Using the Shape of Space for Shortcuts

Speeding up regressions on millions of chess positions

Kenneth W. Regan[1]

FWCG, University at Buffalo, 23 October 2015

[1] Joint with Tamal T. Biswas, AAIM 2014, *Theoretical Computer Science* (in press).

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3. **Cheat.**

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- Correspondence e(s,c)–Elo comes out superbly linear under exact runs on smaller data, so—provided the approximations avoid systematic bias across Elo levels—they will help correct each other.

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6. And we need good approximation to $\mu(\cdots)$ (only) under distributions $D(x)$ controlled by a few model-specific parameters,

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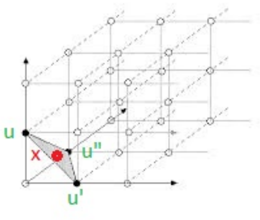

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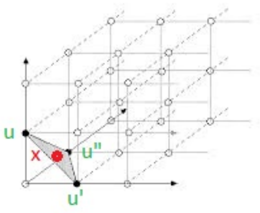

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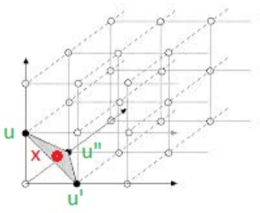

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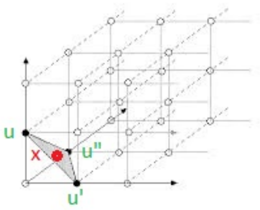

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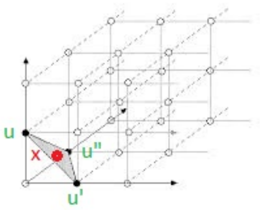

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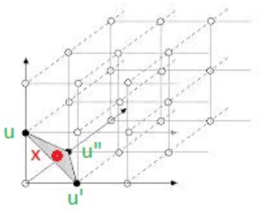

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5. What if the space Γ is warped "similarly" to f? Can we roughly use $\partial\Gamma$ in place of ∂f?

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6. Derived Outputs:
   - Aggregate statistics: *move-match* MM, *average error* AE, ...
   - Projected confidence intervals for those statistics.
   - “Intrinsic Performance Ratings” (IPR’s).

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- Given $(x_1, \dots, x_i, \dots, x_\ell)$, fit subject to $\sum_i p_i = 1$ to find p_1. Other p_i follow by $p_i = h^{-1}(h(p_1)(1 - x_i))$.

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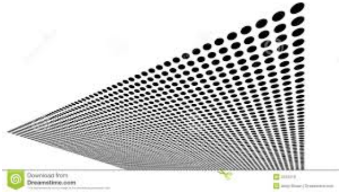

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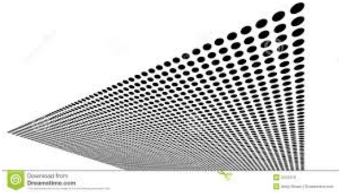

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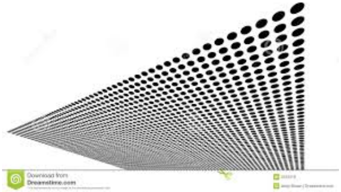

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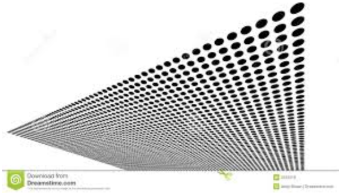

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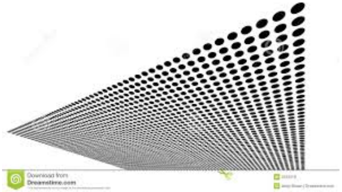

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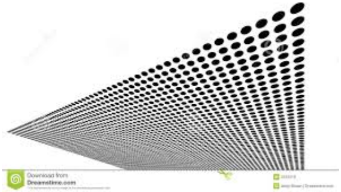

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6. **"Universal Guess"**: In the first Taylor term, use
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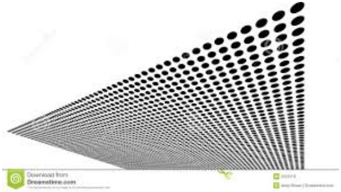
Tailoring the Grid



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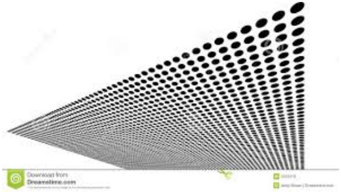

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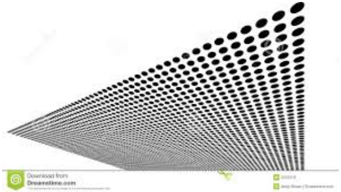

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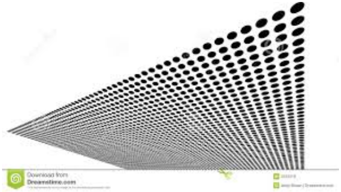

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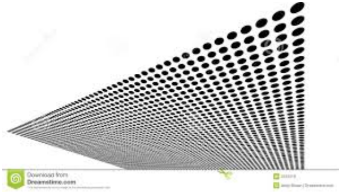

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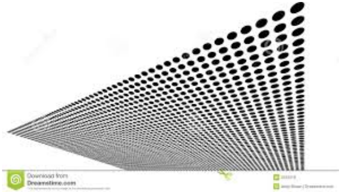

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7. Combine with "universal gradient" idea, or even ignore said idea.

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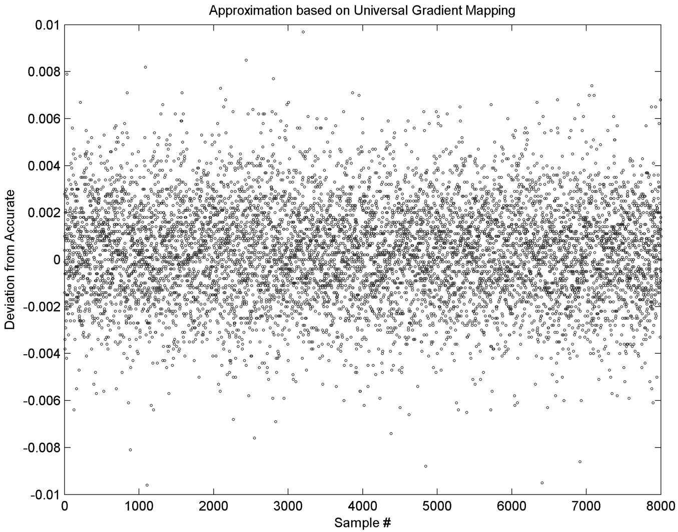
Results for NN+UG



Results for Just NN