}$, $\mathsf{Z}^2 = \mathsf{I} = \mathsf{H}^2$, and $\mathsf{CS}^2 = \mathsf{CZ}$.

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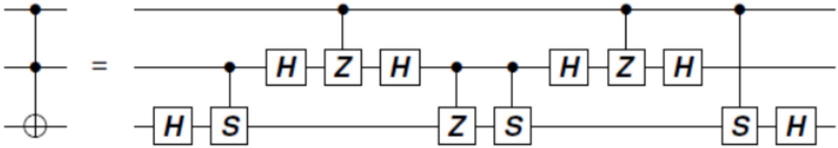

I. Feynman Path Polynomials

Let C have "minphase" $K = 2^k$ and let F embed K-th roots of unity ω.

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- H + CS has $k = 2$, $K = 4$.
- H + CNOT + T has $k = 3$, $K = 8$.

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Theorem (RC 2007-09, extending Dawson et al. (2004) over $\mathbb{Z}_2$)

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I. Feynman Path Polynomials

Let C have "minphase" $K = 2^k$ and let F embed K-th roots of unity ω.

- H + Tof has $k = 1$, $K = 2$.
- H + CS has $k = 2$, $K = 4$.
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where C has h nondeterministic (Hadamard) gates and $y \in \{0,1\}^h$.

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Given C and K, we can efficiently compute a polynomial $Q_C(x_1, \ldots, x_n, y_1, \ldots, y_h, z_1, \ldots, z_n, w_1, \ldots, w_t)$ of degree $O(1)$ over $\mathbb{Z}_K$ and a constant R' such that for all $x, z \in \{0,1\}^n$:

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- T-gate also goes cubic.

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Theorem (C. Guan in RCG 2018)

Given C, n, K, h as above, we can quickly build a Boolean formula ϕ_C in variables $y_1, \ldots, y_h$, together with substituted-for $x_1, \ldots, x_n$, $z_1, \ldots, z_n$, and other "forced" variables such that for all $x, z \in \{0,1\}^n$:

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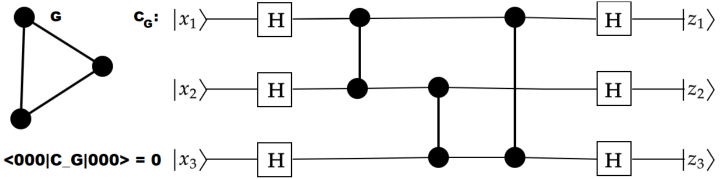
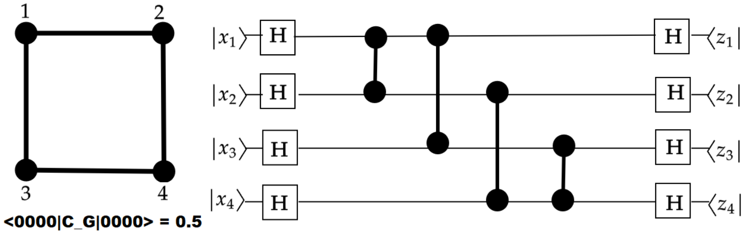
$$|\langle z \mid C \mid x \rangle|^2 = \frac{1}{R} \left(\#sat(\phi_C) - \#sat(\phi'_C) \right).$$

II. Strong Simulation of Graph State Circuits

Computing amplitudes $\langle z \mid C \mid x\rangle$ for Clifford circuits C can be efficiently reduced to computing $\langle 0^n \mid C_G \mid 0^n\rangle$ for **graph-state circuits** C_G of graphs G, using H and CZ gates, as exemplified by:





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For n-qubit stabilizer circuits of size s, $\langle z \mid C \mid x \rangle$ *can be computed in* $O(s + n^\omega)$ *time, where* $\omega \leq 2.37155...$ *is the exponent of multiplying* $n \times n$ *matrices.*

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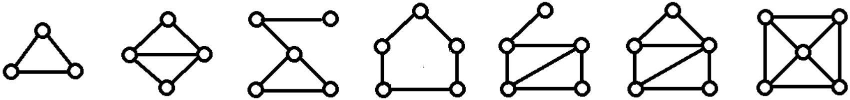

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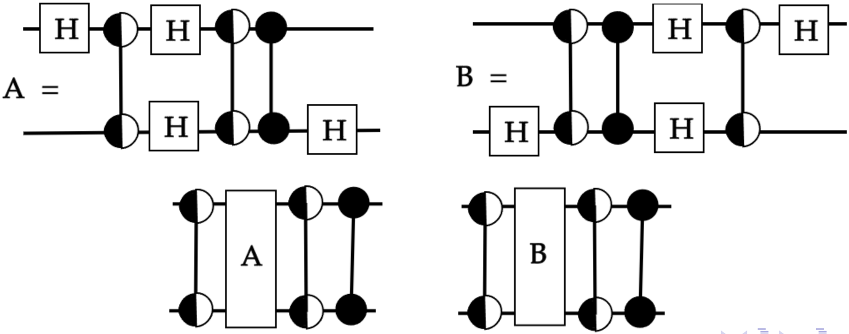

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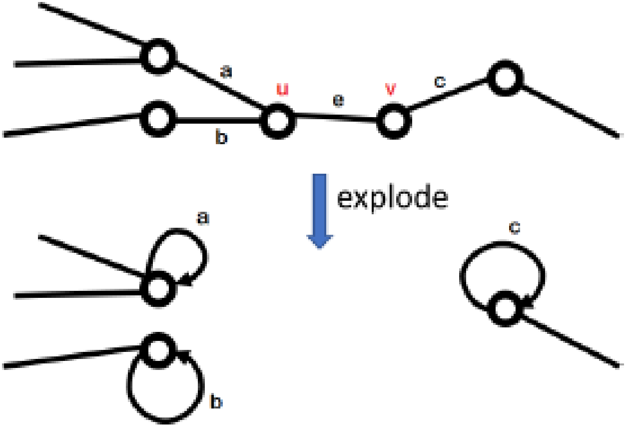
- Else, if G has an edge e between two nodes, define

$$Q_G(x) = Q_{G\setminus e} - \frac{1}{2}Q_{G\setminus\setminus e}.$$

Here $G \setminus e$ means deleting edge e, but $G \setminus \setminus e$ means "**exploding**" e. The recursion is *confluent*—order of choosing e does not matter.

Exploding an Edge



Properties of the Amplitude Polynomial

We connect Q_G to the **rank-generating polynomial** S_G of J. Oxley and G. Whittle, and a variant form S'_G, by

Theorem

$$Q_G(x) = \left(\frac{1}{\alpha}\right)^n S'_G(\alpha x, -\alpha) = \left(\frac{1}{\alpha}\right)^n S_G(\alpha x, -\alpha)(\alpha x)^r,$$

where $\alpha = -i\sqrt{2}$ *and* r *is the number of isolated nodes of* G.

Drawing on their definition of a *generalized Tutte-Grothendieck invariant* (GTGI), we show:

Theorem

Q_G *is a GTGI of graphs* G *and belongs to the first of only two possible families of GTGIs that can arise from G2PMs*

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- Does $\gamma(P_C)$ witness a physical nonlinearity associated with operating quantum circuits C?

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- Thanks for listening. Q & A.