Tracking Quantum Circuits By Polynomials

Oxford University OASIS Seminar

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[1] Includes joint work with Amlan Chakrabarti, U. Calcutta

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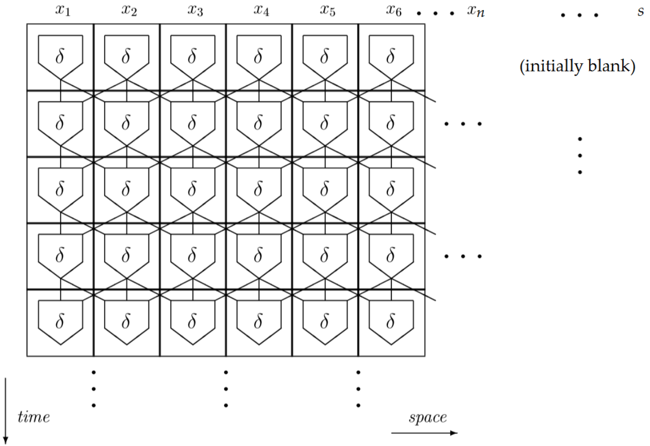
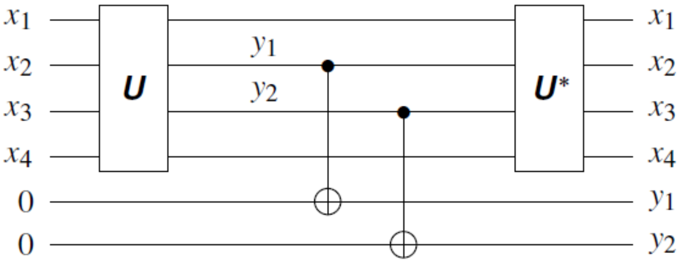
- Bits have no common identity across wires, but they *can*...

Turing "Cue Bits"



Space s, so $n - s$ "ancillary" cells.

Quantum Circuits: similar picture



Example also shows the **copy-uncompute trick**.

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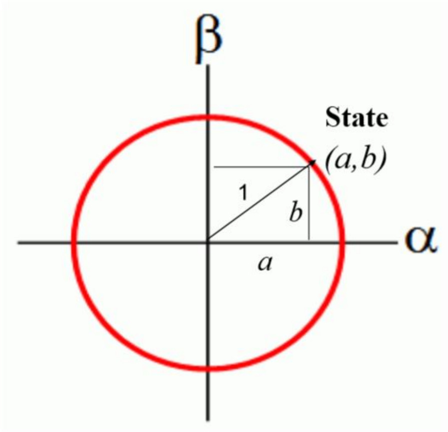
- Under the hood are(??) $S = 2^s$ complex entries of a unit **state vector**.

A Qubit

Quantum Bits, e.g. spins.



Probability of observing Alpha is a-squared, Beta is b-squared. By Pythagoras, these add to 1.

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- But also common: $\mathsf{Y} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$, $\mathsf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\mathsf{S} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$.

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With $k = 2$ qubits, $K = 4$. "Controlled Not" showing quantum coordinates:

$$\mathsf{CNOT} = \begin{array}{c|cccc|} & 00 & 01 & 10 & 11 \\ \hline 00 & 1 & 0 & 0 & 0 \\ 01 & 0 & 1 & 0 & 0 \\ 10 & 0 & 0 & 0 & 1 \\ 11 & 0 & 0 & 1 & 0 \\ \hline \end{array}$$

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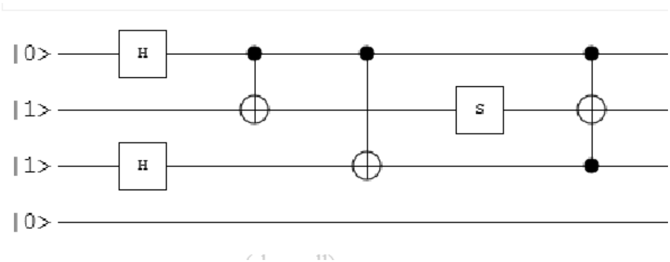
- $\mathsf{H} + \mathsf{CNOT}$ is not quantum universal; it recognizes a proper subclass of P.

Example Quantum Circuit



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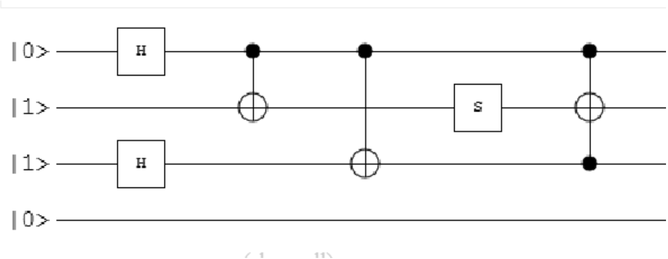

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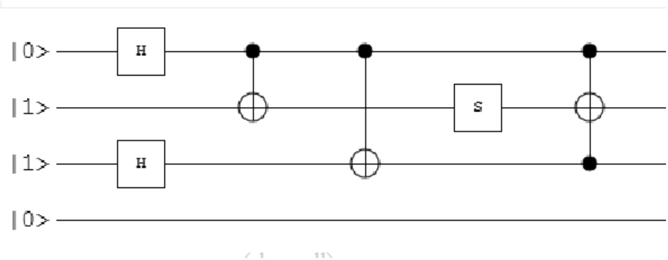

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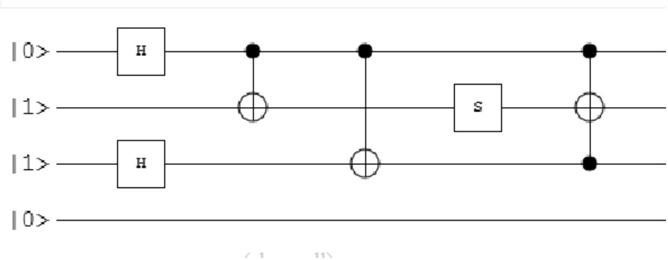

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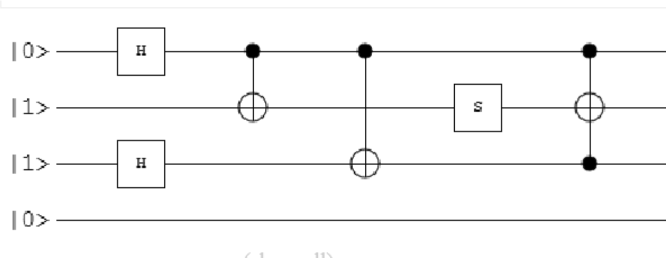

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4. After the S in stage 4, a TOF with controls on 1,3 and target on 2. The whole C computes a unitary U_C.

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- Call this amplitude z_d as $A[C(x) \mapsto d]$.

BQP

Definition

A language L belongs to BQP if there are poly-time uniform quantum circuits C_n for each n such that forall n and inputs $x \in \{0,1\}^n$, designating qubit 1 for yes/no output:

$$\begin{aligned} x \in L &\implies \Pr[C_n(x) \mapsto 1] > \frac{3}{4}, \\ x \notin L &\implies \Pr[C_n(x) \mapsto 1] < \frac{1}{4}, \end{aligned}$$

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- $\mathsf{P} \subseteq \mathsf{BPP} \subseteq \mathsf{BQP} \subseteq \mathsf{PP} \equiv_T^p \#\mathsf{P}$.
- No evidence for $\mathsf{NP} \subseteq \mathsf{BQP}$, nor $\mathsf{BQP} \subseteq \mathsf{PH}$.

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$$p_x = p_C(x_1, \ldots, x_n).$$

Used in quantum query complexity lower bounds, but *building* p_C is hard.

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$$N_{P,x}[a] = |\{y \in \{0,1\}^h : P(x,y) = a\}|.$$

This is a $\#\mathsf{P}$ function.

The Basic Theorem

Theorem (Dawson et al.. 2004, implicitly before?)

Given C *built from* TOF *gates and* h*-many* H *gates, we can efficiently compute a polynomial* $P_C(x_1, \dots, x_n, y_1, \dots, y_h, z_1, \dots, z_r)$ *and a constant* R *(here,* $R = \sqrt{2^h}$*) such that for all* $x \in \{0,1\}^n$ *and* $z \in \{0,1\}^r$*,*

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- Hence cannot simply use Stockmeyer's approximation of counting to get $\mathsf{BQP}\subseteq\Sigma_3^p\cap\Pi_3^p$.

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Theorem

Given any C of minphase K, we can efficiently compute a polynomial $Q_C(x_1,\ldots,x_n,y_1,\ldots,y_h,z_1,\ldots,z_r,w_1,\ldots,w_t)$ over $\mathbb{Z}_K$ and a constant R such that for all $x \in \{0,1\}^n$ and $z \in \{0,1\}^r$,

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where $Q_C = \sum_{gates\ g} q_g + \sum_{constraints\ c} q_c$ has bounded degree.

My trick: Given a **constraint** c with values $0 =$ fail, $1 =$ OK, add

$$q_c = w_0(1-c) + 2w_1(1-c) + 4w_2(1-c) + \cdots + 2^{k-1}w_{k-1}(1-c).$$

Then $c = 0 \implies$ binary assignments to $w_0,\dots,w_{k-1}$ run through all K values $\implies$ the entire sum over y,w cancels. Whereas $c = 1$ zeroes all such terms, so he only effect is to inflate R.

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Open: replace K by $\log K$ in the time? Affirmative for $A[C(x) \mapsto z]$.

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- Possibly quantify the "entangling capacity" of C?